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THE SPECTRA OF ENDOMORPHISMS OF THE DISC ALGEBRA

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In this note the spectra of certain endomorphisms of the disc algebra A are determined. Those endomorphisms T of A given by $Tf = f \circ \varphi$ for some $\varphi \in A$ with φ having a fixed point z_0 in the open unit disc are considered and it is shown that either the spectrum $\sigma(T)$ of T is the closed unit disc, or else $\sigma(T)$ is the closure of $\{(\varphi'(z_0))^n \mid n$ is a positive integer$\} \cup \{1\}$.

By an endomorphism of an algebra B we mean a linear map T of B into itself satisfying $T(fg) = (Tf)(Tg)$ for all $f, g \in B$. We denote by $\sigma(T)$ those complex numbers λ for which $(\lambda - T)^{-1}$ does not exist.

Throughout this note A will denote the sup-norm algebra of functions continuous on the closed unit disc and analytic on the open unit disc. If $T(\neq 0)$ is a endomorphism of A, then there is a function φ in the unit ball of A for which $Tf = f \circ \varphi$ for all $f \in A$. Indeed, if z is the identity function in A, then $\varphi = Tz$. We call T the endomorphism of A induced by φ. Clearly $\sigma(T)$ depends on φ.

We remark that it follows from Schwarz's Lemma that if a function $\varphi \in A$, $\|\varphi\| \leqq 1$ has more than one fixed point in the open unit disc, then $\varphi(z) = z$ for all z, $|z| \leqq 1$. It is well-known, however, that such $\varphi \in A$ can have infinitely many points on the unit circle with $\varphi(z) = z$ and yet φ need not be equal to the identity function z.

We begin by showing that if φ has a fixed point z_0 in the open unit disc, it is no restriction to assume that $z_0 = 0$.

LEMMA 1. *Let $\varphi \in A$, $\|\varphi\| \leqq 1$ and T be the endomorphism of A induced by φ. Suppose $|z_0| < 1$ and $\varphi(z_0) = z_0$. Let g be the linear fractional transformation $g(z) = (z_0 - z)/(1 - \bar{z}_0 z)$ and T' the endomorphism of A induced by $\psi = g \circ \varphi \circ g$. Then $\psi(0) = 0$, $\psi'(0) = \varphi'(z_0)$ and $\sigma(T') = \sigma(T)$.*

Proof. The map $\mathscr{U}: f(z) \to f((z_0 - z)/(1 - \bar{z}_0 z))$ is an isometry of A onto itself, and so $\sigma(T) = \sigma(\mathscr{U}^{-1} T \mathscr{U}) = \sigma(T')$. It is a routine verification that $\psi(0) = 0$ and $\varphi'(0) = \varphi'(z_0)$.

When T is an automorphism (a $1 - 1$ onto endomorphism) of A, then the spectrum of T is easy to determine. Indeed, if φ is the function which induces T, then φ is a schlicht mapping of the disc onto itself. By Lemma 1 we may assume that $\varphi(0) = 0$. Then φ

has the form $\varphi(z) = cz$ for some c, $|c| = 1$. Now, for each positive integer k, c^k is an eigenvalue of T because $T(z^k) = (\varphi(z))^k = (cz)^k = c^k z^k$, and so either $c^n = 1$ for some positive integer n in which case $T^n = I$ and $\sigma(T) = \{1, c, \cdots, c^{n-1}\}$ or else, if c is not a root of unity, then $\sigma(T) = \{\lambda \mid |\lambda| = 1\}$.

DEFINITION. Let $\varphi \in A$ with $||\varphi|| \leqq 1$. If k is a nonnegative integer, we denote the k^{th} iterate of φ by φ_k. That is, $\varphi_0(z) = z$ and $\varphi_k(z) = \varphi(\varphi_{k-1}(z))$, $|z| \leqq 1$. Furthermore, we will call $\bigcap_k$ range (φ_k) the *fixed set of* φ.

Straightforward topological arguments show that the fixed set of φ is a compact, connected subset of the unit disc and that φ maps its fixed set onto itself.

