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COUNTEREXAMPLES TO CONJECTURES OF RYSER AND DE OLIVEIRA

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Let $U(n; k)$ be the set of all $n \times n$ binary matrices with k ones in each row and column. Considering the relation between the permanent and the determinant for matrices in $U(n; k)$, Tinsley established the following result:

THEOREM: **Let $C \in U(7; 3)$ be the cyclic matrix defined by the differences $0, 1, 3 \pmod 7$. Let $A \in U(n; k)$ with $k \geqq 3$. Suppose that there are permutation matrices $P_1, P_2, \cdots, P_k \in U(n; 1)$ such that $A = P_1 + P_2 + \cdots + P_k$ and $P_iP_j = P_jP_i$ $(i, j = 1, \cdots, k)$. Then $\operatorname{per} A = |\det A|$ if and only if $k = 3$, $7 \mid n$, and the rows and columns of A can be permuted in such a way that the resulting matrix is the direct sum of C taken $n/7$ times. Ryser posed**

Conjecture I. **Tinsley's Theorem remains valid when the condition $P_iP_j = P_jP_i (i, j = 1, \cdots, k)$ is dropped.**

Discovery of counterexamples to Conjecture I leads directly to counterexamples to the following conjecture of de Oliveira:

Conjecture II. **Let A be an $n \times n$ doubly stochastic irreducible matrix. If n is even, then $f(z) = \operatorname{per}(zI - A)$ has no real roots; if n is odd, then $f(z) = \operatorname{per}(zI - A)$ has one and only real root.**

2. Preliminary results. Following the terminology of Harary [6, 7, 8] we recall that with every digraph (with loops), D, we may associate a binary matrix, $A(D)$, the (point) adjacency matrix of D. Conversely, with every binary matrix, A, we may associate a digraph, $D(A)$, which has A as its adjacency matrix. Given an $n \times n$ binary matrix, A, let $l_+(l_-)$ denote the number of linear subgraphs of $D(A)$ which contain an even (respectively, odd) number of cycles of even length. Then as shown by Harary [6] $\det A = l_+ - l_-$. Similar reasoning yields the formula $\operatorname{per} A = l_+ + l_-$.

LEMMA 1. *If A is an $n \times n$ binary matrix with ones on the diagonal, then* $\operatorname{per} A = \det A$ *if and only if every cycle of $D(A)$ is of odd length. Moreover, if A is an arbitrary $n \times n$ binary matrix and $D(A)$ has only odd cycles, then* $\operatorname{per} A = \det A$.

Proof. This is an obvious consequence of the relation between the permanent, the determinant, and $D(A)$.

An $n \times n$ matrix, A, is said to be indecomposable if there does not exist a permutation matrix, P, such that $PAP^T = A_1 \oplus A_2$ for some matrices, A_1 and A_2; A is said to be fully indecomposable if there do not exist permutation matrices, P and Q, such that $PAQ = A_1 \oplus A_2$ for some matrices, A_1 and A_2.

LEMMA 2. *Let A be a binary matrix with ones on the diagonal. The following are equivalent:*

(i) *A is indecomposable*

(ii) *A is fully indecomposable*

(iii) *$G(A)$ is weakly connected.*

Proof. This is a simple consequence of a result of Brualdi, Parter, and Schneider [2; Lemma 2.3].

3. Constructions. The counterexamples we require can be generated through the proper use of the following three constructions. In each construction the matrices $A_i \in U(n_i; 3)$ satisfy $\operatorname{per} A = |\det A|$ and have only ones on the diagonal. This later condition is not overly restrictive as any matrix in $U(n; 3)$ can have its rows or columns permuted to put it in this form.

It can easily be verified that in each construction the resulting digraph has only odd cycles, and thus the corresponding matrix has equal permanent and determinant. Furthermore, if the matrices A_i are fully indecomposable, then so is the resulting matrix, as the corresponding digraph is strongly connected.

