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A CLASS OF DIVISIBLE MODULES

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The $\mathscr{T}$-divisible R-modules are defined in terms of a hereditary torsion theory of modules over an associative ring R with identity element. In the special case where $\mathscr{T}$ is the usual torsion class of modules over a commutative integral domain, the class of $\mathscr{T}$-divisible modules is precisely the class of divisible modules M such that every nonzero homomorphic image of M has a nonzero h-divisible submodule. In general, if $\mathscr{T}$ is a stable hereditary torsion class, the class of $\mathscr{T}$-divisible modules satisfies many of the traditional properties of divisible modules over a commutative integral domain. This is especially true when $\mathscr{T}$ is Goldie's torsion class $\mathscr{G}$. For suitable $\mathscr{T}$, the splitting of all $\mathscr{T}$-divisible modules is equivalent to h.d.$Q_{\mathscr{T}} \leqq 1$, where $Q_{\mathscr{T}}$ is the ring of quotients naturally associated with $\mathscr{T}$. Generalizations of Dedekind domains are studied in terms of $\mathscr{T}$-divisibility.

1. Notation, terminology, and preliminary results. In this paper, all rings R are associative rings with identity element, and all modules are unitary left R-modules. ${}_R\mathscr{M}$ denotes the category of all left R-modules. $E(M)$ denotes the injective envelope of $M \in {}_R\mathscr{M}$. In homological expressions, the "R" will be omitted for convenience in printing (e.g. $\mathrm{Ext}_R^1 = \mathrm{Ext}^1$ and h.d. ${}_RQ =$ h.d. Q).

Following S.E. Dickson [4], we call a nonempty subclass $\mathscr{T}$ of ${}_R\mathscr{M}$ a *torsion class* if $\mathscr{T}$ is closed under factors, extensions, and arbitrary direct sums. $\mathscr{T}$ is called *hereditary* if it is closed under submodules. Modules in $\mathscr{T}$ are called *torsion*. Every torsion class $\mathscr{T}$ determines in every $A \in {}_R\mathscr{M}$ a unique maximal torsion submodule $\mathscr{T}(A)$. $\mathscr{T}(A)$ is called the *torsion submodule* of A, and $\mathscr{T}(A/\mathscr{T}(A)) = 0$. Modules in $\mathscr{F} = \{A \in {}_R\mathscr{M} \mid \mathscr{T}(A) = 0\}$ are called *torsionfree*, and the torsionfree class $\mathscr{F}$ is closed under submodules, extensions, and direct products. $\mathscr{T}$ is hereditary if and only if $\mathscr{F}$ is closed under injective envelopes.

Throughout this paper, $\mathscr{T}$ will denote a hereditary torsion class and $\mathscr{F}$ will denote the torsionfree class corresponding to $\mathscr{T}$; hence $(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory [4], [10], and [17]. Each such $(\mathscr{T}, \mathscr{F})$ is uniquely associated with a topologizing and idempotent *filter* $F(\mathscr{T}) = \{I \mid I \subseteqq R$ and $R/I \in \mathscr{T}\}$ of left ideals of R.

In [5] the right derived functors for a hereditary torsion class $\mathscr{T}$ are examined. These derived functors are given by

$$R^0_{\mathscr{T}}(A) = \mathscr{T}(A),\ R^1_{\mathscr{T}}(A) = \mathscr{T}(E(A)/A)\Big/\frac{\mathscr{T}(E(A)) + A}{A}\,,$$

and $R^n_{\mathscr{T}}(A) = R^{n-1}_{\mathscr{T}}(E(A)/A)$ for $n \geqq 2$. If A is injective, then $R^n_{\mathscr{T}}(A) = 0$ for $n \geqq 1$. If $A \in \mathscr{F}$, then $R^1_{\mathscr{T}}(A) = (\mathscr{T}E(A)/A)$. A module M is called *$\mathscr{T}$-injective* ([8], [9], and [15]) if $\mathrm{Ext}^1(T, M) = 0$ for all $T \in \mathscr{T}$. (Note: in [14] $\mathscr{T}$-injective modules are called $F(\mathscr{T})$-divisible.) This happens exactly when $\mathscr{T}(E(M)/M) = 0$ (see [8]). Hence a module $A \in \mathscr{F}$ is $\mathscr{T}$-injective if and only if $R^1_{\mathscr{T}}(A) = 0$. If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is any short exact sequence, then so is the induced sequence $0 \rightarrow \mathscr{T}(A) \rightarrow \mathscr{T}(B) \rightarrow \mathscr{T}(C) \rightarrow R^1_{\mathscr{T}}(A) \rightarrow R^1_{\mathscr{T}}(B) \rightarrow R^1_{\mathscr{T}}(C) \rightarrow R^2_{\mathscr{T}}(A) \rightarrow \cdots$.

A torsion class $\mathscr{T}$ is called *stable* if $\mathscr{T}$ is closed under injective envelopes. $R^1_{\mathscr{T}}(T) = 0$ for all $T \in \mathscr{T}$; and if $\mathscr{T}$ is stable, then $R^n_{\mathscr{T}}(T) = 0$ for all $T \in \mathscr{T}$ and for all $n \geqq 1$. This and the long exact sequence in the preceding paragraph imply that if $\mathscr{T}$ is stable and $B \in {}_R\mathscr{M}$, then $R^n_{\mathscr{T}}(B) = 0$ if and only if $R^n_{\mathscr{T}}(B/\mathscr{T}(B)) = 0$.

The submodule $E_{\mathscr{T}}(M)$ of $E(M)$ is the (unique) largest module satisfying $M \subseteqq E_{\mathscr{T}}(M) \subseteqq E(M)$ and $E_{\mathscr{T}}(M)/M \in \mathscr{T}$. We call $E_{\mathscr{T}}(M)$ the *$\mathscr{T}$-injective envelope* of M. $E_{\mathscr{T}}(M)$ exists [14] and is $\mathscr{T}$-injective. If $R \in \mathscr{F}$, then we will use $Q_{\mathscr{T}}$ to denote $E_{\mathscr{T}}(R)$. $Q_{\mathscr{T}}$ has a ring structure [8] and has received attention in a large number of papers (e.g.[7], [8], [9], and [15]).

A module M is said to *split* if $\mathscr{T}(M)$ is a direct summand of M.

We now state a lemma giving some properties of $\mathscr{T}$-injective modules.

LEMMA 1.1. (1) *$\mathscr{T}$ is stable if and only if every $\mathscr{T}$-injective modules splits.*

(2) *A direct summand of a $\mathscr{T}$-injective module is $\mathscr{T}$-injective.*

(3) *If A is a $\mathscr{T}$-injective module and if $\theta: A \rightarrow B$ is an epimorphism, then $\theta^{-1}(\mathscr{T}(B)) = \{x \in A \mid \theta(x) \in \mathscr{T}(B)\}$ is $\mathscr{T}$-injective.*

(4) *If $\mathscr{T}$ is stable, then every $\mathscr{T}$-injective module in $\mathscr{T}$ is injective.*

Proof. (1) If $\mathscr{T}$ is stable and A is $\mathscr{T}$-injective, then $E(A)/A \in \mathscr{F}$; so $\mathscr{T}(E(A)) \subseteqq A$. By [2, Prop. 2.1], $E(A) = \mathscr{T}(E(A)) \oplus F$ for some $F \in \mathscr{F}$. Hence $A = \mathscr{T}(E(A)) \oplus (F \cap A)$. Since $\mathscr{T}(E(A)) = \mathscr{T}(A)$ in this case, then A splits.