For the endomorphisms we are considering, the spectra will depend on the fixed set of the inducing maps.

LEMMA 2. *Let $\varphi \in A$, $||\varphi|| \leqq 1$, $\varphi(0) = 0$ and T be the endomorphism of A induced by φ. Then $\sigma(T) \supset \{(\varphi'(0))^n \mid n$ is a positive integer$\}$.*

Proof. The assertion is clearly true if T is an automorphism or if $\varphi'(0) = 0$. For the case $0 < |\varphi'(0)| < 1$, we show that for each positive integer k, $((\varphi'(0)^k - T)f \neq z^k$ for all $f \in A$.

For, suppose $f \in A$ and $(\varphi'(0))^k f - f(\varphi) = z^k$. Then

$$(\varphi'(0))^k f'(z) - f'(\varphi(z))\varphi'(z) = kz^{k-1} \tag{*}$$

At $z = 0$, (*) becomes $(\varphi'(0))^k f'(0) - f'(0)\varphi'(0) = 0$, or $f'(0) = 0$.

Further,

$$(\varphi'(0))^k f''(z) - f''(\varphi(z))(\varphi'(z))^2 - f'(\varphi(z))\varphi''(z) = k(k-1)z^{k-2} . \tag{**}$$

At $z = 0$, (**) becomes $(\varphi'(0))^k f''(0) - f''(0)(\varphi'(0))^2 - f'(0)\varphi''(0) = 0$, and since we already have $f'(0) = 0$, we obtain $(\varphi'(0))^k f''(0) - f''(0)(\varphi'(0))^2 = 0$, or $f''(0) = 0$.

Continuing, we obtain for $j < k$,

$$\begin{aligned} &(\varphi'(0))^k f^{(j)}(z) - f^{(j)}(z)(\varphi'(z))^j \\ &\qquad - (\text{terms of derivatives of } f \text{ of degree} < j) \\ &\qquad = k(k-1) \cdots (k-j+1)z^{k-j} , \end{aligned} \tag{***}$$

so that at $z = 0$, $(\varphi'(0))^k f^{(j)}(0) - f^{(j)}(0)(\varphi'(0))^j = 0$, or $f^{(j)}(0) = 0$.

For $j = k$,

$$\begin{aligned} &(\varphi'(0))^k f^{(k)}(0) - f^{(k)}(z)(\varphi'(z))^k \\ &\qquad - (\text{terms of derivatives of } f \text{ of degree} < k) = k! . \end{aligned} \tag{****}$$

The left side of (****) equals 0 at $z = 0$, while the right side equals $k!$. Thus $z^k \notin \text{range}\,((\varphi'(0))^k - T)$ so that $(\varphi'(0))^k \in \sigma(T)$ for all positive integers k.

LEMMA 3. *Let $\varphi \in A$, $\|\varphi\| \leqq 1$, $\varphi(0) = 0$ and T be the endomorphism of A induced by φ. Assume $\lambda \neq (\varphi'(0))^n$ for all positive integers n, and $\lambda \neq 0, 1$. If ν is a positive integer, $f, g \in A$ with $(\lambda - T)f = g$ and $g(0) = g'(0) = \cdots = g^{(\nu)}(0) = 0$, then $f(0) = f'(0) = \cdots = f^{(\nu)}(0) = 0$.*

Proof. Assume $\lambda f - f(\varphi) = g$ and $g(0) = g'(0) = \cdots = g^{(\nu)}(0) = 0$. Evaluating at $z = 0$ gives $\lambda f(0) - f(0) = g(0) = 0$, so that $f(0) = 0$ since $\lambda \neq 1$.

Further, $\lambda f' - f'(\varphi)\varphi' = g'$. At $z = 0$, this becomes $\lambda f'(0) - f'(0)\varphi'(0) = g'(0) = 0$, so that $f'(0) = 0$ since $\lambda \neq \varphi'(0)$.

