PACIFIC JOURNAL OF MATHEMATICS
Vol. 45, No. 2, 1973

COCYCLES WITH RANGE $\{\pm 1\}$

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Let Γ be a subgroup of the real line with the discrete topology and suppose Γ has at least two rationally independent elements. A nontrivial cocycle D whose range $\{\pm 1\}$ consists only of the numbers $+1$ and -1 is constructed on the dual G of Γ using properties of local projective representations.

Cocycles play an important role in harmonic analysis on G and three apparently quite different methods for constructing nontrivial cocycles are known: Helson and Lowdenslager [5] (and extended by Helson and Kahane [4]), Gamelin [2] and the author [7]. In answer to a question raised by Helson [3] Gamelin constructed a nontrivial cocycle with range $\{\pm 1\}$. In this paper we provide a different construction of cocycles with range $\{\pm 1\}$ based upon the method introduced in [7].

2. **Preliminaries.** We will briefly recall a few definitions and will summarize the main idea of [7] to which we refer the reader for further properties of cocycles and projective cocycles.

For the problem at hand it is sufficient to let G be the 2-dimensional torus T^2 realized as the square $[-\pi, \pi] \times [-\pi, \pi]$ with opposite edges identified ([7], p. 559). The open neighborhood $(-\pi, \pi) \times (-\pi, \pi)$ of the identity in T^2 is denoted by $\mathscr{N}$ and $\Lambda = \{e_t \mid t \in \text{Reals}\}$ is the continuous dense one-parameter subgroup of T^2 formed by the winding line of irrational slope passing through the identity of T^2. A (Borel) function φ on T^2 is said to be *unitary* in case $\varphi(x)$ has modulus one a.e. (x) (with respect to Haar measure on T^2). For unitary functions φ and ψ we write $\varphi(\cdot) = \psi(\cdot)$ to mean $\varphi(x) = \psi(x)$ a.e. (x).

It is convenient to view a cocycle A as a family of unitary functions $A(e_t, \cdot)$, $e_t \in \Lambda$, which satisfy the identity

$$A(e_t + e_u, \cdot) = A(e_t, \cdot)A(e_u, \cdot - e_t) \tag{2.1}$$

in addition to a continuity condition which need not be stated here. If $A(e_t, \cdot)$ is a constant unitary function for each $e_t \in \Lambda$ we say that A is a *constant cocycle*; necessarily A is of the form $A(e_t, \cdot) = \exp i\lambda t$ for some real number λ. Cocycles of the form $A(e_t, \cdot) = \varphi(\cdot)\bar{\varphi}(\cdot - e_t)$ for some unitary function φ are called *coboundaries*. We say that a cocycle A is *nontrivial* if A is not the product of a constant cocycle and a coboundary.

We now turn to the main idea of [7]. Given a local projective multiplier ω a projective cocycle A_ω was formed and this induced a cocycle A related to A_ω by

$$A(e_t, \cdot) = q(e_t)A_\omega(e_t, \cdot) \tag{2.2}$$

for some continuous function q on a segment Λ_0 of Λ containing the identity. Moreover, A is unique up to a constant cocycle factor and if the multiplier ω is nontrivial then the cocycle A is nontrivial; this last assertion relies heavily upon the continuity of ω.

Bargmann [1] showed that T^2 has two inequivalent local projective multipliers. We can let ω be the continuous (on $\mathscr{N} \times \mathscr{N}$) nontrivial multiplier so that the cocycle A given by (2.2) is nontrivial.

The idea of this paper is to observe that the continuous multiplier ω^2 must be equivalent to the trivial multiplier 1 and upon taking square roots properly one finds that ω is equivalent to a nontrivial multiplier d with range $\{\pm 1\}$. Now d, though not continuous, is measurable and this essentially allows us to construct a measurable projective cocycle A_d with range $\{\pm 1\}$. Although a cocycle, qA_d, can be induced by A_d it need not have the desired range and it would be somewhat difficult to prove qA_d is nontrivial by the techniques of [7] since d is not continuous.

Fortunately, as is shown in § 4, a simple modification of A_d produces a nontrivial cocycle D with range $\{\pm 1\}$. In fact D is actually induced by A_d but not by the general construction of [7].

Since [7] dealt exclusively with continuous multipliers we will indicate those modifications necessary for constructing the measurable multiplier d and its associated projective cocycle. We attend to these matters in § 3 reserving § 4 for the actual construction of D.

3. Measurable projective multipliers d with range $\{\pm 1\}$ are familiar in the theory of group representations and we will only sketch a construction (Cf. Mackey [6], p. 154).

As mentioned in the preceding section T^2 has only one (up to equivalence) nontrivial continuous local projective multiplier ω defined on $\mathscr{N} \times \mathscr{N}$. It follows that the continuous multiplier ω^2 is either equivalent to ω or is trivial. If ω^2 were equivalent to ω then ω itself would be trivial and so we must assume ω^2 is trivial, i.e.,

$$\omega^2(x, y)(\bar{s}(x)\bar{s}(y)s(x + y)) = 1 \tag{3.1}$$

for some continuous function s of modulus one on $\mathscr{N}$ and for all $x, y \in \mathscr{N}$ such that $x + y \in \mathscr{N}$.

Now let p be a measurable square root of s on $\mathscr{N}$ and define d by

$$d(x, y) = \omega(x, y)(\bar{p}(x)\bar{p}(y)p(x + y)) \tag{3.2}$$

for all $x, y \in \mathscr{N}$ such that $x + y \in \mathscr{N}$. Clearly d is a local projective multiplier with domain $1/2\mathscr{N} \times 1/2\mathscr{N}$, say, and with range $\{\pm 1\}$.