The converse is immediate from [2, Prop. 2.1].

(2) This is well-known and straight forward to prove.

(3) Note that

$$\frac{E_{\mathscr{T}}(\theta^{-1}(\mathscr{T}(B)))}{\theta^{-1}(\mathscr{T}(B))} \subseteqq \frac{A}{\theta^{-1}(\mathscr{T}(B))} \cong \frac{\theta(A)}{\theta\theta^{-1}(\mathscr{T}(B))} = \frac{B}{\mathscr{T}(B)} \in \mathscr{F}\,.$$

So by the definition of $\mathscr{T}$-injective envelope, $E_{\mathscr{T}}(\theta^{-1}(\mathscr{T}(B))) = \theta^{-1}(\mathscr{T}(B))$.

(4) Let $T \in \mathscr{T}$ be $\mathscr{T}$-injective; so $E(T) \in \mathscr{T}$ and $E(T)/T \in \mathscr{F}$. Since $\mathscr{T}$ is closed under homomorphic images, $E(T)/T \in \mathscr{T} \cap \mathscr{F} = \{0\}$; thus $E(T) = T$.

We now turn our attention to an important special torsion theory. The class $\mathscr{G}$ of Goldie torsion modules is the smallest class containing all isomorphic copies of all factor modules A/B, where B is an essential submodule of A (see [1], [2], [16], and [17]). The corresponding Goldie torsionfree class $\mathscr{N}$ is exactly the class of nonsingular modules. $\mathscr{G}$ is hereditary and stable; if $R \in \mathscr{N}$, then $\mathscr{G}$ is precisely the class of singular modules. *So if R is a commutative integral domain, then $\mathscr{G}$ and $\mathscr{N}$ coincide with the usual torsion and torsion-free classes, respectively.* This makes $(\mathscr{G}, \mathscr{N})$ a "natural" torsion theory to consider when one is trying to generalize the usual results about torsion modules over an integral domain. Moreover, the $\mathscr{G}$-injective modules are just the injective modules; and if $R \in \mathscr{N}$, then $Q_{\mathscr{G}} \cong E(R)$ and the filter $F(\mathscr{G})$ is the set of essential left ideals of R.

2. **$\mathscr{T}$-divisible modules.** Historically the class of divisible modules has been very useful in studying modules over a commutative integral domain. Frequently subclasses of the class of divisible modules have also yielded interesting results (e.g. see [11] and [12]). One such subclass studied by Matlis is the class of *h-divisible modules*, i.e., those modules which are homomorphic images of injective modules. A class related to the h-divisible modules is the class of divisible modules M such that every nonzero homomorphic image of M has a nonzero h-divisible submodule. For reasons indicated later, we refer to this latter class as the $\mathscr{G}$-divisible modules. We shall show that not only does the class of $\mathscr{G}$-divisible modules satisfy many interesting properties but also it allows us to "smooth out" some of Matlis' results. For example, Matlis proves [11, Cor. 2.6] that every divisible module splits if and only if (1) h.d. $Q \leqq 1$ (where Q is the quotient field of R) and (2) $T \cong \mathrm{Ext}^1(Q/R, T)$ for every torsion divisible module T with 0 as its only h-divisible submodule. We remove condition (2) by only considering $\mathscr{G}$-divisible modules: every $\mathscr{G}$-divisible module splits if and only if h.d. $Q \leqq 1$ (see Corollary 4.6).

However, we will not limit our investigation to modules over a commutative integral domain. We are able to state our entire theory in terms of hereditary torsion theories $(\mathscr{T}, \mathscr{F})$ of modules over an associative ring R with identity element.

To do this, we begin by defining subclasses $\mathscr{C}_\alpha$ of ${}_R\mathscr{M}$ for each ordinal number α.

$\mathscr{C}_1 = \{M \in {}_R\mathscr{M} \mid M$ is a homomorphic image of a direct sum of $\mathscr{T}$-injective modules$\}$. If α is not a limit ordinal, then $\mathscr{C}_\alpha = \{M \in {}_R\mathscr{M} \mid \exists N \subseteqq M$ such that $N \in \mathscr{C}_{\alpha-1}$ and $M/N \in \mathscr{C}_1\}$. If α is a limit ordinal, then $\mathscr{C}_\alpha = \{M \in {}_R\mathscr{M} \mid M = \cup_{\beta<\alpha} N_\beta$ where $N_\beta \in \mathscr{C}_\beta$ for $\beta < \alpha\}$. This transfinite definition is similar to the construction of a Loewy series [6]. Clearly $\mathscr{C}_\beta \subseteqq \mathscr{C}_\alpha$ for $\beta \leqq \alpha$. It is straight forward to show that each $\mathscr{C}_\alpha$ is closed under homomorphic images and direct sums.

If R is a commutative integral domain and if the torsion class is taken to be $\mathscr{G}$, it follows that each $\mathscr{C}_\alpha$ is a subclass of the (usual) class of divisible modules. Also, over a commutative integral domain, any divisible module in $\mathscr{N}$ is injective.

These two facts, plus the fact that each $\mathscr{C}_\alpha$ depends on $\mathscr{T}$, motivates the terminology, "$\mathscr{T}$-divisible module", for any hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ over any ring.

DEFINITION. A module M is called $\mathscr{T}$-divisible if and only if (i) $M/\mathscr{T}(M)$ is $\mathscr{T}$-injective and (ii) $\mathscr{T}(M) \in \mathscr{C}_\alpha$ for some ordinal α. $\mathscr{D}\mathscr{T}$ will denote the class of all $\mathscr{T}$-divisible modules.

EXAMPLE 2.1. If R is a commutative integral domain, then a module M is $\mathscr{G}$-divisible if and only if (a) M is divisible and (b) every nonzero homomorphic image of M has a nonzero h-divisible submodule.

Let $M \in \mathscr{D}\mathscr{G}$. From the construction of the classes $\mathscr{C}_\alpha$, the closure properties of divisible modules, and the definition of $\mathscr{G}$-divisible, it follows that M is divisible. Let $\theta\colon M \to X$ be an epimorphism. If $\ker\theta$ does not contain $\mathscr{G}(M)$, then it follows from (ii) that $(\mathscr{G}(M) + \ker\theta)/\ker\theta$ contains an h-divisible submodule. If $\ker\theta \supseteqq \mathscr{G}(M)$, then X is a homomorphic image of $M/\mathscr{G}(M)$, and hence X is h-divisible by (i).

