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ISOMETRIC DILATIONS OF CONTRACTIONS ON BANACH SPACES

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This paper is concerned with the dilation, in the case of a Banach space, of operator-valued functions on a group into representations. Banach-space analogues of Sz.-Nagy's theorem and Ando's theorem are obtained.

Throughout this note Z (resp. R, resp. R^+, resp. N, resp. C) is the set of all integer (resp. real, resp. nonnegative real, resp. nonnegative integer, resp. complex) numbers. Also G is a group, $e \in G$ its neutral element: $K: G \rightarrow R^+$ a submultiplicative function (i.e., $K(gh) \leqq K(g)K(h)$ for all $g, h \in G$) with $K(e) = 1$; X a Banach space; $\mathscr{B}(X)$ the Banach algebra of all linear bounded operators on X and $I \in \mathscr{B}(X)$ the identity.

$\mathscr{C}^m(R)$ ($m \in N$, $m = \infty$) being the algebra of all m-times differentiable functions on R with the usual topology and $\Gamma = \{z \in C; |z| = 1\}$, $\mathscr{C}^m(\Gamma)$ is the algebra of all functions $f: \Gamma \rightarrow C$ such that $t \rightarrow f(e^{it})$ belongs to $\mathscr{C}^m(R)$, endowed with the topology induced by $\mathscr{C}^m(R)$. An operator $T \in \mathscr{B}(X)$ is called $\mathscr{C}^m(\Gamma)$-unitary if it is $\mathscr{C}^m(\Gamma)$-scalar ([2], [4]).

THEOREM. (See also [7] Theorem 1). *Let $\phi: G \rightarrow \mathscr{B}(X)$ be a function with the property $||\phi_g|| \leqq K(g)$ for all $g \in G$ and $\phi_e = I$.*

Then there exists a Banach space $\tilde{X}$ containing X (by an isometric isomorphism), a norm one projection P of $\tilde{X}$ onto X and a representation $\tilde{\phi}$ of G as a group of invertible operators on $\tilde{X}$ such that

(0) $1/K(\gamma^{-1}) \leqq ||\tilde{\phi}_\gamma|| \leqq K(\gamma)$ for all $\gamma \in G$ and $\tilde{\phi}_e = \tilde{I}$.

(i) $P\tilde{\phi}_{\gamma|X} = \phi_\gamma$ for any $\gamma \in G$.

(ii) $\tilde{X}$ is the closed vector space spanned by $\{\tilde{\phi}_\gamma x; \gamma \in G, x \in X\}$.

(iii) If ϕ takes its values from the set of contractions on X, then G is represented by $\tilde{\phi}$ as a group of invertible isometries on $\tilde{X}$. Moreover, if G is a topological group and for every $x \in X$, the function $g \rightarrow \phi_g\, x$ is left uniformly continuous, then the representation $\tilde{\phi}$ is strongly continuous.

Proof. Let Y be the vector space of all X-valued functions on G, $y(.)$ with the property

$$||y(g)|| \leqq MK(g) \quad \text{for all} \quad g \in G\,,$$

where M is a positive real constant and K the submultiplicative function from the hypothesis. (In what follows we shall denote elements of Y also by $(y_g)_{g\in G}$.) One sees easily that Y endowed with the norm

$$||y(\cdot)|| = \sup_g ||y(g)||K(g)^{-1}, \text{ is a Banach space .}$$

Let $X^{(G)} = \bigoplus_{g\in G} X^g$ be the direct sum with $X^g = X$ for all $g \in G$. Define a map Θ: $X^{(G)} \to X$ by $(\Theta y)_g = \sum_h \phi_{gh} y_h$ for all $g \in G$ and $y \in X^{(G)}$. Then for every $y \in X^{(G)}$ one has $\Theta y \in Y$ and the set $\hat{X} = \{\Theta y; y \in X^{(G)}\}$ is a subspace of Y. Consider the closure of $\hat{X}$ in Y and denote it by $\tilde{X}$.

