PACIFIC JOURNAL OF MATHEMATICS
Vol. 47, No. 1, 1973

DENDRITIC COMPACTIFICATIONS OF CERTAIN DENDRITIC SPACES

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A *dendritic* space is a connected space in which every two points are separated by a third point. In this paper we describe a very natural method for obtaining a dendritic compactification of any connected space for which a dendritic compactification exists. The method is an extension of the familiar process of compactifying E^1 by adjoining $-\infty$ and $+\infty$.

In what follows, an *arc* is a Hausdorff continuum with only two noncut points. A *ray* is an arc minus one of its noncut points. The space X is *semi-locally connected* at the point p if each open set containing p contains an open set V containing p such that $X-V$ has at most finitely many components.

LEMMA. *If the space X is arcwise connected but is not semi-locally connected at the point p, then there exists an open set U containing p such that if V is an open set containing p and lying in U, then $X-V$ has infinitely many components that intersect both $\bar{V}$ and $X-U$.*

Proof. There exists an open set U containing p such that for each open set V containing p and lying in U, $X-V$ has infinitely many components. Let V be an open set containing p and lying in U, and let $\mathscr{S}$ be the collection of all components of $X-V$ that intersect both $\bar{V}$ and $X-U$. Suppose $\mathscr{S}$ is finite. Let W be the union of V and all components of $X-V$ lying in U. It follows from the arcwise connectivity of X that each component of $X-V$ intersects $\bar{V}$. Therefore $W = X - \cup\mathscr{S}$, so that W is open. But

$$W \subsetneqq U,\ X - W = \cup\mathscr{S}\ ,$$

and C is a component of $X-W$ if and only if $C \in \mathscr{S}$. Therefore $\mathscr{S}$ is infinite.

THEOREM 1. *If the connected space X has a dendritic compactification, then X is arcwise connected and semi-locally connected.*

Proof. Suppose X has a dendritic compactification X^*. Since X^* is a dendritic continuum, it is locally connected, and it then follows from Theorem 7.1 of [3] that the interval ab of X^*, which consists of a and b and the set of all points of X^* separating a from b, is an arc. Suppose ab contains a point x not in X. Then $X^* - \{x\}$ is the union of two disjoint open sets U and V such that $a \in U$ and $b \in V$. But then X is a connected subset of $U \cup V$, so that $X \subseteqq U$ or $X \subseteqq V$. Therefore $ab \subseteqq X$.

Suppose X is not semi-locally connected at p. There exist open sets U and V in X^* such that $p \in V \subseteqq \bar{V} \subseteqq U$ and an infinite net $\{C_\alpha\}$ of distinct components of $X - V$ intersecting both $\bar{V}$ and $X - U$, where the closures are taken in X^*. For each α let $x_\alpha \in C_\alpha \cap \bar{V}$. Some subnet $\{x_{\alpha_n}\}$ of $\{x_\alpha\}$ converges to a point x in X^*. For each n let $y_{\alpha_n} \in C_{\alpha_n} \cap (X - U)$. Since X^* is compact, the net $\{y_{\alpha_n}\}$ has a cluster point y in X^*. Now $x \in \bar{V}$ and $y \in X^* - U$, so that $x \neq y$. But then no point separates x from y in X^*.

THEOREM 2. *If the space X is dendritic, semi-locally connected, and arcwise connected, and each ray in X is a subset of some arc in X, then X is compact.*

Proof. Let $p \in X$. Suppose $\{x_\alpha\}$ is a net of points in $X - \{p\}$ with no cluster point. Suppose that for each α and each point x of the arc px_α different from p there is a $\beta > \alpha$ such that $x \notin px_\beta$. Let U be an open set containing p such that for each α there is a $\beta > \alpha$ such that $x_\beta \notin U$. There is an open set V containing p and lying in U such that $X - V$ has at most finitely many components. There is an α_0 such that $x_{\alpha_0} \in X - V$. Let $x_0 \in px_{\alpha_0}$ such that $px_0 \subseteqq V$. There is an $\alpha_1 > \alpha_0$ such that $x_{\alpha_1} \in X - V$ and $x_0 \notin px_{\alpha_1}$. Let $x_1 \in px_{\alpha_1}$ such that $px_1 \subseteqq V$. There is an $\alpha_2 > \alpha_1$ such that $x_{\alpha_2} \in X - V$, $x_0 \notin px_{\alpha_2}$, and $x_1 \notin px_{\alpha_2}$. Continue this process. There exist m and n such that $m \neq n$ and some component of $X - V$ contains both x_{α_m} and x_{α_n}. But then no point of X separates x_{α_m} from x_{α_n}. This is a contradiction, and hence the set R of all points x in $X - \{p\}$ such that for some α, $x \in px_\beta$ for each $\beta > \alpha$ is nonempty. Let $x, y \in R$. There is an α_1 such that $x \in px_\beta$ for $\beta > \alpha_1$. There is an α_2 such that $y \in px_\beta$ for $\beta > \alpha_2$. Hence if $\beta > \alpha_1, \alpha_2$, then $x, y \in px_\beta$. It follows that if $x, y \in R$, then either $px \subseteqq py$ or $py \subseteqq px$. Therefore there exists a point q of X distinct from p such that $R = pq$ or $R = pq - \{q\}$. For each α let y_α be the last point of px_α on pq. Let U be an open set containing q. There is a point y such that pyq and $yq \subseteqq U$. Since $y \in R$, there is an α such that if $\beta > \alpha$, then $y \in px_\beta$. Hence $y_\beta \in yq$ for $\beta > \alpha$, so that the net $\{y_\alpha\}$ converges to q. It follows that there is a subnet $\{y_{\alpha_n}\}$ of $\{y_\alpha\}$ converging to q such that if $m < n$, then y_{α_m} precedes y_{α_n} on pq and $x_{\alpha_n} \notin pq$. There exists an open set V containing q and lying in U such that $X - V$ has only finitely many components. Hence there is an m such that if $n > m$, then $x_{\alpha_n} \in V$. It follows that q is a cluster point of $\{x_\alpha\}$.