In general, for $k \leqq \nu$,

$$\lambda f^{(k)} - f^{(k)}(\varphi)(\varphi')^k = g^{(k)} + (\text{terms in } f^{(j)}(\varphi), j < k) .$$

Again evaluating at $z = 0$ gives $\lambda f^{(k)}(0) - f^{(k)}(0)(\varphi'(0))^k = 0$, so that $f^{(k)}(0) = 0$ for $k = 0, 1, \cdots, \nu$, since $\lambda \neq (\varphi'(0))^k$, for all positive integers k.

When $g = 0$, Lemma 3 can be restated as

COROLLARY 4. *If $\lambda \neq (\varphi'(0))^n$ for all positive integers n and $\lambda \neq 0, 1$, then λ is not an eigenvalue.*

LEMMA 5. *Let $\varphi \in A$, $\|\varphi\| \leqq 1$ and T be the endomorphism of A induced by φ. If $g \in A$ and $(\lambda - T)f = g$, then*

$$(\,'\,) \quad \lambda^n f = f(\varphi_n) + \lambda^{n-1}g + \lambda^{n-2}g(\varphi) + \cdots + \lambda g(\varphi_{n-2}) + g(\varphi_{n-1}) .$$

Proof. By induction on n. ($'$) is true for $n = 1$.

Assume ($'$) is true for n. Then

$$\lambda^n f = f(\varphi_n) + \lambda^{n-1}g + \lambda^{n-2}g(\varphi) + \cdots + \lambda g(\varphi_{n-2}) + g(\varphi_{n-1}) .$$

Also $\lambda f(\varphi_n) = f(\varphi_{n+1}) + g(\varphi_n)$ by hypothesis. Hence

$$\lambda^{n+1} f = f(\varphi_{n+1}) + \lambda^n g + \lambda^{n-1}g(\varphi) + \cdots + \lambda g(\varphi_{n-1}) + g(\varphi_n)$$

as needed.

LEMMA 6. *Let $\varphi \in A$, $\varphi(0) = 0$ and $\|\varphi\| \leqq 1$. If $|z| < 1$ (or, in fact, if $|\varphi_j(z)| < 1$ for some positive integer j), then $\overline{\lim}_k |\varphi_k(z)|^{1/k} \leqq |\varphi'(0)|$. Furthermore, (1) if $\varphi'(0) = 0$, then given $\varepsilon > 0$, and $r \in [0, 1)$, there exists $B > 0$ so that for each positive integer m, $|\varphi_m(z)| \leqq B\varepsilon^m$ for all z, $|z| \leqq r$; (2) if $0 < |\varphi'(0)| < 1$, then given $\varepsilon > 0$ and $r \in [0, 1)$,* *there exists* $B > 0$ *so that for each positive integer* m, $|\varphi_m(z)| \leqq B((1+\varepsilon)|\varphi'(0)|)^m$ *for all* z, $|z| \leqq r$.

Proof.

(i) $\varphi'(0) = 0$.

By the definition of derivative, given $\varepsilon > 0$, there exists $\delta > 0$ so that $|\varphi(w)| \leqq \varepsilon |w|$ for $|w| < \delta$. Using the fact that Schwarz's lemma implies $|\varphi(w)| < \delta$ when $|w| < \delta$, an induction argument shows that $|\varphi_n(w)| \leqq \varepsilon^n |w|$ for all positive integers n and all w, $|w| < \delta$.

Now if $r \in [0, 1)$, there is a positive integer N with $|\varphi_N(z)| < \delta$ for all z, $|z| \leqq r$. Thus $|\varphi_{n+N}(z)| = |\varphi_n(\varphi_N(z))| \leqq \varepsilon^n |\varphi_N(z)|$ for $|z| \leqq r$, and so $|\varphi_m(z)| \leqq \varepsilon^m |\varphi_N(z)\varepsilon^{-N}|$, $m \geqq N$ when $|z| \leqq r$. Letting $B = (r/\varepsilon)^N$ proves (1).

Furthermore, since $|\varphi_m(z)|^{1/m} \leqq \varepsilon B^{1/m}$, we find $\overline{\lim}_m |\varphi_m(z)|^{1/m} \leqq \varepsilon$. Since ε is an arbitrary positive number, we conclude that $\lim_m |\varphi_m(z)|^{1/m} = 0$.