Now let X_0 be a subspace of $\hat{X}$ of elements

$$y(\cdot) = (\phi_g x)_{g\in G} = (\Sigma_h \phi_{gh}\delta_{eh} x)_{g\in G} \text{ when } x \text{ runs over } X$$
$$(\delta_{gh} = 0 \text{ for } g \neq h \text{ and } \delta_{gh} = 1 \text{ for } g = h). \quad \text{Define a map}$$
$$\varphi\colon X_0 \to X \quad \text{by} \quad \varphi(y(\cdot)) = y(e) \quad \text{for all} \quad y(\cdot) \in X_0 .$$

Then one has

$$||\varphi(y(\cdot))|| = ||y(e)|| \leqq \sup_g ||y(g)||K(g)^{-1} = ||y(\cdot)||$$

and

$$||y(\cdot)|| = \sup_g ||\phi_g x||K(g)^{-1} \leqq ||x|| = ||y(e)|| .$$

Hence φ is an isometric isomorphism of X_0 onto X.

Let Q: $\hat{X} \to X$ be a map defined by

$$Qy(\cdot) = y(e) \quad \text{for all} \quad y(\cdot) \in \hat{X} .$$

Obviously, Q is linear surjective and satisfies $||Qy(\cdot)|| \leqq ||y(\cdot)||$ for all $y(\cdot) \in \hat{X}$. Its extension by continuity to a linear map of $\tilde{X}$ onto X will be denoted by the same symbol. Then $\varphi^{-1}Q$ is a norm one projection of $\tilde{X}$ onto X.

For every $\gamma \in G$, define a map $\hat{\phi}_\gamma$: $\hat{X} \to \hat{X}$ by

$$\hat{\phi}_\gamma \Theta y = ((\Theta y)_{g\gamma})_{g\in G} = (\Sigma_h \phi_{g\gamma h} y_h)_{g\in G} = (\Sigma_d \phi_{gd} z_d)_{g\in G} = \Theta z \in \hat{X}$$

when y runs over $X^{(G)}$. (It is made the notation $d = \gamma h$, $z_d = y_h$ for all $h \in G$; hence z with these components belongs to $X^{(G)}$.) One sees easily that $\tilde{\phi}_\gamma$ is well defined and linear. Moreover, one has

$$\begin{aligned} ||\hat{\phi}_\gamma \Theta y|| &= \sup_g ||\Sigma_h \phi_{g\gamma h} y_h||K(g)^{-1} \\ &= \sup_g ||\Sigma_h \phi_{g\gamma h} y_h||K(g\gamma)^{-1}K(g)^{-1}K(g\gamma) \\ &\leqq K(\gamma) \sup_{g\gamma} ||\Sigma_h \phi_{g\gamma h} y_h||K(g\gamma)^{-1} = K(\gamma)||\Theta y|| . \end{aligned}$$

That is

$$(1) \qquad \|\hat{\phi}_\gamma \Theta y\| \leqq K(\gamma)\|\Theta y\| \quad \text{for all} \quad y \in X^{(G)} .$$

Then $\hat{\phi}_\gamma$ can be extended by continuity to an element of $\mathscr{B}(\tilde{X})$ which will be denoted by $\tilde{\phi}_\gamma$. One sees easily that $\tilde{\phi}_{\alpha\beta} = \tilde{\phi}_\alpha \tilde{\phi}_\beta$ for all $\alpha, \beta \in G$ and $\tilde{\phi}_e = \tilde{I}$. Moreover,

$$(2) \qquad \|\Theta y\| \leqq \|\hat{\phi}_{\gamma^{-1}} \hat{\phi}_\gamma \Theta y\| \leqq K(\gamma^{-1})\|\hat{\phi}_\gamma \Theta y\| \quad \text{for all} \quad y \in X^{(G)} .$$

Also $\hat{\phi}_\gamma$: $\hat{X} \rightarrow \hat{X}$ is surjective since one has

$$\Theta y = \hat{\phi}_\gamma((\Theta y)_{g\gamma^{-1}})_{g \in G} \quad \text{for all} \quad y \in X^{(G)} \quad \text{and} \quad \gamma \in G .$$

Thus the property (0) is proved. To show (i) we see that

$$((\varphi^{-1}Q)\tilde{\phi}_\gamma)\varphi^{-1}(x) = \varphi^{-1}(\phi_\gamma x) \quad \text{for all} \quad x \in X \quad \text{and} \quad \gamma \in G .$$

Identifying X_0 and X via φ and writting P instead of $\varphi^{-1}Q$, this equality reads more naturally as $P\tilde{\phi}_{\gamma|X} = \phi_\gamma$. The property (ii) is immediate noting that every $\Theta y \in \hat{X}$ can be written $\Theta y = \Sigma_h \tilde{\phi}_h \varphi^{-1}(y_h)$. The first assertion of (iii) is immediate because taking $K(g) = 1$ for all $g \in G$, the above inequalities (1) and (2) become

$$(3) \qquad \|\hat{\phi}_\gamma \Theta y\| = \|\Theta y\| \quad \text{for all} \quad y \in X^{(G)} \quad \text{and} \quad \gamma \in G .$$