Conversely, let M satisfy (a) and (b). Since $M/\mathscr{G}(M)$ is torsionfree and divisible, then $M/\mathscr{G}(M)$ is injective by [3, VII, Prop. 1.3]; so (i) holds.

Since any nonzero $\mathscr{G}$-torsionfree injective module is a direct sum of copies of the quotient field Q, then M contains a maximal $\mathscr{G}$-torsionfree injective module B (possibly $B = 0$). Set $M = A \oplus B$. By (b) A contains an h-divisible submodule H_1. From [11, Theorem 1.1] and [3, VII. Prop. 1.3), it follows that $H_1 = \mathscr{G}(H_1) \oplus F$ with $F \in \mathscr{N}$ injective. By the definition of B, $F = 0$. Hence $H_1 \subseteqq \mathscr{G}(A) = \mathscr{G}(M)$. Using similar reasoning on A/H_1, we can find $H_2 \subseteqq A$ such that H_2/H_1 is an h-divisible submodule of $\mathscr{G}(A/H_1) = \mathscr{G}(A)/H_1$. Thus $H_2 \in \mathscr{C}_2 \cap \mathscr{G}$. If β is a limit ordinal and $H_\gamma \in \mathscr{C}_\gamma \cap \mathscr{G}$ has been defined for all $\gamma < \beta$, define $H_\beta = \cup_{\gamma<\beta} H_\gamma$; then $H_\beta \in \mathscr{C}_\beta \cap \mathscr{G}$. Proceeding by transfinite induction, the reader can now easily show that there exists $H_\alpha \in \mathscr{C}_\alpha$ such that $H_\alpha = \mathscr{G}(A) = \mathscr{G}(M)$; so (ii) holds.

EXAMPLE 2.2. The homomorphic image of a $\mathscr{T}$-injective module need not be $\mathscr{T}$-divisible.

Let K be a field, let $R = K[x, y]$ (commutative), and let $M = (x, y)$. Define

$$F(\mathscr{T}_M) = \{I \subseteq R \mid M^n \subseteq I \text{ for some } n\}\,.$$

Then [15, p. 40, Ex. 3] and [8, Theorem 4.5] show that there exists a $\mathscr{T}_M$-injective module $E \in \mathscr{F}_M$ with a homomorphic image $E/E' \in \mathscr{F}_M$ such that E/E' is not $\mathscr{T}_M$-injective. Thus E/E' is not $\mathscr{T}_M$-divisible.

For later use, we note that since R is a commutative Noetherian ring, then $\mathscr{T}_M$ is stable by [15, Prop. 5.12].

Our first result shows that the class of $\mathscr{T}$-divisible modules satisfies some traditional properties [3, p. 128] of modules over a commutative integral domain.

PROPOSITION 2.3. (1) *If $\mathscr{T}$ is stable, then every injective module is $\mathscr{T}$-divisible.*

(2) *If $\mathscr{T}$ is stable, then every $\mathscr{T}$-injective module is $\mathscr{T}$-divisible.*

(3) *Every $\mathscr{T}$-injective module in $\mathscr{T}$ is $\mathscr{T}$-divisible.*

(4) *Every $\mathscr{T}$-divisible module in $\mathscr{F}$ is $\mathscr{T}$-injective.*

Proof. (1) and (2). Since every injective module is $\mathscr{T}$-injective, it is sufficient to show that every $\mathscr{T}$-injective module A is $\mathscr{T}$-divisible. By Lemma 1.1 (1), A splits, and hence $\mathscr{T}(A)$ and $A/\mathscr{T}(A)$ are both $\mathscr{T}$-injective by Lemma 1.1 (2). Thus A is $\mathscr{T}$-divisible.

(3) and (4) follow from the definition of a $\mathscr{T}$-divisible module.

The next three propositions investigate the closure properties of the class $\mathscr{D}\mathscr{T}$ of $\mathscr{T}$-divisible modules.

PROPOSITION 2.4. *If $\mathscr{T}$ is stable, then the following statements are true.*

(1) *$\mathscr{D}\mathscr{T}$ is closed under injective envelopes.*

(2) *$\mathscr{D}\mathscr{T}$ is closed under $\mathscr{T}$-injective envelopes.*

(3) *$\mathscr{D}\mathscr{T}$ is closed under direct summands.*

(4) *$\mathscr{D}\mathscr{T}$ is closed under extensions.*

(5) *$\mathscr{D}\mathscr{T}$ is closed under finite direct sums.*

Proof. (1) follows from Proposition 2.3 (1).

(2) follows from Proposition 2.3 (2).

(3) Suppose that $A = B \oplus C$ and that $A \in \mathscr{D}\mathscr{T}$. Then $\mathscr{T}(B) = \pi(\mathscr{T}(A))$, where π is the projection map from A to B. Since $\mathscr{T}(A) \in \mathscr{C}_\alpha$ for some ordinal α and since $\mathscr{C}_\alpha$ is closed under homomorphic images,

then $\mathscr{T}(B) \in \mathscr{C}_\alpha$ also. Since $A \in \mathscr{D}\mathscr{T}$, then $A/\mathscr{T}(A)$ is $\mathscr{T}$-injective. Since $A/\mathscr{T}(A) \cong (B/\mathscr{T}(B)) \oplus (C/\mathscr{T}(C))$, then $B/\mathscr{T}(B)$ is $\mathscr{T}$-injective by Lemma 1.1 (2). Therefore B is $\mathscr{T}$-divisible.

(4) Let $A, B \in \mathscr{D}\mathscr{T}$, and let

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

be an exact sequence. Since $A, B \in \mathscr{D}\mathscr{T}$, the induced sequence

$$0 = R^1_{\mathscr{T}}(A) \longrightarrow R^1_{\mathscr{T}}(X) \longrightarrow R^1_{\mathscr{T}}(B) = 0$$

is exact. Since $\mathscr{T}$ is stable, it follows from the preceding sequence that $R^1_{\mathscr{T}}(X/\mathscr{T}(X)) = 0$. Hence $X/\mathscr{T}(X)$ is $\mathscr{T}$-injective.

Since $A, B \in \mathscr{D}\mathscr{T}$, then $\mathscr{T}(A) \in \mathscr{C}_\alpha$ and $\mathscr{T}(B) \in \mathscr{C}_\beta$ for some ordinals α and β. Since $R^1_{\mathscr{T}}(A) = 0$, the sequence

$$0 \longrightarrow \mathscr{T}(A) \longrightarrow \mathscr{T}(X) \longrightarrow \mathscr{T}(B \longrightarrow R^1_{\mathscr{T}}(A) = 0$$

is exact. It is straight forward to verify from this sequence that $\mathscr{T}(X) \in \mathscr{C}_{\alpha+\beta}$.

(5) follows from (4) by induction.

