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APPROXIMATION BY HOLOMORPHIC FUNCTIONS ON CERTAIN PRODUCT SETS IN C^N

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In this paper we prove several theorems concerning approximation by holomorphic functions on product sets in C^n where each factor is either a compact plane set or the closure of a strongly pseudoconvex domain. In particular we show that every continuous function which is locally approximable by holomorphic functions on such a set is globally approximable. Our results depend on a generalization of a theorem of Andreotti and Stoll on bounded solutions of the inhomogeneous Cauchy-Riemann equations on certain product domains.

1. Statement of results. If K is a compact set in C^n let $C(K)$ denote the Banach space of continuous complex-valued functions on K with the uniform norm, and let $H(K)$ denote the closure in $C(K)$ of the space of functions which are holomorphic in some neighborhood of K. When $n = 1$, each function in $H(K)$ is the uniform limit of a sequence of rational functions which are finite on K and the spaces $H(K)$ (usually denoted $R(K)$ in this instance) have been extensively studied. In particular, the following properties of $H(K)$ are well-known in the case $n = 1$ (cf. Chapter 3 of [2]):

(1) *If U is a neighborhood of K, $f \in C^1(U)$, and $\partial f/\partial \bar{z} \equiv 0$ on K, then $f|K \in H(K)$.*

(2) *If $f \in C(K)$ and if for each $x \in K$ there is a neighborhood U_x of x in C such that $f \in H(K \cap \bar{U}_x)$, then $f \in H(K)$.*

(3) *If μ is a complex Borel measure on K, then $\mu = \partial\hat{\mu}/\partial\bar{z}$ where*

$$\hat{\mu}(z) = -\frac{1}{\pi}\int_K (\zeta - z)^{-1} d\mu(\zeta)$$

is locally integrable on C. A measure μ is an annihilating measure for $H(K)$ (i.e., $\int f d\mu = 0$ for all $f \in H(K)$) if and only if $\hat{\mu}$ is supported on K.

Properties (1)-(3) are not valid for arbitrary compact sets of C^n, even if one restricts one's attention to holomorphically convex, or even polynomially convex sets. A celebrated example of Kallin [6] shows that (2) fails in general for polynomially convex compact sets. Also, Chirka [3], by modifying her example, has shown that for each positive integer s there is a compact holomorphically convex set K^s in C^3 and a function $f_s \in C^\infty(K^s)$ such that $f_s \notin H(K^s)$ although $\bar{\partial} f_s$

vanishes on K^s to order s.[1]

In this paper we consider compact sets $K \subset \boldsymbol{C}^n$ of the form $K = K_1 \times \cdots \times K_r$ where each K_i is either a compact set in $\boldsymbol{C}$ or the closure of a strongly pseudo-convex domain in $\boldsymbol{C}^{n_i}$. For such K we prove the following theorems:

THEOREM 1.1. *If U is a neighborhood of K, $f \in C^{r+1}(U)$, and $\partial^\alpha f/\partial \bar{z}^\alpha \equiv 0$ on K for all $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\sum \alpha_j \leqq r$, then $f \in H(K)$.*

THEOREM 1.2. *If $f \in C(K)$ and if for each $x \in K$ there is a neighborhood U_x such that $f \in H(K \cap \bar{U}_x)$, then $f \in H(K)$.*

THEOREM 1.3. *A measure μ on K is an annihilating measure for $H(K)$ if and only if there exist distributions $\lambda_1, \cdots, \lambda_n$ of order $\leqq r - 1$ with support in K such that $\mu = \sum \partial\lambda_j/\partial\bar{z}_j$.*

It is possible that Theorem 1.1 remains valid if f is merely required to satisfy $\partial f/\partial \bar{z}_j \equiv 0$ on K, $1 \leqq j \leqq n$. We know of no counterexample.

Theorem 1.2 implies an approximation theorem of the Keldysh-Mergelyan type if, in addition to the above hypotheses, K has the "segment property", i.e., if there is an open cover $\{U_i\}$ of ∂K and corresponding vectors $\{w_i\}$ such that for $0 < t < 1$ $z + tw_i$ lies in the interior of K whenever $z \in K \cap U_i$. In this case every function which is continuous on K and holomorphic in the interior of K satisfies the hypotheses of Theorem 1.2 so lies in $H(K)$. In particular, if K is a product of smoothly bounded domains, then K has the segment property. The case $r = 1$ when K is the closure of a strongly pseudo-convex domain in $\boldsymbol{C}^n$ has been treated by Lieb [8] and Kerzman [7]. We use their method to prove Theorem 1.2.