Construction I. Let $A_1, A_2, \cdots, A_{2m+1}$ be given for some fixed positive integer m. For each i ($i = 1, 2, \cdots, 2m + 1$) select from $D(A_i)$ an edge e_i from u_i to v_i. Form a new digraph G from $G_1 \cup G_2 \cup \cdots \cup G_{2m+1}$ by deleting the edges e_i ($i = 1, 2, \cdots, 2m + 1$) and adding edges from u_i to v_{i+1} ($i = 1, 2, \cdots, 2m$) and from u_{2m+1} to v_1. Clearly $A(G) \in U(n_1 + n_2 + \cdots + n_{2m+1}; 3)$ and $\operatorname{per} A(G) = \det A(G)$.

Construction II. Let A_1, A_2, A_3, and A_4 be given. For each i ($i = 1, 2, 3, 4$) select from $D(A_i)$ an edge e_i from u_i to v_i. Let v_0 be an additional point. Form a new digraph G from $G_1 \cup G_2 \cup G_3 \cup G_4 \cup \{v_0\}$ by deleting the edges e_i for $i = 1, 2, 3, 4$ and adding new edges from u_1 to v_2, from u_3 to v_4, from v_0 to v_1 and v_3, and from u_2 and u_4 to v_0, and a loop at v_0. Clearly $A(G) \in U(n_1 + n_2 + n_3 + n_4 + 1; 3)$ and $\operatorname{per} A(G) = \det A(G)$.

Construction III. Let $A_1, A_2, \cdots, A_{4m+2}$ be given for some fixed positive integer m. For each i $(i = 1, 2, \cdots, 4m + 2)$ select from $D(A_i)$ an edge e_i from u_i to v_i and form a new digraph G_i by deleting e_i and adding two new points u'_i and v'_i together with new edges from u_i to u'_i, from u'_i to v'_i, and from v'_i to v_i. Form the digraph G from $G_1 \cup G_2 \cup \cdots \cup G_{4m+2}$ by identifying the point pairs u'_{2i-1} and u'_{2i} for $i = 1, 2, \cdots, 2m + 1$, v'_{2i} and v'_{2i+1} for $i = 1, 2, \cdots, 2m$, and v'_{4m+1} and v'_1, and adding a loop at each of the resulting points. Clearly $A(G) \in U(n_1 + n_2 + \cdots + n_{4m+2} + 4m + 2; 3)$ and $\operatorname{per} A(G) = \det A(G)$.

4. Conclusions. We are now ready to prove that Conjecture I is false.

THEOREM 1. *Conjecture* I *is false for* $k = 3$. *In fact for every sufficiently large* n *there is a fully indecomposable matrix* $A \in U(n; 3)$ *satisfying* $\operatorname{per} A = \det A$.

Proof. Starting with the matrix C, Constructions I, II, and III may be used to generate a family of fully indecomposable matrices with equal permanent and determinant. It can easily be verified that the family contains matrices of order n for all sufficiently large n.

The question of the existence of matrices in $U(n; k)$ for $k \geqq 4$ with equal permanent and determinant remains open. It should be noted, however, that should one such matrix exist for a given k, then Constructions I, II, and III with the obvious modifications, may be used to construct an infinite family of such matrices. The problem of finding a good characterization of the matrices in $U(n; 3)$ with equal permanent and determinant also remains to be solved.

As to Conjecture II, while Datta [4] has shown that Conjecture II is true for even n if A is symmetric and imprimitive; Hartfiel [9] has produced counterexamples for $n = 4$ and 5; and Csima [3] has produced an infinite family of counterexamples. Counterexamples for all sufficiently large even n follow directly from the results of Theorem 1. However, more can be said as follows:

THEOREM 2. *For each* $n \geqq 3$ *there is an* $n \times n$ *indecomposable doubly-stochastic matrix* A_n *such that* $f(z) = \operatorname{per}(zI - A_n)$ *has* $n - 2$ *distinct real roots in* $(0, 1)$.