If $\mathscr{T}$ is a hereditary torsion class in ${}_R\mathscr{M}$, then a ring R is said to have $\mathscr{T}$-gl. dim $R = n$ if $R^{n+1}_{\mathscr{T}}(M) = 0$ for all $M \in {}_R\mathscr{M}$. In [17] conditions equivalent to $\mathscr{T}$-gl. dim $R = 0$ are given. If $\mathscr{T}$ is stable, then the next result gives a characterization of $\mathscr{T}$-gl. dim $R \leqq 1$.

PROPOSITION 2.5. *If $\mathscr{T}$ is stable, then the following statements are equivalent.*

(1) *$\mathscr{D}\mathscr{T}$ is closed under homomorphic images.*

(2) *$\mathscr{T}$-gl. dim $R \leqq 1$.*

(3) *For any $\mathscr{T}$-injective module A and any epimorphism $\theta: A \to B$ such that $B \in \mathscr{F}$, B is $\mathscr{T}$-injective.*

(4) *For any $\mathscr{T}$-injective module $A \in \mathscr{F}$ and any epimorphism $\theta: A \to B$ such that $B \in \mathscr{F}$, B is $\mathscr{T}$-injective.*

REMARKS. (i) Additional conditions equivalent to (4) are given in [8, Theorem 4.5].

(ii) The equivalence of (2) and (3) is due to C. Megibben, who communicated this equivalence to the author in a personal letter (Jan., 1971). The author wishes to thank Professor Megibben for sending him the result. Moreover, the proof of the equivalence of (2) and (3) does not require that $\mathscr{T}$ be stable.

(iii) The interesting case in Proposition 2.5 is when $\mathscr{T}$-gl. dim $R = 1$. For if $R \in \mathscr{F}$ and $\mathscr{T}$-gl. dim $R = 0$, then by [17, Theorem 3.1 (3)] every module in ${}_R\mathscr{M}$ is also in $\mathscr{F}$.

(iv) In Example 2.2, $\mathscr{T}_M$-gl. dim $R > 1$, and $\mathscr{T}_M$ is stable; so not every hereditary torsion theory satisfies Proposition 2.5.

Proof of Proposition 2.5. (1)$\Rightarrow$(2). It follows from the exactness of the sequence

$$R^1_{\mathscr{T}}(E(M)/M) \longrightarrow R^2_{\mathscr{T}}(M) \longrightarrow R^2_{\mathscr{T}}(E(M)) = 0$$

that it is sufficient to show $R^1_{\mathscr{T}}(E(M)/M) = 0$. Since $\mathscr{T}$ is stable, Proposition 2.3 (1) implies $E(M)$ is $\mathscr{T}$-divisible. Set $X = E(M)/M$. By (1), $X/\mathscr{T}(X) \in \mathscr{F}$ is $\mathscr{T}$-divisible; hence $X/\mathscr{T}(X)$ is $\mathscr{T}$-injective by Proposition 2.3 (4). Thus $R^1_{\mathscr{T}}(X/\mathscr{T}(X)) = 0$, and consequently $R^1_{\mathscr{T}}(E(M)/M) = R^1_{\mathscr{T}}(X) = 0$ as desired.

(2)$\Rightarrow$(3). Let A be a $\mathscr{T}$-injective module, let $B \in \mathscr{F}$, and let $\theta\colon A \to B$ be an epimorphism. By (2) and the $\mathscr{T}$-injectivity of A, the sequence

$$0 = R^1_{\mathscr{T}}(A) \longrightarrow R^1_{\mathscr{T}}(B) \longrightarrow R^2_{\mathscr{T}}(\ker\theta) = 0$$

is exact. Thus $R^1_{\mathscr{T}}(B) = 0$; so $B \in \mathscr{F}$ is $\mathscr{T}$-injective.

(3)$\Rightarrow$(4) is trivial.

(4)$\Rightarrow$(1). Let $D \in \mathscr{D}\mathscr{T}$, and let $\theta\colon D \to M$ be an epimorphism. Now θ naturally induces an epimorphism

$$\bar{\theta}\colon D/\mathscr{T}(D) \longrightarrow M/\theta(\mathscr{T}(D)) .$$

Since $D/\mathscr{T}(D)$ is $\mathscr{T}$-injective, then

$$(M/\theta(\mathscr{T}(D)))/(\mathscr{T}(M)/\theta(\mathscr{T}(D))) \cong M/\mathscr{T}(M) \in \mathscr{F}$$

is also $\mathscr{T}$-injective by (4).

From the definition of $\mathscr{T}$-divisible, it follows that $\theta(\mathscr{T}(D)) \in \mathscr{C}_\alpha$ for some ordinal α. By Lemma 1.1 (3), $\bar{\theta}^{-1}(\mathscr{T}(M/\theta(\mathscr{T}(D))))$ is $\mathscr{T}$-injective. Thus

$$\mathscr{T}(M)/\theta(\mathscr{T}(D)) \cong \mathscr{T}(M/\theta(\mathscr{T}(D))) \in \mathscr{C}_1 ,$$

and consequently $\mathscr{T}(M) \in \mathscr{C}_{\alpha+1}$.

COROLLARY 2.6. *The class $\mathscr{D}\mathscr{T}$ of $\mathscr{T}$-divisible modules is closed under homomorphic images.*

Proof. This follows from Proposition 2.5 and [1, p. 197].

We shall say that the filter $F(\mathscr{T})$ is $\mathscr{T}$-noetherian if the following property holds: if $I_1 \subset I_2 \subset I_3 \cdots$ is a countable ascending chain of left ideals whose union is in $F(\mathscr{T})$, then $I_n \in F(\mathscr{T})$ for some n. The reader is directed to [8] and [15] for an extensive discussion of the $\mathscr{T}$-noetherian property. Our next proposition adds another equivalent condition to the lists given in [8, Theorem 4.4] and [15, Prop. 12.1].

PROPOSITION 2.7. *$\mathscr{D}\mathscr{T}$ is closed under direct sums if and only if $F(\mathscr{T})$ is $\mathscr{T}$-noetherian.*

Proof. Let $\mathscr{A}$ be an index set, and let $A_\alpha \in \mathscr{D}\mathscr{T}$ for each $\alpha \in \mathscr{A}$. Since $\mathscr{T}(\bigoplus \sum_{\alpha \in \mathscr{A}} A_\alpha) = \bigoplus \sum_{\alpha \in \mathscr{A}} \mathscr{T}(A_\alpha)$, it follows that $\mathscr{T}(\bigoplus \sum_{\alpha \in \mathscr{A}} A_\alpha) \in \mathscr{C}_\mu$, where $\mu = \sup\{\beta_\alpha \mid \mathscr{T}(A_\alpha) \in \mathscr{C}_{\beta_\alpha}\}$. Hence $\mathscr{D}\mathscr{T}$ is closed under direct sums if and only if $(\bigoplus \sum_{\alpha \in \mathscr{A}} A_\alpha)/\mathscr{T}(\bigoplus \sum_{\alpha \in \mathscr{A}} A_\alpha) \cong \bigoplus \sum_{\alpha \in \mathscr{A}} (A_\alpha/\mathscr{T}(A_\alpha))$ is $\mathscr{T}$-injective. But the latter condition holds if and only if any direct sum of $\mathscr{T}$-injective modules in $\mathscr{F}$ is $\mathscr{T}$-injective. So the result follows from [8, Theorem 4.4].