If we consider Theorem 1.3 in the case $r = 1$ we conclude that each annihilating measure for $H(K)$ where K is the closure of a strongly pseudo-convex domain is the $\bar{z}$-divergence of an n-tuple of measures supported on K (distributions of order 0). This implies the following localization theorem for annihilating measures which is well-known in case $n = 1$ [2, Lemma 3.2.11] and which is a sort of dual version of Theorem 1.2, which it clearly implies.

THEOREM 1.4. *Let K be the closure of a strongly pseudoconvex domain in $\boldsymbol{C}^n$. Let μ be an annihilating measure for $H(K)$. If $\{U_i\}$ is a finite open covering of K there exist annihilating measures*

[1] The referee has pointed out that Kallin, in an unpublished remark and without knowledge of Chirka's paper, observed that her counterexample could be made to yield this additional property.

μ_i for $H(K \cap \bar{U}_i)$ (in particular each μ_i is supported in U_i) such that $\mu = \sum \mu_i$.

All of our results are derived from an estimate (Theorem 2.2 below) for Cauchy-Riemann operator in certain product domains. This theorem is a generalization of a theorem of Andreotti and Stoll [1] for the case of a polycylinder. Our proof, like theirs, follows the induction procedure used in the proof of the familiar Dolbeaut-Grothendieck lemma, but we make essential use of the representation theorem of Grauert and Lieb [5] for bounded solutions of the Cauchy-Riemann equations in strongly pseudo-convex domains.

2. The basic estimate.

DEFINITION 2.1. *An open set G in C^n is called admissible if* (a) $n = 1$ *and G is a bounded open set or* (b) $n > 1$ *and G is a strongly pseudo-convex domain with C^∞ boundary.*

The following theorem is due to Grauert and Lieb [5] in the case $n > 1$, and is simply a restatement of known properties of the Cauchy kernel when $n = 1$.

THEOREM 2.1. *Let G be an admissible open set in C^n. Then there exists a differential form $\Omega(\zeta, z)$ of type $(n, n-1)$ in ζ, of type $(0, 0)$ in z, defined in a neighborhood of $\bar{G} \times \bar{G}$ such that*

(i) *Ω is of class C^∞ off the diagonal of $G \times G$:*

(ii) *there is a neighborhood of $\partial G \times G$ in $\bar{G} \times G$ in which $\bar{\partial}_z \Omega = 0$;*

(iii) *if g is a bounded $(0, 1)$ form of class C^∞ such that $\bar{\partial} g = 0$ on G, and if*

$$f(z) = -\int_G g(\zeta) \wedge \Omega(\zeta, z)$$

then $f \in C^\infty(G)$ and $\bar{\partial} f = g$ in G;

(iv) *there is a constant $\Delta(G)$, independent of z such that*

$$\int_G |a(\zeta, z)| \, dm\,(\zeta) \leqq \frac{\Delta(G)}{n}$$

where $a(\zeta, z)$ is any coefficient of Ω and dm is Lebesgue measure on G;

(v) *$\bar{G}$ has a sequence $\{G_\nu\}$ of admissible neighborhoods whose intersection is $\bar{G}$ such that $\{\Delta(G_\nu)\}$ is a constant sequence.*

Also, G is the union of an increasing sequence of admissible open sets $\{G_\nu\}$ for which $\{\Delta(G_\nu)\}$ is a constant sequence.

If G is an open set in $\boldsymbol{C}^n$ we denote by $BC^\infty(G)$ the space of functions on G whose derivatives of all orders are bounded and continuous on G. We will need the following corollary of Theorem 2.1.