To prove the second assertion of (iii) we assume still that G is a topological group and $g \rightarrow \phi_g x$ is left uniformly continuous for each $x \in X$. Taking into account of (ii) it is enough to show that for any fixed $\gamma \in G$ and $y(.) \in X_0$, the map $\alpha \rightarrow \tilde{\phi}_\alpha(\tilde{\phi}_\gamma y)(.) = (\tilde{\phi}_{\alpha\gamma} y)(.)$ is continuous. As this map is the composition of $\alpha \rightarrow \alpha\gamma$ and $\alpha\gamma \rightarrow (\tilde{\phi}_\alpha \gamma y)(.)$, we need only show that for each $y(.) \in X_0$, the map $a \rightarrow (\tilde{\phi}_a y)(.)$ is continuous. For this it is sufficient to show the continuity at $a = e$. But this fact is immediate from the left uniform continuity of $g \rightarrow \phi_g x$ for every $x \in X$, because $\|(\tilde{\phi}_a y)(.) - y(.)\| = \sup_g \|\phi_{ga} y(e) - \phi_g y(e)\|$.

COROLLARY 1. *Let $\{T_t\}_{t \in R^+} \subset \mathscr{B}(X)$ be a semigroup of contractions. Then there exists a Banach space $\tilde{X}$ containing X, a norm one projection P of $\tilde{X}$ onto X and a group $\{U_t\}_{t \in R}$ of invertible isometries on $\tilde{X}$ such that:*

(i) *$PU_t x = T_{|t|}x$, for all $x \in X$, $t \in R$.*

(ii) *$\tilde{X}$ is the closed vector space spanned by*

$$\{U_t x; t \in R, x \in X\} .$$

(iii) *If $\{T_t\}_{t \in R^+}$ is strongly continuous, then $\{U_t\}_{t \in R}$ is also strongly continuous.*

Proof. Taking $G = R$, the additive group of real numbers defining ϕ by $\phi_t = T_{|t|}$, and K by $K(t) = 1$, for any $t \in R$, we are in assumptions of the previous theorem.

REMARK 1. An invertible isometry is a $\mathscr{C}^m(\Gamma)$-unitary operator with $m > 1$, ([2], Proposition 5.1.4). Hence Corollary 1 can be understood as a Banach space analogue of Sz.-Nagy's theorem ([9]) about of the dilation of a semigroup of contractions into a group of unitary operators.

COROLLARY 2. (See [9], Theorem IV). *Let $T \in \mathscr{B}(X)$ be a contraction. Then there exists a Banach space $\tilde{X}$ containing X, a norm one projection P of $\tilde{X}$ onto X and an invertible isometry U on $\tilde{X}$ such that:*

(i) *$PU^n x = T^{|n|}x$, for all $x \in X$, $n \in Z$.*

(ii) *$\tilde{X}$ is the closed vector space spanned by*

$$\{U^n x;\ n \in Z,\ x \in X) .$$

Proof. Obviously, for this case one takes $G = Z$ the additive group of integer numbers, ϕ defined by $\phi_n = T^{|n|}$ and K by $K(n) = 1$, for all $n \in Z$.

COROLLARY 3. *Let $\{T_1, T_2, \cdots, T_p\} \subset \mathscr{B}(X)$ be a finite system of not necessarily commuting contractions. Then there exists a Banach space $\tilde{X}$ containing X, a norm one projection P of $\tilde{X}$ onto X and a finite system of commutative invertible isometries $\{U_1, U_2, \cdots, U_p\}$ on $\tilde{X}$ such that:*

(i)
$$PU_1^{n_1}U_2^{n_2}\cdots U_p^{n_p}x = T_1^{|n_1|}T_2^{|n_2|}\cdots T_p^{n_p}x ,$$

for any

$$n_1, n_2, \cdots, n_p \in Z,\ x \in X .$$

(ii) *$\tilde{X}$ is the closed vector space spanned by*

$$\{U_1^{n_1}U_2^{n_2}\cdots U_p^{n_p}x;\ n_1, n_2, \cdots, n_p \in Z,\ x \in X\} .$$

Proof. We take $G = Z_1 \times Z_2 \times \cdots \times Z_p$ with $Z_i = Z$ for $i = 1, 2, \cdots, p$; define ϕ by $\phi(n_1, n_2, \cdots, n_p) = T_1^{|n_1|}T_2^{|n_2|}\cdots T_p^{|n_p|}$ and K by $K(n_1, n_2, \cdots, n_p) = 1$ for any $n_1, n_2, \cdots, n_p \in Z$, then apply the above theorem.