(ii) $0 < |\varphi'(0)| < 1$.

Given $\varepsilon > 0$, there exists $\delta > 0$ so that $|\varphi(w)| \leqq (1+\varepsilon)|\varphi'(0)||w|$ for $|w| < \delta$. Again using Schwarz's lemma to show that $|\varphi(w)| < \delta$ if $|w| < \delta$, we can show by induction that $|\varphi_n(w)| \leqq ((1+\varepsilon)|\varphi'(0)|)^n |w|$ for all positive integers n and all w, $|w| < \delta$.

As before, if $r \in [0, 1)$, there exists a positive integer N for which $|\varphi_N(z)| < \delta$ for all z, $|z| \leqq r$. Thus $|\varphi_{n+N}(z)| \leqq ((1+\varepsilon)|\varphi'(0)|)^n |\varphi_N(z)|$ for $|z| \leqq r$, so that $|\varphi_m(z)| \leqq ((1+\varepsilon)|\varphi'(0)|)^m |\varphi_N(z)| ((1+\varepsilon)|\varphi'(0)|)^{-N}$ for $m \geqq N$, $|z| \leqq r$. Letting $B = ((1+\varepsilon)|\varphi'(0)|)^{-N}$ proves (2).

Also, since $|\varphi_m(z)|^{1/m} \leqq B^{1/m}(1+\varepsilon)|\varphi'(0)|$, we find that $\overline{\lim}_m |\varphi_m(z)|^{1/m} \leqq (1+\varepsilon)|\varphi'(0)|$. Since $\varepsilon > 0$ is arbitrary, we have $\overline{\lim}_m |\varphi_m(z)|^{1/m} \leqq |\varphi'(0)|$.

(iii) If $|\varphi'(0)| = 1$, then $\varphi(z) = cz$ for some c, $|c| = 1$, and for $z \neq 0$, $|\varphi_k(z)| = |z| \neq 0$, for all positive integers k. Clearly, $\lim_k |\varphi_k(z)|^{1/k} = 1 = |\varphi'(0)|$, $|z| \neq 0$.

THEOREM 7. *Let* $\varphi \in A$, $||\varphi|| \leqq 1$ *and* T *be the endomorphism of* A *induced by* φ. *Suppose* φ *has a fixed point in the open unit disc and that the fixed set of* φ *is infinite. If* T *is not an automorphism, then* $\sigma(T) = \{\lambda \mid |\lambda| \leqq 1\}$.

Proof. We may assume that 0 is the fixed point of φ and since T is not an automorphism we have $|\varphi'(0)| < 1$.

Now fix a positive integer ν. We show that $\sigma(T) \supset \{\lambda \mid |\varphi'(0)|^\nu < |\lambda| < 1\}$.

Assume there exists $\lambda_0 \notin \sigma(T)$ with $|\varphi'(0)|^\nu < |\lambda_0| < 1$. Then $(\lambda - T)^{-1}$ exists for λ in a neighborhood of λ_0 which we assume small enough so that each λ in this neighborhood satisfies $|\varphi'(0)|^\nu < |\lambda| < 1$.

Let $g(z) = z^\nu$ and let $f = (\lambda - T)^{-1}g$. By Lemma 5, for each positive integer n, we have

$$(*) \qquad f(z) = f(\varphi_n(z))\lambda^{-n} + \lambda^{-1}\sum_{k=0}^{n-1} g(\varphi_k(z))\lambda^{-k} .$$

Since $g(0) = g'(0) = \cdots = g^{(\nu-1)}(0) = 0$, Lemma 3 implies that $f(0) = f'(0) = \cdots = f^{(\nu-1)}(0) = 0$, and so $|f(\varphi_n(z))| \leqq ||f|| \, |\varphi_n(z)|^\nu$ for all positive integers n. Of course, $|g(\varphi_k(z))| = |\varphi_k(z)|^\nu$ for all positive integers k.