COROLLARY. *Let G be an admissible open set in $\boldsymbol{C}^n$. Let U_k be open in $\boldsymbol{C}^{n_k}$, $k = 1, 2$. Suppose that*

$$g_1(z, x, w), \cdots, g_n(z, x, w) \in BC^\infty(G \times U_1 \times U_2)$$

and that $g = \sum g_j d\bar{z}_j$ satisfies $\bar{\partial}_z g = 0$ in $G \times U_1 \times U_2$. Then there exists $f \in BC^\infty(G \times U_1 \times U_2)$ such that

(i) $\bar{\partial}_z f = g$ *in* $G \times U_1 \times U_2$;

(ii) $\|f\| \leqq \Delta(G) \max_{1 \leqq j \leqq n} \|g_j\|$;

(iii) *if D is a differential operator on $\boldsymbol{C}^{n_1} \times \boldsymbol{C}^{n_2}$ with constant coefficients, then*

$$\|Df\| \leqq \Delta(G) \max_{1 \leqq j \leqq n} \|Dg_j\| .$$

(In particular, if each g_j is holomorphic in U_2 for fixed $(z, x) \in G \times U_1$, then f has the same property.)

Here $\|f\| = \sup_{G \times U_1 \times U_2} |f|$.

Proof. Let $f(z, x, w) = -\int_G g(\zeta, x, w) \wedge \Omega(\zeta, z)$, where Ω is as in Theorem 2.1. Since all the derivatives of g are bounded we may differentiate under the integral sign as often as we wish. The corollary then follows immediately from Theorem 2.1.

Let $G = G_1 \times \cdots \times G_r$ be an open set in $\boldsymbol{C}^n$ where each G_i is an open set in $\boldsymbol{C}^{n_i}$. If f is a function on G and $g = \sum g_j d\bar{z}_j$ is a $(0, 1)$ form on G we will use the following notation:

$$\|f\|_G = \sup_G |f|$$

$$\|f\|_G^{(k,t)} = \max_{\alpha \in A_{k,t}} \|\partial^\alpha f / \partial \bar{z}^\alpha\|_G$$

(where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers, $|\alpha| = \sum \alpha_j$,

$$\partial^\alpha f / \partial \bar{z}^\alpha = \partial^{|\alpha|} f / \partial \bar{z}_1^{\alpha_1} \cdots \partial \bar{z}_n^{\alpha_n} ,$$

and

$$A_{k,t} = \{\alpha = (\alpha_1, \cdots, \alpha_n) : |\alpha| \leqq k \ \text{ and } \ \alpha_j = 0 \ \text{ if } \ j > n_1 + \cdots + n_t\})$$

$$\|f\|_G^{(k)} = \|f\|_G^{(k,r)}$$

$$\|g\|_G^{(k,t)} = \max_{1 \leqq j \leqq n} \|g_j\|^{(k,t)}$$

$$\|g\|_G^{(k)} = \|g\|_G^{(k,r)} .$$

We can now state our basic result.

THEOREM 2.2. *Let* $G = G_1 \times \cdots \times G_r$ *be an open set in* C^n *where each* G_i *is an admissible open set in* C^{n_i}. *Let* g *be a* C^∞ $(0, 1)$ *form in* G *such that* $\bar{\partial} g = 0$ *in* G *and* $\|g\|_G^{(r-1)} < \infty$. *Then there exists* $f \in C^\infty(G)$ *such that*

(i) $\bar{\partial} f = g$ *in* G

(ii) $\|f\|_G \leqq (3\Delta)^r \|g\|_G^{(r-1)}$.

Here Δ *is any number* $\geqq 1$ *such that* $\Delta \geqq \Delta(G_i)$ *as defined in Theorem* 2.1.

Proof. We first prove the theorem in the case when each coefficient of g is in $BC^\infty(G)$.

If g is a $(0, 1)$ form on G, then $g = \sum g^i$ where each g^i is a $(0, 1)$ form on G_i with coefficients depending only on the other variables as parameters. For each k, $1 \leqq k \leqq r$ we consider the following Assertion k:

Let $G = G_1 \times \cdots \times G_r$ *be as above. Let* g *be a* $(0, 1)$ *form on* G *whose coefficients lie in* $BC^\infty(G)$ *and such that* $\bar{\partial} g = 0$ *in* G. *Suppose that*

$$g = \sum_{i=1}^{k} g^i$$

where each g^i *is a* $(0, 1)$ *form on* G_i, (*with coefficients depending also on the other variables*). *Then there exists* $f \in BC^\infty(G)$ *such that*

(i) $\bar{\partial} f = g$

(ii) $\|f\|_G \leqq (3\Delta)^k \|g\|^{(k-1,k-1)}$.

