THE DECIDABILITY OF A CLASS OF AE SENTENCE IN THE ISOLS

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The isols are a recursive analogue of the Dedekind finite cardinals originally developed by Dekker. In this paper a metatheorem is proved which shows that for certain sentences $\mathscr{A}$ about addition in which no existential quantifier precedes a universal quantifier, the truth of $\mathscr{A}$ in the natural numbers is sufficient to ensure the truth of $\mathscr{A}$ in the isols. A more general class of sentences is also considered and it is seen that the applicability of the metatheorem is also necessary for the truth of any sentence in this class. It follows that there exists a decision procedure for that class of sentences. Extensions of these results to the case of the cosimple isols are also considered.

Careful definitions and exact statements of the results will be given in the next section. The remainder of this section will be devoted to a brief indication of some applications of the metatheorem and to a brief discussion of the relationship of the results in this paper to other work in the theories of addition of the isols, the recursive equivalence types (RETs), and cardinals in set theory without choice. Section 3 contains a proof of the metatheorem. Section 4 contains a proof of the necessity of the metatheorem in the class of sentences referred to above. In §5 the correctness of the two main results in the cosimple isols is proved.

A typical application of the metatheorem proves the well known refinement property [4, Theorem 14] for the isols.

(1.1) If $A_1 + \cdots + A_n = B_1 + \cdots + B_m$, then there exist isols $C_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m$ such that

$$
\begin{array}{ccccccc}
C_{1,1} & + & C_{2,1} & + & \cdots & + & C_{n,1} & = & B_1 \\
+ & & + & & & & + \\
C_{1,2} & + & C_{2,2} & + & \cdots & + & C_{n,2} & = & B_2 \\
+ & & + & & & & + \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
+ & & + & & & & + \\
C_{1,m} & + & C_{2,m} & + & \cdots & + & C_{n,m} & = & B_m \\
\| & & \| & & & & \| \\
A_1 & & A_2 & & \cdots & & A_n \, .
\end{array}
$$

Other examples are the truth in the isols of (1.2) and (1.3).

(1.2) For finite positive n and any A and B, $nA \leq nB$ implies

$A \leq B$.

(1.3)   For finite positive n and any A, B, and C, $nA \leq B + nC$ implies $A \leq B + C$.

The truth in the isols of the finite part of Tarski's theorem on the decomposition of linear forms (See [12, Theorem 7] and [13, p. 242].) also follows from the metatheorem.   A precise statement of this theorem is given in the next section.   The last example is especially interesting because it cannot be extended to the RETs (See Manaster [8].).

The proof of the metatheorem is a generalization of earlier proofs by Dekker-Myhill.   It was motivated by these proofs and modifications of them by Friedberg and Nerode.   The basic idea of a "chain", which plays a crucial role in the proof of the metatheorem, can be found in Tarski [15] applied to cardinal arithmetic.   This type of proof is applied to the isols by Dekker-Myhill in [4] in proving that an isolated set represents an isol and again, in a more complex manner, in proving (1.2) [*ibid.*, Theorems 39 and 40].   Friedberg [6] and Nerode [11] have extended this type of argument to the RETs.   It is not possible to extend directly Nerode's results for universal sentences in the RETs to the types of sentences considered here because of the failure of the decomposition of linear forms theorem in the RETs.

An example of an application of the decision procedure is the falsity in the isols of the following sentence, even though it is true in the natural numbers: for every x_1, x_2, x_3 there exist y_1 and y_2 such that if $5x_1 = x_2 + 3x_3$ then $3x_1 = 5y_1 + 2y_2$ or $3x_2 = 5y_1 + 2y_2$.   Nerode-Manaster [12] shows that while the decision procedure here may be capable of some extension, it can not be extended to a decision procedure for the complete theory of addition of the isols.

The metatheorems proved below were developed at almost the same time as very similar metatheorems were discovered in cardinal arithmetic without the axiom of choice (See Bradford [1].).   A significant difference is the restriction to the Dedekind-finite case here. Bradford did not make such a restriction.

2.   Preliminaries and summary.   The following notational conventions will be used.   Some of the notions referred to in this paragraph will be defined later in this section. E denotes the set of natural numbers, $\{0, 1, 2, \cdots\}$.   Lower case English letters, except f and g, with or without subscripts, denote natural numbers. $gcd(x_1, \cdots, x_n)$ denotes the greatest common divisor of $x_1, \cdots x_n$.   Lower case Greek letters denote sets of natural numbers. $|\xi|$ denotes the cardinality of ξ.   All applications of pairing functions are implicit so that we identify

an n-tuple of natural numbers with a natural number. $\xi^{j,i}$ denotes $\xi \times \{j\} \times \{i\} = \{(x, j, i) \mid x \in \xi\}$. The letters X, Y, Z, V, with or without subscripts or superscripts, denote formal variables and, at times, isols or natural numbers. The letter A, with or without subscripts, denotes an indecomposable isol. Vector notation may be used; thus X denotes $(X_1, \cdots, X_n)$ for some n. Any conjunction, disjunction, or summation is finite even if the range of the operator is not explicitly indicated. The union over an empty index set is empty; thus $\bigcup_{0 < k \leq 0} \delta_k = \varnothing$. The matrix of a formula is the formula remaining after the initial string of quantifiers is removed.