Lemma 6 asserts that $\overline{\lim}_n |\varphi_n(z)|^{1/n} \leqq |\varphi'(0)|$ for all z, $|z| < 1$, so that for such z,

$$\overline{\lim}_n |f(\varphi_n(z))\lambda^-|^{1/n} \leqq \overline{\lim}_n (||f|| \, |\varphi_n(z)^\nu\lambda^{-n}|)^{1/n} \, |(\varphi'(0))^\nu\lambda^{-1}| < 1 .$$

Hence the first term of the right hand side of (*) approaches 0 as $n \rightarrow \infty$.

Furthermore, $\overline{\lim}_k |g(\varphi_k(z))\lambda^{-k}|^{1/k} = \overline{\lim}_k |\varphi_k(z)^\nu\lambda^{-k}|^{1/k} \leqq |\varphi'(0)|^\nu |\lambda|^{-1} < 1$ so that $\sum_{k=0}^\infty g(\varphi_\kappa(z))\lambda^{-k}$ converges for all z, $|z| < 1$. Thus for λ in some neighborhood of λ_0 with $|\varphi'(0)|^\nu < |\lambda| < 1$,

$$f(z) = (\lambda - T)^{-1}g(z) = \lambda^{-1}\sum_{k=0}^\infty g(\varphi_k(z))\lambda^{-k} \text{ for all } z, |z| < 1 .$$

Now let S be the fixed set of φ. Since φ is analytic on the open unit disc, $|\varphi'(0)| < 1$ and φ maps S onto itself, we can construct a sequence $\{x_n\}_{-\infty}^\infty$ in S satisfying

(i) $0 < |x_0| < 1$, (ii) $\varphi(x_n) = x_{n+1}$, and (iii) the x_n's are distinct. If x_0 is fixed, then $x_n = \varphi_n(x_0)$ are uniquely determined for $n > 0$, but unless φ is $1 - 1$ on S, there may be many choices for $x_{-1}, x_{-2}, \cdots$.

Let B be the Banach algebra of bounded functions on $\{x_n\}$ with component-wise addition and multiplication and sup-norm. The map φ induces an isometric automorphism $\tilde{T}$, say, on B, by $\tilde{T}h(x_n) = h(\varphi(x_n)) = h(x_{n+1})$, for $h \in B$. For convenience, define φ_{-k} on $\{x_n\}$ by $\varphi_{-k}(x_n) = x_{n-k}$.

Now, $\sigma(\tilde{T}) = \{\lambda \mid |\lambda| = 1\}$ so that if $|\lambda| < 1$, then $(\lambda - \tilde{T})^{-1}$ exists on B and $F = (\lambda - \tilde{T})^{-1}g$ (on $\{x_n\}$) satisfies

$$\begin{aligned} F(x_0) &= -\tilde{T}^{-1}[(I - \lambda\tilde{T}^{-1})^{-1}g](x_0) = -\tilde{T}^{-1}\sum_{k=0}^\infty \lambda^k\tilde{T}^{-k}g(x_0) = -\sum_{k=0}^\infty \lambda^k\tilde{T}^{-(k+1)}g(x_0) \\ &= -\sum_{k=0}^\infty \lambda^k g(\varphi_{-(k+1)}(x_0)) = -\lambda^{-1}\sum_{k=1}^\infty \lambda^k g(\varphi_{-k}(x_0)) \\ &= -\lambda^{-1}\sum_{k=-\infty}^{-1} g(\varphi_k(x))\lambda^{-k} . \end{aligned}$$

Therefore, for each λ in some ball about λ_0, with $|\varphi'(0)|^\nu < |\lambda| < 1$, we have

$$(**) \qquad \lambda^{-1}\sum_{k=0}^{\infty} g(\varphi_k(x_0))\lambda^{-k} = -\lambda^{-1}\sum_{k=-\infty}^{-1} g(\varphi_k(x_0))\lambda^{-k}\,,$$

since both expressions represent $((\lambda - \tilde{T})^{-1}g)(x_0)$.