We shall prove Assertion 1 and then show that for $k = 1, 2, \cdots, r - 1$, Assertion k implies Assertion $k + 1$. (Of course Assertion r implies Theorem 2.2 in the case g is of class BC^∞.)

If g satisfies the hypotheses of Assertion 1, then g is a $\bar{\partial}$-closed $(0, 1)$ form in G_1 whose coefficients are holomorphic functions in $G_2 \times \cdots \times G_r$ for fixed z in G_1. Assertion 1 thus follows directly from the Corollary to Theorem 2.1.

Suppose now that Assertion k is true and that g satisfies the hypotheses of Assertion $(k + 1)$. Then

$$g = \sum_{i=1}^{k+1} g^i .$$

Notice that $\bar{\partial} g = 0$ implies $\bar{\partial}_{k+1} g^{k+1} = 0$ (where $\bar{\partial}_{k+1}$ differentiates only with respect to the variables from G_{k+1}) and that the coefficients of g are holomorphic in $G_{k+2} \times \cdots \times G_r$. Applying the corollary to Theorem 2.1 we conclude the existence of $u \in BC^\infty(G)$ such that $\bar{\partial}_{k+1} u = g^{k+1}$ and such that u is holomorphic in $G_{k+2} \times \cdots \times G_r$. Let $s = g - \bar{\partial} u$.

Then $\bar\partial s = \bar\partial g = 0$ and s is clearly a sum of (0, 1)-forms involving only differentials in the variables from $G_1, \cdots, G_k$. By Assertion k there exists $t \in BC^\infty(G)$ such that $\bar\partial t = s$.

Let $f = u + t$. Then $\bar\partial f = g$. Also,

$$\begin{aligned} \|f\| &\leqq \|u\| + \|t\| \\ &\leqq \Delta\|g\| + (3\Delta)^k\|s\|^{(k-1,k-1)} . \end{aligned}$$

But

$$\begin{aligned} \|s\|^{(k-1,k-1)} &\leqq \left\|\sum_{i=1}^k g^i\right\|^{(k-1,k-1)} \\ &\quad + \max_{1\leqq j\leqq n_k} \|\partial u/\partial \bar z_j\|^{(k-1,k-1)} \\ &\leqq \|g\|^{(k-1,k-1)} + \|u\|^{(k,k)} \\ &\leqq \|g\|^{(k-1,k-1)} + \Delta\|g\|^{(k,k)} \\ &\leqq (2\Delta)\|g\|^{(k,k)} . \end{aligned}$$

Hence

$$\begin{aligned} \|f\| &\leqq (\Delta + (2\Delta)(3\Delta)^k)\|g\|^{(k,k)} \\ &\leqq (3\Delta)^{k+1}\|g\|^{(k,k)} . \end{aligned}$$

This concludes the proof in the case when g is of class BC^∞.

Suppose now that g satisfies the hypotheses of Theorem 2.2. By Theorem 2.1 and what has been proved so far, we can find a sequence of open sets $\{G_\nu\}$ and a constant C independent of ν such that

(i) $\bar G_\nu \subset G_{\nu+1} \subset G$

(ii) $G = \bigcup G_\nu$

(iii) there exists $f_\nu \in BC^\infty(G_\nu)$

such that $\bar\partial f_\nu = g$ on G_ν and

$$\|f_\nu\|_{G_\nu} \leqq C\|g\|_G^{(r-1)} .$$

For each μ let $S_\mu = \{f_\nu | G_\mu : \nu \geqq \mu\}$. Since $f_\nu - f_\mu$ is holomorphic on G_μ for each ν, and $\{f_\nu - f_\mu | G_\mu\}$ is uniformly bounded, S_μ is relatively compact by Montel's theorem. Thus we may choose, for each μ, a subsequence $\{f_{\nu,\mu}\}$ of S_μ which converges uniformly on G_μ, such that $\{f_{\nu,\mu+1}\}$ is a subsequence of $\{f_{\nu,\mu}\}$. Then the diagonal sequence $\{f_{\mu,\mu}\}$ converges uniformly on each G_μ to a continuous function f defined on all of G. But f is in fact in $C^\infty(G)$ since, if $\nu > \mu$, $f_{\nu\nu} - f_{\mu\mu}$ is holomorphic on G_μ and $f_{\nu\nu} - f_{\mu\mu} \rightarrow f - f_{\mu\mu}$ on G_μ. Thus $f - f_{\mu\mu}$ is holomorphic, hence in $C^\infty(G_\mu)$ so $f \in C^\infty(G_\mu)$ for each μ. This also shows that $\bar\partial f = g$ on G. Finally, if $z \in G$ then there exists μ such that $z \in G_\nu$ for $\nu \geqq \mu$. This means that