Two sets α and β are called recursively equivalent if there exists a 1-1 partial recursive function f such that the domain of f includes α and the image of α under f is β. Recursive equivalence is an equivalence relation on the power set of E. The equivalence classes under this relation are called recursive equivalence types (RETs). The RET to which the set α belongs is denoted $\langle \alpha \rangle$.

A set ξ is isolated if it does not contain any infinite recursively enumerable (r.e.) subset. An RET X is called an isol if any $\xi \in X$ is isolated. Δ denotes the set of isols. ξ and η are called recursively separated if there exist disjoint r.e. sets ω and θ such that $\xi \subseteq \omega$ and $\eta \subseteq \theta$. The sum of X and Y is defined to be $\langle \xi \cup \eta \rangle$ where $\xi \in X$, $\eta \in Y$, and ξ and η are recursively separated. If ξ and η are recursively separated we may write $\xi \cup \eta$ as $\xi + \eta$. X is said to be less than or equal to Y ($X \leq Y$) if there is a Z such that $X + Z = Y$. Basic properties of the RETs and the isols may be found in [4]. In particular, the isols satisfy the additive cancellation law,

$$(2.1) \qquad (\forall X)(\forall Y)(\forall Z)[X + Y = X + Z \longrightarrow Y = Z] \ .$$

An infinite isol, A, is called indecomposable if $A = X + Y$ implies X is finite or Y is finite. The existence of uncountably many pairwise incomparable indecomposable isols is well known (See Dekker [3, T1] or Dekker-Myhill [4, p. 103].). Define $X =_1 Y$ to mean there exist finite m and n such that $X + m = Y + n$. Generalizations of indecomposability and $=_1$ are considered in Manaster [9]. The following properties will be used in §4.

(2.2) If A is indecomposable and $A = \sum X_i$, then there is exactly one i for which X_i is infinite; moreover for that i, $X_i =_1 A$ and so X_i is indecomposable.

(2.3) If $X =_1 Y$, then X and Y are comparable.

(2.4) If $A, A_1, \cdots, A_n$, are indecomposable and $A \leq \sum A_i + p$, then for some i, $A =_1 A_i$.

(2.5) If $A_1, \cdots, A_m$ are pairwise incomparable indecomposable isols,

then there is no isol V and $j, 1 \leqq j \leqq m$, such that $2V \leqq \sum A_k$ & $A_j \leqq 3V$.

Proofs. (2.2) is proved by a routine induction on the number of summands. To prove (2.3) suppose $X =_1 Y$. Let $X + m = Y + n$. If $m \geqq n$, then (2.1) shows $X + (m - n) = Y$ and hence $X \leqq Y$. A symmetric argument shows that if $m < n$, $Y \leqq X$.

For (2.4) suppose $A, A_1, \cdots, A_n$ are indecomposable and $A \leqq \sum A_i + p$. The refinement property gives the existence of $X_1, Y_1, \cdots, X_{n+1}, Y_{n+1}$ such $X_i + Y_i = A_i$ for $i \leqq n$, $X_{n+1} + Y_{n+1} = p$, $A = \sum X_i$, and $Y = \sum Y_i$. (2.2) gives the existence of a unique i such that $A =_1 X_i$. Since X_i is infinite, $i \leqq n$. A further application of (2.2) to $X_i + Y_i = A_i$ gives $X_i =_1 A_i$. Thus $A =_1 A_i$.

To prove (2.5) suppose $A_1, \cdots, A_m$ are pairwise incomparable indecomposable isols, $2V \leqq \sum A_k$, and $A_j \leqq 3V$ for some $j, 1 \leqq j \leqq m$. Since $A_j \leqq 3V = V + V + V$, the refinement property and (2.2) combine to show that there is a natural number q such that $A_j \leqq V + q$. Thus $2A_j \leqq 2V + 2q \leqq \sum A_k + 2q$. (2.1) shows $A_j \leqq \sum_{k \neq j} A_k + 2q$. (2.4) then shows $A_j =_1 A_k$ for some $k \neq j$. Thus, by (2.3), A_j is comparable with A_k.