Actually, our interest lies with the unitary function

$$d(e_t, \cdot) = \omega(e_t, \cdot)(\bar{p}(e_t)\bar{p}(\cdot)p(e_t + \cdot)) \tag{3.3}$$

defined for each $e_t \in \Lambda \cap \mathscr{N}$. Notice that $d(e_t, \cdot)$ has essential range $\{\pm 1\}$.

For each $x \in \mathscr{N}$, $A_\omega(x, y) = \omega(x, y - x)$ defines a unitary function $A_\omega(x, \cdot)$ since ω is continuous on $\mathscr{N} \times \mathscr{N}$ ([7], p. 563). If d were continuous then $A_d(x, y) = d(x, y - x)$ formally defines a projective cocycle which satisfies

$$A_\omega(x, y)\bar{A}_d(x, y) = p(x)B(x, y) \tag{3.4}$$

where $B(x, y) = \bar{p}(y)p(y - x)$ ([7], p, 562).

However, for our purposes, we need not define $A_d(x, \cdot)$ for all $x \in \mathscr{N}$ nor obtain (3.4) for measurable multipliers. Rather, let B be the coboundary $B(e_t, \cdot) = \bar{p}(\cdot)p(\cdot - e_t)$ (which is defined for all $e_t \in \Lambda$) and let

$$A_d(e_t, \cdot) = \bar{p}(e_t)\bar{B}(e_t, \cdot)A_\omega(e_t, \cdot) \tag{3.5}$$

which defines $A_d(e_t, \cdot)$ as a unitary function for each $e_t \in \Lambda \cap \mathscr{N}$.

A straightforward computation using (3.3), (3.5) and the defining expressions for B and A_ω shows that $A_d(e_t, \cdot) = d(e_t, \cdot - e_t)$ for all $e_t \in \Lambda \cap \mathscr{N}$ and we conclude that $A_d(e_t, \cdot)$ has essential range $\{\pm 1\}$.

4. The construction. We can eliminate A_ω from (2.2) and (3.5) to obtain

$$A_d(e_t, \cdot) = \overline{pq}(e_t)\bar{B}A(e_t, \cdot) \tag{4.1}$$

for all $e_t \in \Lambda_0$.

With the exception of q all the terms in (4.1) are defined, at least, for all $e_t \in \Lambda \cap \mathscr{N}$. Hence the unitary function $P(e_t, \cdot)$ given by

$$P(e_t, \cdot) = \bar{A}_d(e_t, \cdot)\bar{B}A(e_t, \cdot) \tag{4.2}$$

is defined for all $e_t \in \Lambda \cap \mathscr{N}$ and coincides with the constant unitary function $pq(e_t)$ for $e_t \in \Lambda_0$.

Disregarding the fact that $A_d(e_t, \cdot)$ is not defined for all $e_t \in \Lambda$ the function A_d is a cocycle only if P is a cocycle. Now P^2 but not necessarily P is a cocycle and $D = \bar{r}\bar{B}A$ where r is a cocycle square root of P^2 is the desired nontrivial cocycle with range $\{\pm 1\}$. To see this first square both sides of (4.2) to obtain

$$P^2(e_t, \cdot) = (\bar{B}A)^2(e_t, \cdot) \tag{4.3}$$

for all $e_t \in \Lambda \cap \mathscr{N}$.

We can use (4.3) to extend P^2 to Λ since $(\bar{B}A)^2$ is a cocycle and as such $\bar{B}A(e_t, \cdot)$ is a unitary function for all $e_t \in \Lambda$. Thus, retaining the same notation, (4.3) is valid for all $e_t \in \Lambda$ and we see that P^2 is a cocycle.

Now $P^2(e_t, \cdot) = (pq)^2(e_t)$ for $e_t \in \Lambda_0$ and a routine application of the cocycle identity (2.1) shows that $P^2(e_t, \cdot)$ is a constant unitary function for all $e_t \in \Lambda$. Hence P^2 is a constant cocycle and we have $P^2(e_t, \cdot) = \exp(i2\lambda t)$ for some real number 2λ. The constant cocycle r given by $r(e_t) = \exp(i\lambda t)$ is evidently a square root of P^2.

Let D be defined for all $e_t \in \Lambda$ by

$$D(e_t, \cdot) = \bar{r}(e_t)\bar{B}A(e_t, \cdot)\ . \tag{4.4}$$

Clearly D is a cocycle and since D is a square root of $\bar{P}^2\bar{B}^2A^2 = 1$ it follows that the essential range of $D(e_t, \cdot)$ is contained in $\{\pm 1\}$ for each $e_t \in \Lambda$. Moreover, D is nontrivial because A is nontrivial.

5. **Remarks.** In [3] Helson showed that any cocycle A can be written as the product

$$A = CRD' \tag{5.1}$$

where C is a coboundary, D' a cocycle with range $\{\pm 1\}$ and R is a regular cocycle given by

$$R(e_t, x) = \exp\left(i\int_0^t m(x - e_u)du\right), \text{a.e.}(x)\ , \tag{5.2}$$

for some real Borel function m on T^2. It was the factoring (5.1) which led to the question if nontrivial cocycles with range $\{\pm 1\}$ exist.

If we apply the factoring (5.1) to the cocycle A induced by A_ω and substitute into (4.4) we obtain

$$D\bar{D}'(e_t, \cdot) = \bar{r}(e_t)(\bar{B}C)(e_t, \cdot)R(e_t, \cdot)\ . \tag{5.3}$$

Notice that $D\bar{D}'$ is trivial if and only if R is trivial. However, nothing is known about the regular factor R of the cocycle A induced by A_ω. In particular, if R were trivial then projective multipliers would give rise to a class of nontrivial cocycles quite distinct from the nontrivial regular cocycles produced in [4] and [5].

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Received February 29, 1972.

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