$$|f_{\mu\mu}(z)| \leqq C\|g\|_G^{(r-1)}$$

but $f(z) = \lim f_{\mu\mu}(z)$ so

$$||f||_G \leqq C||g||_G^{(r-1)} .$$

3. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We may suppose that f has compact support in U. Choose $\phi \in C^\infty(\boldsymbol{C}^n)$ such that $\phi \geqq 0$, $\int \phi = 1$ and $\phi = 0$ outside the closed unit ball. For each $\delta > 0$ define f_δ by

$$f_\delta(z) = \int f(z - \delta w)\phi(w)dw .$$

Then $f_\delta \in C^\infty(\boldsymbol{C}^n)$, $f_\delta \rightarrow f$ uniformly as $\delta \rightarrow 0$ and for each α,

$$(\partial^\alpha f_\delta/\partial\bar{z}^\alpha)(z) = \int(\partial^\alpha f_\delta/\partial\bar{z}^\alpha)(z - \delta w)\phi(w)dw$$

so if G is an open set in $\boldsymbol{C}^n$,

$$||f_\delta||_G^{(s)} \leqq ||f||_{G^\delta}^{(s)} \qquad s = 1, 2, \cdots$$

where $G^\delta = \{z - \delta w : z \in G, |w| \leqq 1\}$.

For each i, $1 \leqq i \leqq r$ we can find a sequence $\{G_i^\nu\}$ of admissible neighborhoods such that $K_i = \cap G_i^\nu$ and such that $\{\Delta(G_i^\nu)\}$ is a constant sequence. Let us denote the constant by Δ_i. Choose $\Delta \geqq 1$ such that $\Delta \geqq \Delta_i$ for $1 \leqq i \leqq r$.

Let $\varepsilon > 0$ be given. Choose δ_0 such that $||f - f_\delta||_K < \varepsilon/2$ if $\delta < \delta_0$. Choose ν such that if $G = G_1^\nu \times \cdots \times G_r^\nu$, then

$$||f||_G^{(r)} < (3\Delta)^{-r} \cdot \frac{\varepsilon}{4} .$$

Then there exists $\delta < \delta_0$ such that

$$||f||_{G^\delta}^{(r)} < (3\Delta)^{-r}\frac{\varepsilon}{2} .$$

By Theorem 2.2 we can choose $u \in C^\infty(G)$ such that $\bar{\partial}u = \bar{\partial}f_\delta$ and

$$||u||_G \leqq (3\Delta)^r||\bar{\partial}f_\delta||_G^{(r-1)} .$$

Then $h = f_\delta - u$ is holomorphic in a neighborhood of K and

$$\begin{aligned} ||f - h||_K &\leqq ||f - f_\delta||_K + ||f_\delta - h||_K \\ &< \varepsilon/2 + ||u||_K \\ &< \varepsilon/2 + (3\Delta)^r||f_\delta||_G^r \\ &< \varepsilon/2 + (3\Delta)^r||f||_{G^\delta}^r \\ &< \varepsilon . \end{aligned}$$

Proof of Theorem 1.2. (Here we follow Lieb [8].) Since K is compact we can choose finitely many neighborhoods $U_{x_1}, \cdots, U_{x_m}$ which cover K. Let $U = U_{x_1} \cup \cdots \cup U_{x_m}$. Choose sequences $\{G_i^\nu\}$ of admissible neighborhoods of K_i, $1 \leqq i \leqq r$ as in the proof of Theorem 1.1 and let Δ be as above.

Let $\varepsilon > 0$ be given. Choose ν such that $G^\nu = G_1^\nu \times \cdots \times G_r^\nu$ lies in U, and such that there exist holomorphic functions h_j on $U_{x_j} \cap G^\nu$ with $\|f - h_j\|_{U_{x_j} \cap K} < \varepsilon$. Notice that $|h_i - h_j| \leqq |f - h_i| + |f - h_j| < 2\varepsilon$ on $U_{x_i} \cap U_{x_j} \cap K$. By choosing ν larger if necessary we may suppose that $|h_i - h_j| < 4\varepsilon$ on $U_{x_i} \cap U_{x_j} \cap G^\nu$.