An *AE* sentence is a sentence with a prenex form in which no existential quantifier precedes any universal one. "$E \models \mathscr{A}$" asserts that $\mathscr{A}$ is true in the natural numbers. "$\Lambda \models \mathscr{A}$" asserts that $\mathscr{A}$ is true in the isols. The following result is the main result of this paper:

AE metatheorem. Let $\mathscr{F}$ be any sentence of the following form:

(2.6)
$$(\forall X_1) \cdots (\forall X_p)(\exists Y_1) \cdots (\exists Y_r)\left[\underset{j=1}{\overset{n}{\&}} \left(\sum_{i=1}^{p} a'_{j,i} X_i = \sum_{i=1}^{p} a''_{j,i} X_i \right) \right.$$
$$\left. \longrightarrow \underset{j=1}{\overset{m}{\&}} \left(\sum_{i=1}^{p} b'_{j,i} X_i + \sum_{i=1}^{r} c'_{j,i} Y_i = \sum_{i=1}^{p} b''_{j',i} X_i + \sum_{i=1}^{r} c''_{j,i} Y_i \right) \right].$$

If $E \models \mathscr{F}$, then $\Lambda \models \mathscr{F}$.

An isol analogue of a theorem originally proved by Tarski for the theory of cardinal numbers without the axiom of choice follows. Its proof (See [8].) led to the discovery of the *AE* metatheorem.

Theorem on the decomposition of linear forms. Let $X_1, \cdots, X_m$ be isols satisfying the system of linear equations:

(2.7)
$$\underset{k=1}{\overset{n}{\&}} \left(\sum_{i=1}^{m} a'_{k,i} X_i = \sum_{i=1}^{m} a''_{k,i} X_i \right).$$

There exist isols $Y_{1,1}, Y_{1,2}, Y_{2,1}, Y_{2,2}, Y_{3,1}, Y_{3,2}, \cdots, Y_{m,1}, Y_{m,2}$ such that $X_i = Y_{i,1} + Y_{i,2}$ for $i = 1, \cdots, m$, $Y_{1,1} \leqq Y_{2,1}$, $Y_{2,2} \leqq Y_{1,2}$, and both $Y_{1,1}, Y_{2,1}, \cdots, Y_{m,1}$ and $Y_{1,2} Y_{2,2}, \cdots, Y_{m,2}$ satisfy the original system of equations (i.e., both $\&(\sum a'_{k,i} Y_{i,1} = \sum a''_{k,i} Y_{i,1})$ and $\&(\sum a'_{k,i} Y_{i,2} = \sum a''_{k,i} Y_{i,2})$).

The conclusion of the theorem on the decomposition of linear forms can be strengthened.

Extension of linear forms decomposition theorem. Let $X_1, \cdots, X_m$ satisfy (2.7). Let $g_1, \cdots, g_{m!}$ be the set of permutations on $\{1, \cdots, m\}$. Then there exist isols $X_{i,j}$ for $1 \leqq i \leqq m, 1 \leqq j \leqq m!$ such that

$$X_i = \sum_{j=1}^{m!} X_{i,j} \quad \text{for} \quad i = 1, \cdots, m \,,$$

$$\&(\sum a'_{k,i} X_{i,j} = \sum a''_{k,j} X_{i,j}) \quad \text{for} \quad j = 1, \cdots, m! \,,$$

and $g_j(i) < g_j(h)$ implies $X_{i,j} \leqq X_{h,j}$ for $i, h \in \{1, \cdots, m\}$ and $1 \leqq j \leqq m!$.

The other metatheorem, providing a partial converse to the *AE* metatheorem, is most easily stated after two definitions are given.

DEFINITION. A *special AE sentence* is any sentence in the form:

$$(2.8) \quad \begin{aligned} &(\forall X_1) \cdots (\forall X_p)(\exists Y_1) \cdots (\exists Y_r)\Big[\mathop{\&}_{k=1}^{n} \Big(\sum_{i=1}^{p} a'_{k,i} X_i = \sum_{i=1}^{p} a''_{k,i} X_i\Big) \\ &\longrightarrow \mathop{\bigvee}_{j=1}^{m} \Big(\mathop{\&}_{h=1}^{q_j} \Big(\sum_{i=1}^{p} b'_{j,h,i} X_i + \sum_{i=1}^{r} c'_{j,h,i} Y_i = \sum_{i=1}^{p} b''_{j,h,i} X_i + \sum_{i=1}^{r} c''_{j,h,i} Y_i\Big)\Big)\Big]. \end{aligned}$$

A *Horn-reduct* of the sentence (2.8) is one of the m sentences obtained by deleting all but one of the disjuncts in the consequent of the matrix of the (2.8); thus it is one of the m formulas

$$(2.9) \quad \begin{aligned} &(\forall X_1) \cdots (\forall X_p)(\exists Y_1) \cdots (\exists Y_r)\Big[\&(\sum a'_{k,i} X_i = \sum a''_{k,i} X_i) \\ &\longrightarrow \mathop{\&}_{h=1}^{q_j} (\sum b'_{j,h,i} X_i + \sum c'_{j,h,i} Y_i = \sum b''_{j,h,i} X_i + \sum c''_{j,h,i} Y_i)\Big]. \end{aligned}$$

A converse to the *AE* metatheorem. A special *AE* sentence is true in $\varLambda$ only if one of its Horn-reducts is true in E.