COROLLARY 2.8. *$\mathscr{D}\mathscr{G}$ is closed under direct sums if and only if $F(\mathscr{G})$ has a cofinal subset of finitely generated left ideals. In particular, if R has a semi-simple maximal left quotient ring, then $\mathscr{D}\mathscr{G}$ is closed under direct sums.*

REMARK. Corollary 2.8 adds another condition to the list given in [16, Theorem 2.1].

COROLLARY 2.9. *If $F(\mathscr{G})$ has a cofinal subset of finitely generated left ideals, then every module A has a (necessarily unique) largest submodule $\mathscr{D}\mathscr{G}(A)$ in $\mathscr{D}\mathscr{G}$. In particular, if R has a semi-simple maximal left quotient ring, then every module A has a largest submodule $\mathscr{D}\mathscr{G}(A)$ in $\mathscr{D}\mathscr{G}$.*

Proof. This follows from Proposition 2.4 (4), Corollary 2.6, Corollary 2.8, and [4, Theorem 2.3].

3. Analogs of the Dedekind domain case. A commutative integral domain D is called Dedekind if any one of the following three equivalent properties hold: (a) every ideal of D is projective; (b) every divisible module is injective; and (c) the homomorphic image of an injective module is injective. In this section, we look at the analogs of these conditions in terms of hereditary torsion theories $(\mathscr{T}, \mathscr{F})$ over more general rings: (a) every left ideal in $F(\mathscr{T})$ is projective; (b) every $\mathscr{T}$-divisible module is $\mathscr{T}$-injective; and (c) the homomorphic image of a $\mathscr{T}$-injective module is $\mathscr{T}$-injective. Our goal is to obtain characterizations of these conditions in terms of the class $\mathscr{D}\mathscr{T}$ of $\mathscr{T}$-divisible modules.

We start our investigation by considering the following condition:

$$(*) \qquad \text{h.d. } E_{\mathscr{T}}(F) \leqq 1 \text{ for all free modules } F \in {}_R\mathscr{M}\,.$$

Clearly any torsion theory over a left hereditary ring satisfies (*). In the commutative integral domain case, Matlis [11] and [12] considered rings for which h.d. $Q \leqq 1$, where Q is the quotient field of the domain. Since $Q = Q_{\mathscr{F}}$, the following result shows that (*) is a "natural" generalization of the condition, "h.d. $Q \leqq 1$."

PROPOSITION 3.1. *Let $R \in \mathscr{F}$, and let $F(\mathscr{T})$ be $\mathscr{T}$-noetherian. Then* (*) *holds if and only if* h.d. $Q_{\mathscr{T}} \leqq 1$.

Proof. The "only if" part is trivial since $Q_{\mathscr{T}} = E_{\mathscr{T}}(R)$.

Let $\mathscr{A}$ be an index set, let $R^{(\alpha)} \cong R$ for each $\alpha \in \mathscr{A}$, and let $F = \bigoplus \sum_{\alpha \in \mathscr{A}} R^{(\alpha)}$. Since $F(\mathscr{T})$ is $\mathscr{T}$-noetherian, then $E_{\mathscr{T}}(F) = \bigoplus \sum_{\alpha \in \mathscr{A}} Q_{\mathscr{T}}^{(\alpha)}$, where $Q_{\mathscr{T}}^{(\alpha)} \cong Q_{\mathscr{T}}$ for each $\alpha \in \mathscr{A}$. So if h.d. $Q_{\mathscr{T}} \leqq 1$, then

$$\mathrm{Ext}^2(E_{\mathscr{T}}(F), \_) = \mathrm{Ext}^2(\bigoplus \textstyle\sum_{\alpha \in \mathscr{A}} Q_{\mathscr{T}}^{(\alpha)}, \_) \cong \Pi_{\alpha \in \mathscr{A}} \mathrm{Ext}(Q_{\mathscr{T}}^{(\alpha)}, -) = 0 .$$

Hence h.d. $E_{\mathscr{T}}(F) \leqq 1$.

LEMMA 3.2. *If every homomorphic image of a $\mathscr{T}$-injective module is $\mathscr{T}$-injective, then* (*) *holds.*

Proof. Let A be a $\mathscr{T}$-injective module, let F be a free module, and let $L = E_{\mathscr{T}}(F)/F \in \mathscr{T}$. By the hypothesis, the sequence

$$0 = \mathrm{Ext}^1(L, E(A)/A) \longrightarrow \mathrm{Ext}^2(L, A) \longrightarrow \mathrm{Ext}^2(L, E(A)) = 0$$

is exact. Hence $\mathrm{Ext}^2(L, A) = 0$.

Now let $M \in {}_R\mathscr{M}$, and set $A = E_{\mathscr{T}}(M)$. By the hypothesis and the preceding paragraph, the sequence

$$0 = \mathrm{Ext}^1(L, A/M) \longrightarrow \mathrm{Ext}^2(L, M) \longrightarrow \mathrm{Ext}^2(L, A) = 0$$

is exact. Hence $\mathrm{Ext}^2(L, M) = 0$. Since F is a free module, the sequence

$$0 = \mathrm{Ext}^2(L, M) \longrightarrow \mathrm{Ext}^2(E_{\mathscr{T}}(F), M) \longrightarrow \mathrm{Ext}^2(F, M) = 0$$

is exact. Thus $\mathrm{Ext}^2(E_{\mathscr{T}}(F), M) = 0$ for all $M \in {}_R\mathscr{M}$, and hence (*) holds.

LEMMA 3.3. *If* (*) *holds, then the following statements are equivalent.*

(1) $\mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, H) = 0$ *for all free modules* F.

(2) *Every homomorphism from a free module F into H can be extended to a homomorphism from $E_{\mathscr{T}}(F)$ into H.*

(3) *H is a homomorphic image of a $\mathscr{T}$-injective module.*

Proof. (1) $\Rightarrow$ (2) follows from the exact sequence

$$\mathrm{Hom}(E_{\mathscr{T}}(F), H) \longrightarrow \mathrm{Hom}(F, H) \rightarrow \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, H).$$

(2) $\Rightarrow$ (3) is trivial since every module is a homomorphic image of a free module.

(3) $\Rightarrow$ (1). Let A be a $\mathscr{T}$-injective module, and let $\theta: A \to H$ be an epimorphism. It follows from (*) that h.d. $E_{\mathscr{T}}(F)/F \leqq 1$ for any free module F. Using this fact, we have the exact sequence

$$\begin{aligned}\mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, A) &\longrightarrow \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, H)\\ &\longrightarrow \mathrm{Ext}^2(E_{\mathscr{T}}(F)/F, \ker\theta) = 0\,.\end{aligned}$$

Since A is $\mathscr{T}$-injective, the left end of this sequence is also 0, and hence (1) follows from the exactness of the sequence.