Choose a C^∞ partition of unity $\phi_1, \cdots, \phi_m$ subordinate to the cover $\{U_{x_i}\}$. Let $g_k = \sum_i \phi_i(h_k - h_i)$. Notice that $\|g_k\|_{U_{x_k} \cap K} < 2\varepsilon$. Also $g_j - g_k = h_j - h_k$ and $\bar\partial g_j - \bar\partial g_k = 0$ on $U_{x_j} \cap U_{x_k} \cap G^\nu$. Thus $\{\bar\partial g_j\}$ defines a (0, 1) form g on G^ν which satisfies $\bar\partial g = 0$ so by Theorem 2.2 we can find $u \in C^\infty(G^\nu)$ that $\bar\partial u = g$ on G^ν and

$$\|u\|_{G^\nu} \leqq (3\Delta)^r \|g\|_{G^\nu}^{(r-1)} .$$

But

$$\|g\|_{G^\nu}^{(r-1)} \leqq \max_{1 \leqq k \leqq n} \|g_k\|_{G^\nu \cap U_{x_k}}^{(r)}$$

and

$$\partial^\alpha g_k / \partial \bar z^\alpha = \sum_i (\partial^\alpha \phi_i / \partial \bar z^\alpha)(h_k - h_i) .$$

Hence $\|g\|_{G^\nu}^{(r-1)} \leqq 4\varepsilon C$ where C is a constant depending only on the partition of unity $\{\phi_i\}$.

Let $b_j = g_j - u$. Then $\bar\partial b_j = 0$ on $U_{x_j} \cap G^\nu$ and $b_j - b_k = g_j - g_k = h_j - h_k$. Thus $\{h_j - b_j\}$ defines a holomorphic function h on G^ν and

$$\begin{aligned} \|f - h\|_K &\leqq \max_{1 \leqq j \leqq m} (\|f - h_j\|_K + \|g_j\|_K) + \|u\|_K \\ &\leqq \varepsilon + 2\varepsilon + 4C(3\Delta)^r \varepsilon , \end{aligned}$$

where C and Δ are independent of ε. Since ε was arbitrary, this completes the proof.

4. Annihilating Measures for $H(K)$. If G is an open set in C^n we denote by $A^{(0,1)}(G)$ the space of differential forms of type (0, 1) on G of class C^∞. We topologize $A^{(0,1)}(G)$ as the direct sum of n copies of the Frechet space $C^\infty(G)$ with the topology of uniform convergence on compact subsets of G of derivatives of all orders. The dual space of $C^\infty(G)$ is the space of distributions on C^n whose support is a compact subset of G. We will identify the dual space of $A^{(0,1)}(G)$ with the space of distributions with compact support in G.

LEMMA 4.1. *Let G be a domain of holomorphy in $\boldsymbol{C}^n$, K a compact subset of G, and μ an annihilating measure for $H(K)$. Then there exist distributions $\lambda_1, \cdots, \lambda_n$ with compact support in G such that $\mu = -\sum \partial\lambda_j/\partial\bar{z}_j$.*

Proof. $C^\infty(G)$ and $A^{(0,1)}(G)$ are Frechet spaces. Since G is a domain of holomorphy, $\bar{\partial}$ maps $C^\infty(G)$ onto the closed subspace of (0, 1) forms g satisfying $\bar{\partial}g = 0$. The measure μ, considered as a continuous linear functional on $C^\infty(G)$, annihilates the kernel of $\bar{\partial}$, so by a theorem of Dieudonné and Schwartz [4], μ is in the image of the adjoint, $\bar{\partial}^*$, of $\bar{\partial}$. But if $\lambda = (\lambda_1, \cdots, \lambda_n)$ is an n-tuple of distributions with compact support in G and $f \in C^\infty(G)$, then

$$\begin{aligned}(\bar{\partial}^*\lambda)(f) = \lambda(\bar{\partial}f) &= \sum \lambda_j(\partial f/\partial\bar{z}_j)\\ &= -\sum (\partial\lambda_j/\partial\bar{z}_j)(f) \ .\end{aligned}$$

Thus for some $\lambda_1, \cdots, \lambda_n$ we have $\mu = -\sum \partial\lambda_j/\partial\bar{z}_j$.