On the other hand, $\sum_{k=-\infty}^{\infty} g(\varphi_k(x_0))w^{-k}$ is the Laurent expansion of a function analytic in the annulus $\{w \mid |\varphi'(0)|^\nu < |w| < 1\}$ since $\overline{\lim}_k |g(\varphi_k(x_0))|^{1/k} = \overline{\lim}_k |(\varphi_k(x_0))^\nu|^{1/k} \leqq |\varphi'(0)|^\nu$ and $\overline{\lim}_k |g(\varphi_{-k}(x_0))|^{1/k} \leqq 1$. But (**) implies that $\sum_{k=-\infty}^{\infty} g(\varphi_k(x_0))\lambda^{-k} = 0$ in a ball about λ_0 and so the analytic function $\sum_{k=-\infty}^{\infty} g(\varphi_k(x_0))w^{-k}$ is identically zero in the entire annulus $\{w \mid |\varphi'(0)|^\nu < |w| < 1\}$. Thus $g(\varphi_k(x_0)) = 0$ for all integers k. Since $\{\varphi_k(x_0)\}$ is infinite and $\varphi_k(x_0) \to 0$ as $k \to \infty$, the analytic function g vanishes on an infinite set with 0 as a limit point. Hence $g = 0$. But this contradicts the assumption that $g(z) = z^\nu$.

Therefore, the assumption that there exists $\lambda_0 \notin \sigma(T)$ with $|\varphi'(0)|^\nu < |\lambda_0| < 1$ is false. Hence $\sigma(T) \supset \{\lambda \mid |\varphi'(0)|^\nu < |\lambda| < 1\}$. Since ν is arbitrary, $\sigma(T) = \{\lambda \mid |\lambda| \leqq 1\}$.

LEMMA 8. *Let $\varphi \in A$, $\|\varphi\| \leqq 1$, $\varphi(0) = 0$ and T be the endomorphism of A induced by φ. Let ν be a positive integer. Suppose every function in A with a zero of order at least $(\nu + 1)$ at 0 is in the range of $(\lambda - T)$, where $\lambda \neq 0, 1, (\varphi'(0))^n$, n a positive integer. Then $1, z, z^2, \cdots, z^\nu$ are in the range of $\lambda - T$.*

Proof. Let g be defined on the unit disc by $g(z) = (\varphi(z))^\nu - (\varphi'(0))^\nu z^\nu$. Then $g \in A$ and has a zero of order at least $(\nu + 1)$ at 0. By hypothesis we can find $h \in A$ with $(\lambda - T)h = g$. Let $f = (\lambda - (\varphi'(0))^\nu)^{-1}(h + z^\nu)$. Then

$$\begin{aligned}(\lambda - T)f &= (\lambda - (\varphi'(0))^\nu)^{-1}[(\lambda - T)h + (\lambda - T)z^\nu] \\ &= (\lambda - (\varphi'(0))^\nu)^{-1}[g + \lambda z^\nu - (\varphi(z))^\nu] \\ &= (\lambda - (\varphi'(0))^\nu)^{-1}[(\varphi(z))^\nu - (\varphi'(0))^\nu z^\nu + \lambda z^\nu - (\varphi(z))^\nu] = z^\nu\,.\end{aligned}$$

Thus if range $(\lambda - T)$ contains all functions with a zero of order at least $(\nu + 1)$ at 0, and if $\lambda \neq 0, 1, (\varphi'(0))^n$ for all positive integers n, then $z^\nu \in$ range $(\lambda - T)$.

In the same way we can conclude, successively, that $z^{\nu-1}, z^{\nu-2}, \cdots, z \in$ range $(\lambda - T)$. Also $(\lambda - T)(\lambda - 1)^{-1} = 1$ showing that the constants are in range $(\lambda - T)$.

THEOREM 9. *Let $\varphi \in A$, $\|\varphi\| \leqq 1$ and T be the endomorphism of A induced by φ. Let z_0 be a fixed point of φ in the open unit disc and suppose $\{z_0\}$ is the fixed set of φ. Then $\sigma(T) = \{(\varphi'(z_0))^n \mid n$ is a positive integer$\} \cup \{0, 1\}$.*

Proof. By Lemma 1 we may assume that $z_0 = 0$ and Lemma 2 implies that $\sigma(T) \supset \{(\varphi'(0))^n \mid n \text{ is a positive integer}\}$. Certainly 0 and 1 are in $\sigma(T)$.