Since the *AE* metatheorem implies the converse, a special *AE* sentence is true in $\varLambda$ iff one of its Horn-reduct is true in E. Presburger [13] gives an effective procedure for determining the truth or falsity of each of the Horn-reducts of (2.8) in E. Thus Presburger's result together with the equivalence just stated gives a decision procedure for special *AE* sentences in the theory of addition of isols.

3. **The AE metatheorem.** Let $\mathscr{F}$ be the sentence (2.6). The antecedent, or hypothesis, of the matrix of $\mathscr{F}$ will be denoted by H, the consequent by C. Since both E and $\varLambda$ satisfy the additive concellation law, we may assume with no loss of generality that

$$\min (a'_{j,i}, a''_{j,i}) = \min (b'_{j,i}, b''_{j,i}) = \min (c'_{j,i}, c''_{j,i}) = 0 \,.$$

(Explicit mention of the range of subscripts willl be omitted; the range

is always finite and determined by $\mathscr{F}$ in an appropriate way.)

Assume $E \models \mathscr{F}$. Assume further that $\bar{X}$ is a vector of isols satisfying H. For each i let ξ_i be a representative of X_i where there exist pairwise disjoint r.e. sets ω_i such that each $\xi_i \subseteq \omega_i$ and $\bigcup \omega_i \subseteq \{2x \mid x \in E\}$. Since $\bar{X}$ satisfies H, there exists a set of 1-1 partial recursive functions f_j such that for each j the domain of f_j includes $\sum_i (\sum_{0 < k \leq a'_{j,i}} \xi_i^{j,k})$ and

$$(3.1) \qquad f_j \left(\sum_i \sum_{0 < k \leq a'_{j,i}} \xi_i^{j,k} \right) = \sum_i \sum_{0 < k \leq a''_{j,i}} \xi_i^{j,k} \, .$$

To prove the theorem a method of enumerating 1-1 partial recursive functions g_j must be described and sets η_i must be defined such that for each j the domain of g_j includes $\sum_i (\sum_{0 < k \leq b''_{j,i}} \xi_i^{j,k}) + \sum_i \sum_{0 < k \leq c'_{j,i}} \eta_i^{j,k}$ and

$$g_j \left(\sum_i \left(\sum_{0 < k \leq b'_{j,i}} \xi_i^{j,k} \right) + \sum_i \sum_{0 < k \leq c'_{j,i}} \eta_i^{j,k} \right) = \sum_i \left(\sum_{0 < k \leq b''_{j,i}} \eta_i^{j,k} \right) + \sum_i \sum_{0 < k \leq c''_{j,i}} \eta_i^{j,k} \, .$$

Let

$$a_{j,i} = \max \, (a'_{j,i}, a''_{j,i}), \quad b_{j,i} = \max \, (b'_{j,i}, b''_{j,i}), \quad c_{j,i} = \max \, (c'_{j,i}, c''_{j,i}) \, .$$

If $x \in \omega_i$, (x, j, k) is called an *H-version* of x if $0 < k \leq a_{j,i}$ and (x, j, k) is called a *C-version* of x if $0 < k \leq b_{j,i}$. If $x \in \omega_i$, $y \in \omega_{i'}$, (x, j, k) is an H-version of x and (y, j, k') an H-version of y, (x, j, k) is said to be *H-satisfied* by (y, j, k') and conversely in case $a'_{j,i} > 0$, $a''_{j,i'} > 0$, and $f_j(x, j, k) = (y, j, k')$. x and y are *directly H-linked* if there are H-versions of x and y which satisfy each other. x is said to be *H-linked* to y if $x \in \sum \omega_i$ and $x = y$ or if there is a finite sequence $x = x_1, \cdots, x_n = y$ such that each x_i is directly H-linked to x_{i+1}. An *H-chain* is a subset of $\sum \omega_i$ any two elements of which are H-linked to each other. A *complete H-chain* is an H-chain in which all H-versions of the elements of the chain are H-satisfied by H-versions of elements of the chain.