We will also consider, for a bounded open set G in $\boldsymbol{C}^n$, the Banach space $C^r(\bar{G})$ of continuous functions on $\bar{G}$ whose derivatives of order $\leqq r$ are bounded and continuous on G, with the norm

$$||f||_G^r = \max_{|\alpha| \leqq r} ||D^\alpha f||_G$$

where α is a $2n$-tuple of non-negative integers and

$$D^\alpha = (\partial/\partial z_1)^{\alpha_1} \cdots (\partial/\partial z_n)^{\alpha_n}(\partial/\partial\bar{z}_1)^{\alpha_{n+1}} \cdots (\partial/\partial\bar{z}_n)^{\alpha_{2n}} \ .$$

A continuous linear functional on $C^r(\bar{G})$ is easily seen to define a distribution on $\boldsymbol{C}^n$ with support in $\bar{G}$. In addition, we will denote by $B_r^{(0,1)}(\bar{G})$ the space of (0, 1) forms on G with coefficients in $C^r(\bar{G})$, topologized as the direct sum of n copies of $C^r(\bar{G})$ with the norm

$$||\sum g_j d\bar{z}_j||_G^r = \max_{1 \leqq j \leqq n} ||g_j||_G^r \ .$$

If $G_1 \subset G_2$ are two bounded open sets in $\boldsymbol{C}^n$ let $R: B_r^{(0,1)}(G_2) \to B_r^{(0,1)}(G_1)$ be the operator which restricts forms in G_2 to G_1. Then R^* is a norm-decreasing embedding of the dual space of $B_r^{(0,1)}(G_1)$ into $B_r^{(0,1)}(G_2)^*$ since, if $\lambda \in B_r^{(0,1)}(G_1)^*$, and $g = \sum g_j d\bar{z}_j$ is in $B_r^{(0,1)}(G_2)$,

$$\begin{aligned}|(R^*\lambda)(g)| = |\lambda(Rg)| &\leqq ||\lambda|| \, ||Rg||_{G_1}^r\\ &\leqq ||\lambda|| \, ||g||_{G_2}^r \ ,\end{aligned}$$

so $||R^*\lambda|| \leqq ||\lambda||$.

With these preliminaries we can proceed to the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $\{G_\nu\}$ be a sequence of bounded open neighborhoods of K such that

(i) $\bar{G}_\nu \subset G_{\nu-1}$

(ii) $K = \cap G_\nu$

(iii) there exists a constant C independent of ν such that if $g \in A^{(0,1)}(G_\nu)$, $\bar\partial g = 0$ in G_ν, and $\|g\|_{G^\nu}^{(r-1)} < \infty$, then there exists $f \in C^\infty(G_\nu)$ such that $\bar\partial f = g$ and

$$\|f\|_{G_\nu} \leqq C\|g\|_{G^\nu}^{(r-1)} .$$

If μ is an annihilating measure for $H(K)$ we can apply Lemma 4.1 to obtain, for each ν, an n-tuple $\lambda^\nu = (\lambda_1^\nu, \cdots, \lambda_n^\nu)$ of distributions with compact support in G_ν such that $\mu = -\sum \partial\lambda_j^\nu/\partial\bar{z}_j$. Let W_ν be the subspace of $C^{r-1}(\bar{G}_\nu)$ consisting of restrictions to G_ν of C^∞ functions on $\boldsymbol{C}^n$. If $f \in W_\nu$ we can find $h \in C^\infty(G_\nu)$ such that $f - h$ is holomorphic on G_ν and $\|h\|_{G_\nu} \leqq C\|\bar\partial f\|_{G_\nu}^{(r-1)}$ where C is the constant in (iii) above. Thus

$$\begin{aligned}|\lambda^\nu(\bar\partial f)| &= \left|\int f d\mu\right| = \left|\int h d\mu\right| \\ &\leqq \|\mu\|\,\|h\|_K \leqq C\|\mu\|\,\|\bar\partial f\|_{G^\nu}^{(r-1)}\end{aligned}$$

where $\|\mu\|$ is the total variation of μ. This means that λ^ν defines a continuous linear functional on the subspace $\bar\partial W_\nu$ of $B_{r-1}^{(0,1)}(G_\nu)$ of norm $\leqq C\|\mu\|$. By the Hahn-Banach theorem there is an n-tuple, which we will continue to denote by λ^ν, of continuous linear functionals on $C^{r-1}(\bar{G}_\nu)$ such that

(a) $\lambda^\nu(\bar\partial f) = \int f d\mu$ for all $f \in C^\infty(\boldsymbol{C}^n)$

(b) $\|\lambda^\nu\| \leqq C\|\mu\|$.