We prove that $\sigma(T) = \{(\varphi'(0))^n \mid n$ is a positive integer$\} \cup \{0, 1\}$ for the case $0 < |\varphi'(0)| < 1$. The case $\varphi'(0) = 0$ is entirely similar.

Since the fixed set of φ is $\{0\}$, given $r \in (0, 1)$, there exists a positive integer m with $|\varphi_m(z)| < r$ for all $z, |z| \leqq 1$. Choose $\varepsilon > 0$ so that $(1 + \varepsilon)|\varphi'(0)| < 1$. Let ν be an arbitrary positive integer and consider λ satisfying $((1 + \varepsilon)|\varphi'(0)|)^{\nu+1} < |\lambda|$.

By Lemma 6, there exists $B_1 > 0$ so that

$|\varphi_k(\varphi_m(z))| < B_1((1 + \varepsilon)|\varphi'(0)|)^k$ for all $z, |z| \leqq 1$, and all positive integers k. Hence

$|\varphi_k(z)| < B((1 + \varepsilon)|\varphi'(0)|)^k$ for all $z, |z| \leqq 1$, and all positive integers k, where $B = B_1((1 + \varepsilon)|\varphi'(0)|)^{-m}$.

Now let $g \in A$ with $g(0) = g'(0) = \cdots = g^{(\nu)}(0) = 0$. We claim that $g \in \text{range}\ (\lambda - T)$. To see this, we observe first that $\sum_{k=0}^{\infty} g(\varphi_k(z))\lambda^{-k}$ converges uniformly in z. Indeed, $|g(z)| \leqq ||g||\,|z|^{\nu+1}$ and

$$(*) \qquad \left|\sum_{k=N}^{M} g(\varphi_k(z))\lambda^{-k}\right| \leqq ||g|| \sum_{k=N}^{M} |\varphi_k(z)|^{\nu+1}|\lambda|^{-k}$$
$$\leqq ||g||\,B^{\nu+1} \sum_{k=N}^{M} [((1 + \varepsilon)|\varphi'(0)|)^{\nu+1}|\lambda|^{-1}]^k .$$

Since $((1 + \varepsilon)|\varphi'(0)|)^{\nu+1} < |\lambda|$, the right most term of (*) goes to 0 as $N, M \to \infty$.

Define f on the closed unit disc by $f(z) = \lambda^{-1}\sum_{k=0}^{\infty} g(\varphi_k(z))\lambda^{-k}$. Then $f \in A$ and $\lambda f(z) - f(\varphi(z)) = \sum_{k=0}^{\infty} g(\varphi_k(z))\lambda^{-k} - \lambda^{-1}\sum_{k=0}^{\infty} g(_{k+1}(z))\lambda^{-k} = g(z)$.

Hence, if $((1 + \varepsilon)|\varphi'(0)|)^{\nu+1} < |\lambda| < 1$ and g has a zero of order at least $(\nu + 1)$ at 0, then $g \in \text{range}\ (\lambda - T)$. By the preceding lemma, if λ also is not equal to $0, 1, (\varphi'(0))^n$ for positive integers n, then $1, z, \cdots, z^\nu$ also belong to range $(\lambda - T)$.

Now, every $h \in A$ may be written as

$$h(z) = \left(h(0) + h'(0)z + \cdots + \frac{h^{(\nu)}(0}{\nu!}z^\nu\right) + g(z)$$

where

$$g(z) = \left(h(z) - h(0) - h'(0)z - \cdots - \frac{h^{(\nu)}(0)}{\nu!}z^\nu\right).$$

Clearly, $g(0) = g'(0) = \cdots = g^{(\nu)}(0)$. As we have shown, $g \in \text{range}$ $(\lambda - T)$ when $|\lambda| > ((1 + \varepsilon)|\varphi'(0)|)^{\nu+1}$. Also, if $\lambda \neq 0, 1, (\varphi'(0))^n$, n a positive integer, then $1, z, \cdots, z^\nu \in \text{range}\ (\lambda - T)$. Thus, for these λ, every h in A is in the range of $(\lambda - T)$, so $(\lambda - T)$ is onto. Also, by Corollary 4, $(\lambda - T)$ is $1 - 1$ if $\lambda \neq 0, 1, (\varphi'(0))^n$, n a positive integer.