Observe that the relation of being H-linked is an equivalence relation on $\sum \omega_i$ so that the set of maximal H-chains partitions $\sum \omega_i$. Since each f_j is partial recursive and each ω_i is r.e., each maximal H-chain is r.e. To enumerate the set of elements linked to a given x, $x \in \sum \omega_i$, simultaneously enumerate the sets ω_i and the functions f_j. Whenever sufficient information has been generated to show that y is H-linked to x, place y in the H-chain defined by x. It can be proved by induction on the number of direct H-links required to link y to x that any y which is H-linked to x will be placed in the enumerated H-chain. If x belongs to a finite complete H-chain, there is

a finite stage of the enumeration at which this complete H-chain is enumerated. At some, possibly later, finite stage of the process all information needed to show that the H-chain is complete will have been generated since all H-versions of all elements of the chain will have been satisfied by H-versions of other elements of the chain.

Suppose that γ is a finite complete H-chain. Let $\gamma_i = \gamma \cap \omega_i$ and let $x_i = |\gamma_i|$. The functions f_j show that $\bar{x}$ satisfies H. Since $\mathscr{F}$ is true in E, there exist natural numbers $\bar{y}$ such that $(\bar{x}, \bar{y})$ satisfies C. Moreover a sequence of numbers $\bar{y}$ with this property can be found effectively. Let $\bar{\delta}$ be a sequence of pairwise disjoint sets of odd numbers (so that $\bigcup \delta_i \cap \bigcup \gamma_i = \varnothing$) such that $|\delta_i| = y_i$ for each i. $\bar{\delta}$ may also be chosen in an effective way. For the remainder of this paragraph let the notions of C-versions of elements of $\bigcup \delta_i$ be defined analogously to that for elements of $\bigcup \omega_i$: in detail, (x, j, k) is a C-version of $x \in \delta_i$ if $0 < k \leqq c_{j,i}$. There is an effective procedure for defining 1-1 partial functions g_j such that, with the obvious definitions relative to the functions g_j, $\bigcup \gamma \cup \bigcup \delta_i$ is a finite complete C-chain. This effective procedure exists because, for each j,

$$= \left| \sum_i \left(\sum_{0 < k \leqq b'_{j,i}} \gamma_i^{j,k} \right) + \sum_i \sum_{0 < k \leqq c'_{j,i}} \delta_i^{j,k} \right|$$

$$= \sum_i b'_{j,i} x_i + \sum_i c'_{j,i} y_i = \sum_i b''_{j,i} x_i + \sum_i c''_{j,i} y_i$$

$$= \left| \sum_i \left(\sum_{0 < k \leqq b''_{j,i}} \gamma_i^{j,k} \right) + \sum_i \sum_{0 < k \leqq c''_{j,i}} \delta_i^{j,k} \right|.$$

The constructions of the preceding paragraphs may be combined to enumerate sets ζ_i and functions g_j in the following way. By simultaneously enumerating all maximal H-chains and checking to see when a finite complete H-chain has been enumerated, an effective enumeration of all finite complete H-chains is given. Whenever a finite complete H-chain is enumerated and identified, find sets $\bar{\delta}$ and functions g_j as in the preceding paragraph and, further, such that $\bigcup \delta_i$ is disjoint from the union of all the sets chosen as δ_i's at earlier stages of the enumeration. For each i, let ζ_i be the union of all the sets δ_i enumerated in the process. For each j let g_j be the union of all the finite functions enumerated as g_j's. Each g_j is a 1-1 partial recursive function and $\bar{\zeta}$ is a sequence of pairwise disjoint r.e. sets. Now define C-chain and related notions analogously to the corresponding definitions for H; here elements of $\sum \omega_i + \sum \zeta_i$ will be considered and the functions g_j will be used. The construction has the property that any finite complete H-chain is contained in a finite complete C-chain formed by adjoining elements of $\bigcup \delta_i$ for sets δ_i such that for each i, $\delta_i \subseteqq \zeta_i$.

For each i define η_i to be the subset of ζ_i consisting of elements

C-linked to some element of $\sum \xi_i$. Observe that if $x \in \sum \xi_i$ and y is H-linked to x, then $y \in \sum \xi_i$. This may be proved by an induction on the number of links in an H-chain from x to y using (3.1). Therefore, since the even elements of any complete C-chain are precisely the elements of some finite complete H-chain, the even elements of any complete C-chain are either all in $\sum \xi_i$ or none in $\sum \xi_i$. Finally a finite complete C-chain is either a subset of $\sum \xi_i + \sum \eta_i$ or disjoint from this set.