REMARK. Lemma 3.3 is a generalization of [11, Prop. 2.1] (see also [13, Prop. 3.1]). If $R \in \mathscr{F}$ and $F(\mathscr{T})$ is $\mathscr{T}$-noetherian, then condition (2) in Lemma 3.3 can be replaced by the following condition: every homomorphism from R into H can be extended to a homomorphism from $Q_{\mathscr{T}}$ into H.

The next lemma gives us conditions which are sufficient to insure that every $\mathscr{T}$-divisible module is a homomorphic image of a $\mathscr{T}$-injective module.

LEMMA 3.4. *If $R \in \mathscr{F}$, if $F(\mathscr{T})$ is $\mathscr{T}$-noetherian, and if (*) holds, then every $\mathscr{T}$-divisible module is a homomorphic image of a $\mathscr{T}$-injective module.*

Proof. Let $D \in \mathscr{D}\mathscr{T}$; so $D/\mathscr{T}(D)$ is $\mathscr{T}$-injective. Temporarily assume that $\mathscr{T}(D)$ is a homomorphic image of a $\mathscr{T}$-injective module. Then by Lemma 3.3, the sequence

$$\begin{aligned}0 = \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, \mathscr{T}(D)) &\longrightarrow \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, D)\\ &\longrightarrow \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, D/\mathscr{T}(D)) = 0\end{aligned}$$

is exact for every free module F. Hence $\mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, D) = 0$ by exactness. Thus Lemma 3.3 implies D is a homomorphic image of a $\mathscr{T}$-injective module. So in order to prove the theorem, it is sufficient to prove that $\mathscr{T}(D)$ is a homomorphic image of a $\mathscr{T}$-injective module.

Since $D \in \mathscr{D}\mathscr{T}$, then $\mathscr{T}(D) \in \mathscr{C}_\alpha$ for some ordinal α. Let $\{\mathscr{D}_\beta \mid \beta \in \mathscr{B}\}$ be the set of submodules of $\mathscr{T}(\mathscr{D})$ such that each D_β is a homomorphic image of a direct sum of $\mathscr{T}$-injective modules. It is easy to see that there exists $\gamma \in \mathscr{B}$ such that $D_\gamma = \sum_{\beta \in \mathscr{B}} D_\beta$. Now each $\mathscr{T}$-injective module is a homomorphic image of $E_{\mathscr{T}}(F)$ for some free module F. Since $F(\mathscr{T})$ is a $\mathscr{T}$-noetherian, then by [8, Theorem 4.4] each $E_{\mathscr{T}}(F)$ is a direct sum of copies of $Q_{\mathscr{T}}$, and any direct sum of copies of $Q_{\mathscr{T}}$ is $\mathscr{T}$-injective. Hence D_γ is a homomorphic image of a $\mathscr{T}$-injective module. To complete the proof, we wish to show $D_\gamma = \mathscr{T}(D)$.

If $D_\gamma \neq \mathscr{T}(D) \in \mathscr{C}_\alpha$, then there exists a $\mathscr{T}$-injective module A such that $\mathrm{Hom}(A, \mathscr{T}(D)/D_\gamma) \neq 0$. Let $\phi: A \to \mathscr{T}(D)/D_\gamma$ be a nonzero homomorphism, and let $\mathrm{im}\,\phi = B/D_\gamma$, where $B \subseteqq \mathscr{T}(D)$. By Lemma 3.3, the sequence

$$\begin{aligned} 0 = \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, D_\gamma) &\longrightarrow \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, B) \\ &\longrightarrow \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, B/D_\gamma) = 0 \end{aligned}$$

is exact. Hence $\mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, B) = 0$; so B is a homomorphic image of a $\mathscr{T}$-injective module by Lemma 3.3. Hence there exists $\sigma \in \mathscr{B}$ such that $B = D_\sigma \subseteqq \sum_{\beta \in \mathscr{B}} D_\beta = D_\gamma$. But this contradicts our choice of B; so $D_\gamma = \mathscr{T}(D)$ as desired.

PROPOSITION 3.5. *Suppose that $R \in \mathscr{F}$ and that $F(\mathscr{T})$ is $\mathscr{T}$-noetherian. If every homomorphic image of a $\mathscr{T}$-injective module is $\mathscr{T}$-injective, then every homomorphic image of a $\mathscr{T}$-divisible module is $\mathscr{T}$-injective.*

Proof. By Lemma 3.2 and the hypothesis, (*) holds. Since $F(\mathscr{T})$ is $\mathscr{T}$-noetherian, the Lemma 3.4 implies every $\mathscr{T}$-divisible is a homomorphic image of a $\mathscr{T}$-injective module. Consequently, every homomorphic image of a $\mathscr{T}$-divisible module is also a homomorphic image of a $\mathscr{T}$-injective module; so the result follows from the hypothesis.

PROPOSITION 3.6. *Suppose that $\mathscr{T}$ is stable. If every homomorphic image of a $\mathscr{T}$-divisible module is $\mathscr{T}$-injective, then every homomorphic image of a $\mathscr{T}$-injective module is $\mathscr{T}$-injective.*

Proof. Since $\mathscr{T}$ is stable, Proposition 2.1 (2) implies that every $\mathscr{T}$-injective module is $\mathscr{T}$-divisible; so the result now follows from the hypothesis.

Collecting our previous results, we obtain:

THEOREM 3.7 *Suppose that $\mathscr{T}$ is stable and that $F(\mathscr{T})$ is $\mathscr{T}$-noetherian. If $R \in \mathscr{F}$, then the following statements are equivalent.*

(1) *Every homomorphic image of a $\mathscr{T}$-divisible module is $\mathscr{T}$-injective.*

(2) *Every homomorphic image of a $\mathscr{T}$-injective module is $\mathscr{T}$-injective.*

(3) *Every left ideal in $F(\mathscr{T})$ is projective.*

Proof. $(1) \Rightarrow (2)$ is immediate from Proposition 3.6; $(2) \Rightarrow (1)$ is immediate from Proposition 3.5; and $(2) \Leftrightarrow (3)$ is [9, Prop. 3.3].

COROLLARY 3.8. *Suppose that R has a semi-simple left maximal quotient ring. Then every homomorphic image of a $\mathscr{G}$-divisible module is injective if and only if R is hereditary.*

As a special case of Corollary 3.8, we obtain the following characterization of a Dedekind domain.

COROLLARY 3.9. *Let R be a commutative integral domain. Then every homomorphic image of a $\mathscr{G}$-divisible module is injective if and only if R is a Dedekind domain.*

REMARKS. From Corollary 3.9 we see that if R is a Dedekind domain, then every $\mathscr{G}$-divisible module is injective, and hence every divisible module is $\mathscr{G}$-divisible. However, the classes of divisible and $\mathscr{G}$-divisible modules may coincide without R being a Dedekind domain. In particular, if R is a commutative integral domain with quotient field Q that is a countably generated R-module, then every divisible module is $\mathscr{G}$-divisible by [11, Theorem 1.3] and Corollary 2.6.