Hence, $(\lambda - T)^{-1}$ exists for all λ, $|\lambda| > ((1 + \varepsilon)|\varphi'(0)|)^{\nu+1}$ and $\lambda \neq 0, 1$, $(\varphi'(0))^n$, n a positive integer. Since ν is arbitrary, we conclude that if $\lambda \neq 0, 1, (\varphi'(0))^n$, n a positive integer, then $\lambda \notin \sigma(T)$.

As we noted, Lemma 2 shows that for each positive integer n, $(\varphi'(0))^n$ is in $\sigma(T)$. Since 0 and 1 are in $\sigma(T)$, $\sigma(T) = \{\varphi'(0))^n \mid n$ is a positive integer$\} \cup \{0, 1\}$.

The case when $\varphi'(0) = 0$ is similar. We just replace $(1 + \varepsilon)|\varphi'(0)|$ by ε.

To summarize, we have shown that if T is the endomorphism of A induced by $\varphi \in A$, $||\varphi|| \leqq 1$, and if there is a fixed point z_0 of φ in the open unit disc, then the spectrum of T is determined as follows.

(1) If φ is schlicht and onto, then T is an automorphisms and $\sigma(T)$ is the closure of $\{(\varphi'(z_0))^n \mid n$ is a positive integer$\}$. We have seen that $\sigma(T)$ is contained in the unit circle and that $\sigma(T)$ may be finite.

(2) If T is not an automorphism, but the fixed set of φ is infinite, then Theorem 7 shows that $\sigma(T) = \{\lambda \mid |\lambda| \leqq 1\}$.

(3) If the fixed set of φ consists of the single point z_0 in the open unit disc, then Theorem 9 shows that $\sigma(\sigma) = \{(\varphi'(z_0))^n \mid n$ is a positive integer$\} \cup \{0, 1\}$.

Some simple examples of the various types of endomorphisms we have discussed are (i) T is induced by a linear fractional transformation φ of the unit disc onto itself. Then T is an automorphism and $\sigma(T) =$ closure $\{(\varphi'(z_0))^n \mid n$ is a positive integer$\}$ where $|z_0| < 1$ and $\varphi(z_0) = z_0$. If φ is normalized to have $z_0 = 0$, then φ has the form $\varphi(z) = cz$, $|c| = 1$. Here, $\sigma(T) = \{\lambda \mid |\lambda| = 1\}$ if c is not a root of unity; otherwise, if $c^n = 1$, $\sigma(T) = \{1, c, \cdots c^{n-1}\}$.

(ii) T is induced by $\varphi(z) = (z + z^2)/2$. Here one can show that the fixed set is infinite and hence $\sigma(T) = \{\lambda \mid |\lambda| \leqq 1\}$.

(iii) T is induced by $\varphi(z) = (z + z^2)/4$. Here the fixed set of φ is $\{0\}$, $\varphi'(0) = 1/4$, and so $\sigma(T) = \{4^{-n} \mid n$ is a nonnegative integer$\} \cup \{0\}$.

(iv) T is induced by $\varphi(z) = cz^k$, k a positve integer > 1. If $|c| = 1$, then the fixed set of φ is the entire disc and $\sigma(T) = \{\lambda \mid |\lambda| \leqq 1\}$, while if $|c| < 1$, then the fixed set of φ is $\{0\}$ and $\sigma(T) = \{0, 1\}$.

As a final remark, the question of determining the spectra of endomorphisms induced by $\varphi \in A$ with fixed points only on the unit circle is still open. Again the spectra seem to depend on the fixed set of φ, but only partial results have been obtained.

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