Each η_i is isolated. Since any element of η_i is C-linked to at least one element of $\sum \xi_i$, this follows from the existence of a simultaneous effective enumeration of all complete C-chains (in fact, the procedure defining $\bar{g}$ and $\bar{\zeta}$ is, essentially, such an enumeration). Thus, if η were an infinite r.e. subset of η_j, we could obtain from its enumeration and the enumeration of all complete C-chains an infinite r.e. subset of $\sum \xi_i$ contradicting the assumption that each ξ_i is isolated.

The critical observation now is that if $x \in \sum \xi_i$, then x belongs to a finite complete H-chain. An induction shows that if $x \in \sum \xi_i$, then x does belong to a complete H-chain. If this complete H-chain were infinite, it would be an infinite r.e. subset of $\sum \xi_i$, again contradicting the assumption that each ξ_i is isolated. Since any finite complete H-chain is contained in a finite complete C-chain, any $x \in \sum \xi_i$ belongs to a finite complete C-chain which is a subset of $\sum \xi_i + \sum \eta_i$. This completes the proof.

The two theorems on the decomposition of linear forms are immediate consequences of the AE metatheorem. For each system of equations (2.7), each theorem can be expressed as a sentence true in E to which the metatheorem applies.

4. A converse to the AE metatheorem. The proof of the converse to the AE metatheorem described in §2 begins with an observation about the natural numbers which was exploited by Bradford in [1].

LEMMA.

$$E \models (\forall X)(\forall Y_1) \cdots (\forall Y_n)\left[\not> \left(X = 1 \,\&\, \underset{i=1}{\overset{n}{\&}} \, Y_i = 0 \right) \right.$$
$$\left. \longleftrightarrow (\exists V)\left[2V \leqq X + \sum_{i=1}^{n} Y_i \,\&\, X \leqq 3V \right] \right].$$

Proof. Let $X, Y_1, \cdots, Y_n$ be any natural numbers. Suppose first there is a V satisfying the matrix of the second part of the equivalence. If $V = 0$, then $X = 0$ and the first part of the equivalence is also satisfied. If $V > 0$, then $X + \sum Y_i \geqq 2$. Thus $X \geqq 2$ or $\sum Y_i >$

0 so $X \geqq 2$ or at least one $Y_i > 0$. In any event, the first part of the equivalence is satisfied.

For the converse consider first X even. In this case $V = (1/2)X$ will satisfy the matrix of the second part of the equivalence. If X is odd but different from 1, then $V = 1/2(X - 1)$ satisfies the matrix since $2V = X - 1 \leqq X \leqq 3/2(X - 1)$. Finally if X is 1 but some $Y_i > 0$, $V = 1$ satisfies the matrix.

Let $\mathscr{F}$ be the formula (2.8) and let $\mathscr{F}_j$ be the formula (2.9) for $1 \leqq j \leqq m$. Let $\mathscr{H}(X_1, \cdots, X_p)$ be the hypothesis of the matrix of (2.8) and let $\mathscr{C}_j(X_1, \cdots, X_p, Y_1, \cdots, Y_r)$ be the conclusion of the matrix of $\mathscr{F}_j$. To prove the converse, assume $E \not\models \mathscr{F}_j$ for $j = 1, \cdots, m$. For each $j = 1, \cdots, m$, let $\bar{x}_j = (x_{j,1}, \cdots, x_{j,p})$ be a sequence of natural numbers such that

$$(4.1) \qquad\qquad E \models \mathscr{H}(x_{j,1}, \cdots x_{j,p})$$

but

$$(4.2) \qquad E \models 7 (\exists Y_1) \cdots (\exists Y_r) \mathscr{C}_j(x_{j,1}, \cdots, x_{j,p}, Y_1, \cdots, Y_r) .$$

Since $\mathscr{H}$ is a conjunction of homogeneous linear equations and any linear combination of solutions to a system of homogeneous equations is itself a solution to the system,

$$(4.3) \qquad E \models (\forall Z_1) \cdots (\forall Z_m) \mathscr{H}\left(\sum_{j=1}^{m} x_{j,1} Z_j, \cdots, \sum_{j=1}^{m} x_{j,p} Z_j\right) .$$

An immediate consequence of (4.2) is

$$(4.4) \quad \begin{aligned} E \models (\forall Z_1) &\cdots (\forall Z_m)(\forall Y_1) \cdots \\ (\forall Y_r) &\left[\mathscr{C}_j\left(\sum_{k=1}^{m} x_{k,1} Z_k, \cdots, \sum_{k=1}^{m} x_{k,p} Z_k, Y_1, \cdots, Y_r\right) \right. \\ &\longrightarrow \left. 7\left(Z_j = 1 \,\&\, \underset{k \neq j}{\&} Z_k = 0\right)\right] . \end{aligned}$$