Now, by composing with the adjoint of the appropriate restriction operator we may consider each λ^ν so obtained as a continuous linear functional on $B_{r-1}^{(0,1)}(G_1)$. Then the sequence $\{\lambda^\nu\}$ constitutes a bounded sequence of elements in the dual space of a separable Banach space. Consequently, there is a subsequence $\{\lambda^{\nu'}\}$ and an n-tuple $\lambda = (\lambda_1, \cdots, \lambda_n)$ of continuous linear functionals on $C^{r-1}(\bar{G}_1)$ such that $\lambda^{\nu'} \to \lambda$ in the weak star topology. Since each $\lambda_j^{\nu'}$ is supported, as a distribution, on $\bar{G}_\nu$, and since $K = \cap G_{\nu'}$ it follows that the support of each λ_j, as a distribution, lies in K. Moreover, if f is a C^∞ function on $\boldsymbol{C}^n$ then

$$\lambda(\bar\partial f) = \lim \lambda^{\nu'}(\bar\partial f) = \int f d\mu ,$$

i.e., $\mu = -\sum \partial\lambda_j/\partial\bar{z}_j$. Finally, it is clear that each λ_j is of order $\leqq r - 1$.

Conversely, suppose μ is a measure on K, and $\mu = \sum \partial\lambda_j/\partial\bar{z}$, where $\lambda_1, \cdots, \lambda_n$ are distributions with compact support on K. If f is holomorphic in a neighborhood of K, then $\partial f/\partial \bar{z}_j$, $1 \leqq j \leqq n$, are identically 0 in a neighborhood of K so

$$\begin{aligned}\int f d\mu &= \sum (\partial \lambda_j/(\partial \bar{z}_j)(f) \\ &= -\sum \lambda_j(\partial f/\partial \bar{z}_j) \\ &= 0\end{aligned}$$

since the λ_j are supported on K.

Proof of Theorem 1.4. Let $\{\phi_i\}$ be a C^∞ partition of unity subordinate to the open covering $\{U_i\}$, i.e., suppose $\phi_i \in C^\infty(U_i)$, ϕ_i has compact support, $0 \leqq \phi_i \leqq 1$, and $\sum \phi_i = 1$ on a neighborhood of K. If μ is an annihilating measure for $H(K)$, then by Theorem 1.3 there exist measures $\lambda_1, \cdots, \lambda_n$ on K such that

$$\begin{aligned}\mu = -\sum \partial\lambda_j/\partial\bar{z}_j &= -\sum_j \partial\Big(\sum_i \phi_i\lambda_j\Big)\Big/\partial\bar{z}_j \\ &= -\sum_j \sum_i \phi_i(\partial\lambda_j/\partial\bar{z}_j) - \sum_j\sum_i (\partial\phi_i/\partial\bar{z}_j)\lambda_j \\ &= \sum_i \Big\{\phi_i\mu - \sum_j (\partial\phi_i/\partial\bar{z}_j)\lambda_j\Big\} \\ &\equiv \sum_i \mu_i\end{aligned}$$

where each μ_i is a measure compactly supported on $U_i \cap K$. If h is holomorphic on $U_i \cap K$, then

$$\partial(h\phi_i)/\partial\bar{z}_j = h(\partial\phi_i/\partial\bar{z}_j)$$

for $1 \leqq j \leqq n$. Thus

$$\begin{aligned}\int h d\mu_i &= \int h\phi_i d\mu - \sum_j \int h(\partial\phi_i/\partial\bar{z}_j) d\lambda_j \\ &= \int h\phi_i d\mu - \sum_j \int (\partial/\partial\bar{z}_j)(\phi_i h) d\lambda_j \\ &= 0\,.\end{aligned}$$

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