Proof. Start with $A_3 = J_3$, which is clearly satisfactory, and continue inductively.

Suppose A_{n-1} satisfies the conditions of the theorem. The required matrix, A_n, is constructed as follows. Let $A_n(\lambda) = \lambda J_n + (1 - \lambda)\,((1) \oplus A_{n-1})$, where J_n is the $n \times n$ matrix each of whose entries is $1/n$. Clearly $A_n(\lambda)$ is doubly-stochastic for $0 \leqq \lambda \leqq 1$ and indecomposable for $\lambda \neq 0$. Let $B_n(\lambda, z) = zI - A_n(\lambda)$, and let $g_n(\lambda, z) = \operatorname{per} B_n(\lambda, z)$. Then

$$\frac{\partial g_n(\lambda, z)}{\partial \lambda} = \sum_{i,j} - \frac{d(A_n(\lambda))_{ij}}{d\lambda} \operatorname{per}(B_n(\lambda, z))(i|j)$$

where $(A_n(\lambda))_{ij}$ is the entry of $A_n(\lambda)$ in row i and column j, and $(B_n(\lambda, z))(i|j)$ is the matrix obtained from $B_n(\lambda, z)$ by deleting row i and column j. Observe that $B_n(0, 1) = (0) \oplus (I - A_{n-1})$; hence for $\lambda = 0$, $z = 1$ all of the terms in the summation above, except the term for $i = j = 1$, vanish. Thus

$$\frac{\partial g_n(0, 1)}{\partial \lambda} = \left(1 - \frac{1}{n}\right) \operatorname{per}(I - A_{n-1}) \neq 0 .$$

It follows that for $\lambda > 0$ sufficiently small $\operatorname{per}(I - A_n(\lambda)) \neq 0$, so that for such λ, $z = 1$ is not a root of $\operatorname{per}(zI - A_n(\lambda))$. As the roots of $\operatorname{per}(zI - A_n(\lambda))$ are continuous, and, as shown by Brenner and Brualdi [1], the real roots lie on $(0, 1]$, it must be the case that for some $\lambda_0 > 0$ $\operatorname{per}(zI - A_n(\lambda_0))$ has $n - 2$ real roots in $(0, 1)$. Thus the matrix $A_n = A_n(\lambda_0)$ is as required, and the theorem is proved. We believe that this result may be best possible in the sense that no doubly-stochastic matrix other than the identity yields only real roots.

References

1. J. L. Brenner and R. A. Brualdi, *Properties of the permanent function*, Notices Amer. Math. Soc., **14** (1967), 87.
2. R. A. Brualdi, S. V. Parter, and H. Schneider, *The diagonal equivalence of a non-negative matrix to a stochastic matrix*, J. Math. Anal. Appl., **16** (1966), 31-50.
3. J. Csima, *A class of counterexamples on permanents*, Pacific J. Math., **37** (1971), 655-656.
4. B. N. Datta, *On a conjecture of de Oliveira*, Notices Amer. Math. Soc., **17** (1970), 1068.
5. Q. N. de Oliveira, *A conjecture and some problems on permanents*, Pacific J. Math., **32** (1970), 495-499.
6. F. Harary, *The determinant of the adjacency matrix of a graph*, SIAM Review, **4** (1962), 202-209.
7. ———, *Graphs and matrices*, SIAM Review, **9** (1967), 83-90.
8. ———, *Graph Theory*, Addison-Wesley, Reading, Mass., 1969.
9. D. J. Hartfiel, *Counterexamples to a conjecture of G. N. de Oliveira*, Pacific J. Math., **38** (1971), 67-68.
10. H. Ryser, *Matrices of zeros and ones*, Bull. Amer. Math. Soc., **66** (1960), 442-463,
11. M. Tinsley, *Permanents of cyclic matrices*, Pacific J. Math., **10** (1960), 1067-1082.

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