Applying the equivalence of Lemma 4.1 to the conclusion of the matrix, using the definition of $\leqq$, and extracting quantifies gives

$$(4.5) \qquad E \models (\forall Z_1) \cdots (\forall Z_m)(\forall Y_1) \cdots (\forall Y_r)(\exists V)(\exists V_0)(\exists V_1)[\mathscr{M}]$$

where $\mathscr{M}$ is

$$\begin{aligned} \mathscr{C}_j(\textstyle\sum x_{k,1} Z_k, &\cdots, \sum x_{k,p} Z_k, Y_1, \cdots, Y_r) \\ &\longrightarrow (2V + V_0 = \textstyle\sum Z_k \,\&\, Z_j + V_1 = 3V) . \end{aligned}$$

Now apply the *AE* metatheorem to see that the formulas in (4.3) and (4.5) are true in the isols.

Let $A_1, \cdots, A_m$ be pairwise incomparable indecomposable isols. Let

$X_1 = \sum_{k=1}^{m} x_{k,1} A_k, \cdots, X_p = \sum_{k=1}^{m} x_{k,p} A_p$. By (4.3) in Λ, $\Lambda \models \mathscr{H}(X_1, \cdots, X_p)$. Assume now that $\Lambda \models \mathscr{F}$. Since $\bar{X}$ satisfies the hypothesis of $\mathscr{F}$ there must be a $j, 1 \leq j \leq m$, and isols $Y_1, \cdots, Y_r$ such that $\Lambda \models \mathscr{C}_j(X_1, \cdots, X_p, Y_1, \cdots, Y_r)$. Recalling the definition of X,

$$\Lambda \models \mathscr{C}_j(\sum x_{k,1} A_k, \cdots, \sum x_{k,p} A_k, Y_1, \cdots, Y_r) .$$

By (4.5) in Λ, $\Lambda \models (\exists V)[2V \leq \sum A_k \,\&\, A_j \leq 3V]$. Since $A_1, \cdots, A_m$ are pairwise incomparable, this contradicts (2.5). Thus if $E \not\models \mathscr{F}_j$ for $j = 1, \cdots, m$, then $\Lambda \not\models \mathscr{F}$.

It is perhaps of interest to point out an alternative to the last paragraph. Using the truth in Λ of (4.3), (4.5), and $\mathscr{F}$, it is possible to show $\Lambda \models (\forall Z_1) \cdots (\forall Z_m)(\exists V)[\bigvee_{j=1}^{m}(2V \leq \sum Z_k \,\&\, Z_j \leq 3V)]$. However this is not the case since if $Z_1, \cdots, Z_m$ is a sequence of pairwise incomparable indecomposable isols, there is no V satisfying the matrix.

5. The AE metatheorem for cosimple isols. An isol X is cosimple if it has a representative ξ with r.e. complement. We denote the cosimple isols by Λ_z. By Theorem 56 of [4] Λ_z is closed under isolic addition and predecessors (i.e., is an ideal in Λ) so that the notions of the previous sections are applicable to Λ_z. Here we show that if $\mathscr{F}$ is the sentence (2.6) and $E \models \mathscr{F}$ then $\Lambda_z \models \mathscr{F}$.

Except for minor differences we use the terminology of §3. Let $\bar{X} = (X_1, \cdots, X_p)$ be a vector of cosimple isols satisfying the hypothesis H of $\mathscr{F}$. Let ω be the even integers, and $\omega_i, 1 \leq i \leq p$ a partition of ω into infinite recursive subsets. Choose $\xi \subset \omega$ such that $\xi_i = \xi \cap \omega_i$ represents X_i and $\omega_i - \xi_i$ is r.e.. It is clear that this choice can be made since if ξ_i' is a representative of X_i with r.e. complement and f is a one-to-one recursive function mapping E onto ω_i then $\xi_i = f(\xi_i')$ satisfies our requirements. Define complete H-chain as in §3 and let S be the set of all finite complete H-chains. As mentioned in §3, S is r.e. and hence can be enumerated by a recursive function $\gamma(m)$.

Let θ be the odd integers, and $\theta_i, 1 \leq i \leq r$ a partition of θ into infinite recusive subsets. We define a partial recursive function h by the following induction. Suppose $h(\gamma(m))$ has been defined for $m < n$. Let $x_i = |\gamma(m) \cap \omega_i|$. Clearly the vector $\bar{x} = (x_1, \cdots, x_p)$ satisfies H so there must be a vector of integers $\bar{y} = (y_1, \cdots, y_r)$ such that $(\bar{x}, \bar{y})$ satisfies the conclusion C of $\mathscr{F}$. Let $\delta \subset \theta$ be the first finite set in the canonical enumeration of finite sets such that $|\delta \cap \theta_i| = y_i$ and δ is disjoint from $h(\gamma(m))$ for $m < n$. Set $h(\gamma(n)) = \delta$. Since δ can be effectively be found, h is partial recursive. As in §3, finite functions g_i can be effectively found such that $\gamma \cup h(\gamma)$ is a finite complete C-chain. Let $\eta = \bigcup\{h(\gamma) \,|\, \gamma \in S \text{ and } \gamma \subset \xi\}$ and $\eta_i = \eta \cap \theta_i$.