THEOREM 3.10. *If $\mathscr{T}$ is stable, then the following statements are equivalent.*

(1) *A module is $\mathscr{T}$-divisible if and only if it is $\mathscr{T}$-injective.*

(2) *Every $\mathscr{T}$-divisible module is $\mathscr{T}$-injective.*

(3) *Every $\mathscr{T}$-divisible module in $\mathscr{T}$ is injective.*

Moreover, each of the above three statements implies that the following statements are true.

(a) *$F(\mathscr{T})$ satisfies the ascending chain condition.*

(b) $\operatorname{inj}\dim M \leqq 1$ *whenever* $E(M)/M \in \mathscr{T}$; *in particular,* $\operatorname{inj}\dim T \leqq 1$ *for all* $T \in \mathscr{T}$.

Proof. $(1) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (3)$. Let $T \in \mathscr{T} \cap \mathscr{D}\mathscr{T}$. By (2), T is $\mathscr{T}$-injective. Since $\mathscr{T}$ is stable, Lemma 1.1 (4) implies T is injective.

$(3) \Rightarrow (1)$. Since $\mathscr{T}$ is stable, any $\mathscr{T}$-injective module is $\mathscr{T}$-divisible by Proposition 2.3 (2). So we need to show that any $\mathscr{T}$-divisible module is $\mathscr{T}$-injective. Let $D \in \mathscr{D}\mathscr{T}$; then $D/\mathscr{T}(D)$ is $\mathscr{T}$-injective. Since $D \in \mathscr{D}\mathscr{T}$, then also $\mathscr{T}(D) \in \mathscr{D}\mathscr{T}$; so $\mathscr{T}(D)$ is injective by (3). Therefore $D \cong \mathscr{T}(D) \oplus (D/\mathscr{T}(D))$ is $\mathscr{T}$-injective.

$(3) \Rightarrow (a)$. Let $I_1 \subseteqq I_2 \subseteqq I_3 \subseteqq \cdots$ be an ascending chain with each $I_n \in F(\mathscr{T})$. Since $\mathscr{T}$ is stable, each $E(R/I_n) \in \mathscr{T}$, and hence $\bigoplus \sum_{n=1}^{\infty} E(R/I_n) \in \mathscr{T}$. Now any direct sum of injective modules in $\mathscr{T}$ is $\mathscr{T}$-divisible and hence injective by (3). Therefore $\bigoplus \sum_{n=1}^{\infty} E(R/I_n)$ is injective. We now can complete the proof of (a) by a standard argument. Let $I = \bigcup_{n=1}^{\infty} I_n$, and define $\theta\colon I \to \bigoplus \sum_{n=1}^{\infty} E(R/I_n)$ via $x \to (x + I_n)$ for all $x \in I$. By injectivity, θ extends to a map from R to

$\bigoplus \sum_{n=1}^{\infty} E(R/I_n)$. Since R has an identity element, it follows that $I = I_m$ for some integer m.

(3) $\Rightarrow$ (b). If $E(M)/M \in \mathscr{T}$, then $E(M)/M$ is injective by (3). So the sequence

$$0 = \operatorname{Ext}^1(\_, E(M)/M) \to \operatorname{Ext}^2(\_, M) \to \operatorname{Ext}^2(\_, E(M)) = 0 .$$

is exact, and hence $\operatorname{Ext}^2(\_, M) = 0$. Therefore, inj dim $M \leqq 1$.

COROLLARY 3.11. *The following statements are equivalent.*

(1) *A module is $\mathscr{G}$-divisible if and only if it is injective.*

(2) *Every $\mathscr{G}$-divisible module is injective.*

(3) *Every $\mathscr{G}$-divisible module in $\mathscr{G}$ is injective.*

(4) *$F(\mathscr{G})$ satisfies the ascending chain condition, and l.* gl. dim $R \leqq 1$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) is immediate from Theorem 3.10.

(4) $\Rightarrow$ (3). Let $G \in \mathscr{G} \cap \mathscr{D}\mathscr{G}$. Let $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ be the set of submodules of G which are homomorphic images of injective modules. Set $A = \sum_{\alpha \in \mathscr{A}} A_\alpha$. If $\theta_\alpha: E_\alpha \to A_\alpha$ is an epimorphism of an injective module E_α, then by (4) A_α is injective. We wish to show that $E = \bigoplus \sum_{\alpha \in \mathscr{A}} A_\alpha$ is injective.

To see E is injective, it is sufficient to show that, for each essential left ideal I of R, any homomorphism $\varphi: I \to E$ can be extended to a homomorphism $\varphi': R \to E$. Since $E \in \mathscr{G}$, then $I/\ker \varphi \in \mathscr{G}$; so since $I \in F(\mathscr{G})$, it follows that $\ker \varphi \in F(\mathscr{G})$ also. Hence $R/\ker \varphi$ has a.c.c. on submodules by (4). But then $I/\ker \varphi$ is finitely generated, and hence im $\varphi \subseteqq \bigoplus \sum_{\alpha \in \mathscr{B}} A_\alpha$, where $\mathscr{B}$ is a finite subset of $\mathscr{A}$. Since a direct sum of finitely many injective modules is injective, then φ can be extended to $\varphi': R \to \bigoplus \sum_{\alpha \in \mathscr{B}} A_\alpha \subseteqq E$.

Now E is injective; so A is also injective by (4). Hence $G = A \oplus X$ for some $X \subseteqq G$. Since $X \in \mathscr{G}$ is $\mathscr{G}$-divisible by Proposition 2.4 (3), then $X = 0$ by the construction of A and the definition of $\mathscr{G}$-divisible. Hence $G = A$ is injective.

An immediate consequence of Corollary 3.11 is the following characterization of a Dedekind domain.

COROLLARY 3.12. *Let R be a commutative integral domain. Every $\mathscr{G}$-divisible module is injective if and only if R is a Dedekind domain.*

4. The splitting of $\mathscr{T}$-divisible modules and h.d. $Q_{\mathscr{T}} \leqq 1$. In this section we show that, for a wide class of torsion theories with $R \in \mathscr{F}$, h.d. $Q_{\mathscr{T}} \leqq 1$ if and only if every $\mathscr{T}$-divisible module splits.

We start by considering the condition:

(I) If F is a free module and A is $\mathscr{T}$-injective, then

$$\mathrm{Ext}^2(E_{\mathscr{T}}(F), A) = 0.$$

For the Goldie torsion theory, the class of injective modules coincides with the class of $\mathscr{G}$-injective modules; so (I) always holds for the Goldie torsion theory. The first result of this section relates (I) to $Q_{\mathscr{T}}$ when $R \in \mathscr{F}$.