REMARK 2. Corollary 3 is a Banach space analogue of Ando's theorem ([1]). We remark that it is not necessarily to assume any property of commutativity also we can take a number of more than two contractions, (in a Hilbert space this is not true, see [5]).

REMARK 3. The above theorem also asserts that for any sequence $\{T_n\}_{n\in Z}\subset \mathscr{B}(X)$ of contractions with $T_0 = 1$, there exists a Banach space $\tilde{X}\supset X$, a norm one projection P of $\tilde{X}$ onto X and a invertible isometry U on $\tilde{X}$ such that $T_n = PU^n{}_{|X}$ for any $n\in Z$. Also $\tilde{X}$ is the closed vector space spanned by $\{U^n x;\, n\in Z,\, x\in X\}$. (This fact is true in a Hilbert space if and only if T_n is a positive definite sequence.)

COROLLARY 4. *Let $\{T_t\}_{t\in R^+}\subset \mathscr{B}(X)$ be a semigroup of operators such that $\|T_t\|\leqq Me^{at}$ (resp. $\|T_t\|\leqq t^\alpha + 1$, with $0\leqq \alpha\leqq 1$) for all $t\in R^+$, where a and M are real positive constants. Then there exists a Banach space $\tilde{X}\supset X$, a norm one projection P of $\tilde{X}$ onto X and a group of invertible (resp. $\mathscr{C}^m(\Gamma)$-unitary with $m > \alpha + 1$) operators on $\tilde{X}$, $\{U_t\}_{t\in R}$ such that:*

(0) *$M^{-1}e^{-a|t|}\leqq \|U_t\|\leqq Me^{a|t|}$ for all $t\in R$, if $M > 1$, or $e^{-a|t|}\leqq \|U_t\|\leqq e^{a|t|}$ for all $t\in R$, if $M\leqq 1$, (resp. $(|t|^\alpha+1)^{-1}\leqq \|U_t\|\leqq |t|^\alpha + 1$ for all $t\in R$).*

(i) *$PU_t x = T_{|t|}x$ for all $t\in R$, $x\in X$,*

(ii) *$\tilde{X}$ is the closed vector space spanned by $\{U_t x;\, t\in R,\, x\in X\}$.*

Proof. Taking $G = R$ the additive group of real numbers, defining ϕ by $\phi_t = T_{|t|}$ for all $t\in R$ and K thus: if $M > 1$, $K(t) = Me^{a|t|}$ for $t\neq 0$, and $K(0) = 1$; or if $M\leqq 1$, $K(t) = e^{a|t|}$ for $t\neq 0$ and $K(0) = 1$, (resp. $K(t) = |t|^\alpha + 1$ for any $t\in R$), we have the hypothesis of the theorem.

Moreover, for the second case we obtain

$$\|U_{nt}\| = \|(U_t)^n\|\leqq |n|^\alpha(|t|^\alpha + 1)$$

for all $|n| > 1$, $t\in R$. Then applying Proposition 5.1.4 from [2], it follows that U_t is a $\mathscr{C}^m(\Gamma)$-unitary operator with $m > \alpha + 1$, for each $t\in R$.

COROLLARY 5. *Let $T\in\mathscr{B}(X)$, satisfying $\|T^n\|\leqq n^\alpha + 1$ for all $n\in N$, with $0\leqq\alpha\leqq 1$. Then there exists a Banach space $\tilde{X}\supset X$, a norm one projection P of $\tilde{X}$ onto X and a $\mathscr{C}^m(\Gamma)$-unitary operator, with $m > \alpha + 1$, U on $\tilde{X}$ such that:*

(0) *$(|n|^\alpha + 1)^{-1}\leqq \|U^n\|\leqq |n|^\alpha + 1$ for all $n\in Z$.*

(i) *$PU^n x = T^{|n|}x$ for all $n\in Z$, $x\in X$.*

(ii) *$\tilde{X}$ is the closed vector space spanned by*

$$\{U^n x;\, n\in Z,\, x\in X\}\,.$$

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