Then as in §3, if η_i is a representative of Y_i, and $\bar{Y} = (Y_1, \cdots, Y_r)$ then $\bar{Y}$ is a vector of isols such that $(\bar{X}, \bar{Y})$ satisfies C. It only remains to show that each Y_i is cosimple. Let $\zeta = \bigcup\{h(\gamma)\,|\,\gamma \in S\}$. ζ is r.e. and since $\omega - \xi$ is r.e. so is $\zeta - \eta = \bigcup\{h(\gamma)\,|\,\gamma \in S$ and $\gamma \cap (\omega - \xi) \neq \varnothing\}$. Let q be a one-to-one recursive function mapping E onto $\zeta_i = \zeta \cap \theta_i$. Then $E = q^{-1}(\eta_i) \cup q^{-1}(\zeta_i - \eta_i)$ and $q^{-1}(\eta_i)$ is a representative of Y_i whose complement $q^{-1}(\zeta_i - \eta_i)$ is r.e.. Thus $Y_i \in \varLambda_z$ which completes our proof.

There is also a converse to the *AE* metatheorem for cosimple isols. Here we show that if $\mathscr{F}$ is the sentence (2.8) and $\varLambda_z \models \mathscr{F}$ then some Horn reduct of $\mathscr{F}$ is true in E.

By the methods of §4 it will suffice to find a countable collection of pairwise incomparable indecomposable cosimple isols. Although it is possible to give a direct construction of such a collection, we prefer to obtain the result from existing literature. A set $\xi \subset E$ is *maximal* (cf. [14]) if ξ is r.e. and for any r.e. set ω, exactly one of $\omega \cap (E - \zeta)$ and $(E - \omega) \cap (E - \zeta)$ is infinite. Clearly if ξ is maximal then its complement is a representative of an indecomposable cosimple isol. By [5] maximal sets exist. By theorem 5.1 of [7] the Turing degree of a maximal set contains a countably infinite collection of maximal sets having pairwise incomparable many-one degrees. By theorem 58 of [4] if α and β are infinite sets with r.e. complements, and the RET of α is $\leq$ the RET of β then α is one-to-one reducible to β. Combining those facts gives the desired result.

REFERENCES

1. R. Bradford, *Cardinal addition and the axiom of chioce*, Ph. D. Thesis, University of California, Berkeley, California, 1965, also Ann. Math. Logic, **3** (1971), 111-196.

2. J. C. E. Dekker, *A non-constructive extension of the number system*, Parts I and II, J. Symbolic Logic, **20** (1955), 204-205.

3. ———, *Congruences in isols with a finite modulus*, Mathematische Zeitschrift, **70** (1958), 357-373.

4. J. C. E. Dekker and J. Myhill, *Recursive equivalence types*, University of California Publications in Mathematics (N.S.) 3, No. 3 (1960), 67-214.

5. R. M. Friberg, *Three theorems on recursive enumeration*, J. Symbolic Logic, **23** (1958), 309-316.

6. ———, *The uniqueness of finite division for recursive equivalence types*, Mathematische Zeitschrift, **75** (1961), 3-7.

7. M. Lerman, *Recursive functions modulo comaximal sets*, Ph. D. Thesis, Cornell University, 1968.

8. A. B. Manaster, *A theorem on the decomposition of linear forms for the isols*, Notices of the Amer. Math. Soc. II (1964), 527.

9. ———, *Higher-order indecomposable isols*, Trans. Amer. Math. Soc., **125** (1966), 363-383.

10. ———, *The failure of Tarski's decomposition of linear forms theorem in the RETs*, in preparation.

11. A. Nerode, *Additive relations among recursive equivalence types*, Mathematische

Annalen, **159** (1965), 329-343.

12.   A. Nerode and A. B. Manaster, *A universal embedding property of the RETs*, J. Symbolic Logic, **35** (1970), 51-59.

13.   M. Presburger, *Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt*, Sprawozdania z I Kongresu Matematykow Krajów Słowianskich, Warszawa (1929), 92-101 and 395.

14.   H. Rogers Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York, 1967.

15.   A. Tarski, *Cancellation laws in the arithemetic of cardinals*, Fundamenta Mathematicae, **36** (1949), 77-92.

16.   ————, *Cardinal algebras*, Oxford University Press, New York, 1949.

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