PROPOSITION 4.1. *If $R \in \mathscr{F}$ and if $F(\mathscr{T})$ is $\mathscr{T}$-noetherian, then* (I) *holds if and only if* $\mathrm{Ext}^2(Q_{\mathscr{T}}, A) = 0$ *for all $\mathscr{T}$-injective modules A.*

Proof. Since $Q_{\mathscr{T}} = E_{\mathscr{T}}(R)$, the "only if" part is trivial. Since $F(\mathscr{T})$ is $\mathscr{T}$-noetherian, $E_{\mathscr{T}}(F) \cong \bigoplus \sum_{\alpha \in \mathscr{A}} Q_{\mathscr{T}}^{(\alpha)}$ for any free module F ($Q_{\mathscr{T}}^{(\alpha)} \cong Q_{\mathscr{T}}$ for all $\alpha \in \mathscr{A}$). So if $\mathrm{Ext}^2(Q_{\mathscr{T}}, A) = 0$ for all $\mathscr{T}$-injective modules A, then

$$\mathrm{Ext}^2(E_{\mathscr{T}}(F), A) \cong \mathrm{Ext}^2(\bigoplus \textstyle\sum_{\alpha \in \mathscr{A}} Q_{\mathscr{T}}^{(\alpha)}, A) \cong \Pi_{\alpha \in \mathscr{A}} \mathrm{Ext}^2(Q_{\mathscr{T}}^{(\alpha)}, A) = 0 .$$

The following result generalizes [11, Theorem 1.2] (see also [13, Prop. 2.2]).

LEMMA 4.2. *Assume that* (I) *holds and that $R \in \mathscr{F}$. Suppose that any $\mathscr{T}$-divisible module D splits whenever $\mathscr{T}(D)$ is a homomorphic image of a $\mathscr{T}$-injective module. Then* (*) *holds; i.e.,* h.d. $E_{\mathscr{T}}(F) \leqq 1$ *for all free modules $F \in {}_R\mathscr{M}$.*

Proof. Let $A \in {}_R\mathscr{M}$, and let F be a free module. Then consider the exact sequence

$$0 \longrightarrow E_{\mathscr{T}}(A)/A \xrightarrow{\alpha} D \longrightarrow E_{\mathscr{T}}(F) \longrightarrow 0 .$$

Since $\mathscr{T}$ is closed under homomorphic images and since $E_{\mathscr{T}}(F) \in \mathscr{F}$, it follows that im $\alpha = \mathscr{T}(D)$. Since $\mathscr{T}(D) \cong E_{\mathscr{T}}(A)/A$ and $E_{\mathscr{T}}(F) \in \mathscr{F}$, then D is $\mathscr{T}$-divisible. By hypothesis $\mathscr{T}(D)$ is a direct summand of D. But then the above sequence must split because im $\alpha = \mathscr{T}(D)$; thus $\mathrm{Ext}^1(E_{\mathscr{T}}(E), E_{\mathscr{T}}(A)/A) = 0$. So by (I), the sequence

$$\begin{aligned} 0 = \mathrm{Ext}^1(E_{\mathscr{T}}(F), E_{\mathscr{T}}(A)/A) &\longrightarrow \mathrm{Ext}^2(E_{\mathscr{T}}(F), A) \\ &\longrightarrow \mathrm{Ext}^2(E_{\mathscr{T}}(F), E_{\mathscr{T}}(A)) = 0 \end{aligned}$$

is exact. From the exactness it follows that $\mathrm{Ext}^2(E_{\mathscr{T}}(F), A) = 0$, and hence h.d. $E_{\mathscr{T}}(F) \leqq 1$.

Next we need the following condition:
(II) A $\mathscr{T}$-divisible module splits if it is a homomorphic image of a $\mathscr{T}$-injective module.

Matlis [11, Theorem 1.1] showed that (II) holds for the usual torsion theory over a commutative integral domain. Armendariz [2, Theorem 2.5] extended Matlis' result by showing that the Goldie torsion submodule of a homomorphic image of a quasi-injective module splits off. (Armendariz's result is for any ring.) A consequence of Armendariz's result is that (II) always holds for Goldie's torsion theory.

LEMMA 4.3. *Assume* (*) *and* (II) *hold. Let D be a $\mathscr{T}$-divisible module, and let H be a homomorphic image of a $\mathscr{T}$-injective module such that $H \subseteqq \mathscr{T}(D)$. Then D splits if and only if D/H splits.*

Proof. If D splits, then $D = \mathscr{T}(D) \oplus S$, where $S \in \mathscr{F}$. Then $D/H \cong (\mathscr{T}(D)/H) \oplus S$. Since $\mathscr{T}(D)/H = \mathscr{T}(D/H)$, then D/H splits.

Conversely, if D/H splits, then $D/H = (\mathscr{T}(D)/H) \oplus (G/H)$, where $H \subseteqq G \subseteqq D$. Since D is $\mathscr{T}$-divisible, then $G/H \cong (D/H)/(\mathscr{T}(D)/H) \cong D/\mathscr{T}(D) \in \mathscr{F}$ is $\mathscr{T}$-injective. Thus (*) and Lemma 3.3 imply that the sequence

$$0 = \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, H) \to \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, G) \to \mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, G/H) = 0$$

is exact. Hence $\mathrm{Ext}^1(E_{\mathscr{T}}(F)/F, G) = 0$. By Lemma 3.3, G is a homomorphic image of a $\mathscr{T}$-injective module. Since $G/H \in \mathscr{F}$ is $\mathscr{T}$-injective, the definition of H enables us to see that G is $\mathscr{T}$-divisible. By (II) G splits; thus $G = H \oplus L$ (as $H = \mathscr{T}(G)$). Hence $D = \mathscr{T}(D) \oplus L$, i.e., D splits.

THEOREM 4.4. *Assume (I) and (II) hold. If $F(\mathscr{T})$ is $\mathscr{T}$-noetherian and $R \in \mathscr{F}$, then the following statements are equivalent.*

(1) h.d. $Q_{\mathscr{T}} \leqq 1$.

(2) *Every $\mathscr{T}$-divisible module splits.*

Proof. (1) $\Rightarrow$ (2). By Proposition 3.1, (*) holds; so by Lemma 3.4, every $\mathscr{T}$-divisible module D is a homomorphic image of a $\mathscr{T}$-injective module. Thus $\mathscr{T}(D)$ is also a homomorphic image of a $\mathscr{T}$-injective module by Lemma 1.1 (3). So it follows trivially from Lemma 4.3 that D splits.

(2) $\Rightarrow$ (1) follows immediately from Lemma 4.2 and Proposition 3.1.

As an immediate consequence of Theorem 4.4, we have the following result.

COROLLARY 4.5. *If R has a semi-simple maximal left quotient ring, then the following statements are equivalent.*

(1) h.d. $E(R) \leqq 1$.

(2) *Every $\mathscr{G}$-divisible module splits.*

A special case of this result is the following corollary which the reader may wish to compare with [11, Cor. 2.6] (see also [13, Prop. 4.2]).

COROLLARY 4.6 *If R is an integral domain with quotient field $Q \neq R$, then the following statements are equivalent.*

(1) h.d. $Q = 1$.

(2) *Every $\mathscr{G}$-divisible module splits.*

Finally, we note that Corollary 4.6 adds to the list given in [12, Theorem 10.1] of conditions equivalent to h.d. $Q = 1$.

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