An incorrect version of the following lemma is stated as Lemma 3 to Theorem 3 in [1]. The lemma stated here may be used as a substitute without altering the proof of that theorem.

LEMMA. *If H and K are two separated connected sets in the arcwise connected dendritic space X, then some point of X separates H from K.*

Proof. Let $a \in H$ and $b \in K$. Since H and K are separated, there exists a point p of the arc ab not in $H \cup K$. Let U be the set of all points $x \neq p$ such that px contains a point of $ap - \{p\}$, and let $V = X - (U \cup \{p\})$. Suppose the point x of U is a limit point of V. Then there exists a net $\{x_\alpha\}$ of points in $V - pb$ converging to x and a net $\{y_\alpha\}$ of points in pb such that for each α, y_α is the last point of pb on px_α. Some point y of pb is a cluster point of $\{y_\alpha\}$, and hence no point of X separates x from y. This is a contradiction. Therefore U is open, and it follows by a similar argument that V is open. Since H and K are connected, $H \subseteqq U$ and $K \subseteqq V$. Therefore p separates H from K.

THEOREM 3. *The dendritic space X has a dendritic compactification if and only if X is arcwise connected and semi-locally connected.*

Proof. Suppose X is arcwise connected and semi-locally connected. Let $p \in X$, and let X^* be the union of X and the collection of all maximal rays in X starting from p. Let $\mathscr{S}$ be the collection of all open sets U in X such that $X - U$ has at most finitely many components. For each U in $\mathscr{S}$ let U^* be the union of U and the collection of all maximal rays starting from p and having a subray lying in U. Let $\mathscr{S}^* = \{U^* \mid U \in \mathscr{S})$. It is easily seen that if $U, V \in \mathscr{S}$, then $U \cap V \in \mathscr{S}$ and $(U \cap V)^* = U^* \cap V^*$. Therefore $\mathscr{S}^*$ is a base for a topology of X^*, and X with its original topology is a subspace of X^*. Now for each maximal ray R in X starting from p the point R of X^* is a limit point of the point set R. Therefore X is dense in X^*. Furthermore $R \cup \{R\}$ is an arc from p to R. Therefore X^* is arcwise connected. Suppose $U \in \mathscr{S}$ and C is a component of $X^* - U^*$ containing a point R of $X^* - X$. If R has a subray in U, then $R \in U^*$. Hence there is a point x in $R - U$. Let K be the component of $X - U$ containing x. Since

$$X - U \subseteqq X^* - U^* ,$$

it follows that $K \subseteqq C$. Hence each component of $X^* - U^*$ contains a component of $X - U$. Therefore X^* is semi-locally connected. Let a and b be points of X. There is a point x of X such that $X - \{x\}$ is the union of two disjoint open sets U and V in X such that $a \in U$ and $b \in V$. Since the only component of $X - U$ is $V \cup \{x\}$, it follows that $U \in \mathscr{S}$ and similarly that $V \in \mathscr{S}$. Since $U \cap V = \emptyset$, it follows that $U^* \cap V^* = \emptyset$. If $R \in X^* - X$, then there is a subray S of R such that $x \notin S$. Hence $S \subseteqq U$ or $S \subseteqq V$, so that $R \in U^*$ or $R \in V^*$. Therefore $X^* - \{x\} = U^* \cup V^*$. It follows that x separates a from b in X^*. Now let $a \in X$ and $R \in X^* - X$. There is a subray S of R such that $a \notin S$. Some point x of X separates a from S in X, and it follows as before that x separates a from R in X^*. Finally, let P and R be two elements of $X^* - X$. There exist disjoint rays Q and S such that $Q \subseteqq P$ and $S \subseteqq R$. Some point x of X separates Q from S in X. It follows that x separates P from R in X^*. Therefore X^* is dendritic. It remains to be proved that X^* is compact. Suppose R is a ray in X^* starting from a point q in X^*. Now $X^* - X$ is totally disconnected since each two points of $X^* - X$ are separated by a point of X, and if x and y are points of X, then the arc xy in X^* is a subset of X. It follows that $R - \{q\} \subseteqq X$. Hence there is a maximal ray S in X starting from p and containing a subray of R. Since $S \cup \{S\}$ is an arc in X^*, R is contained in some arc in X^*. It now follows from Theorem 2 that X^* is compact. This completes the proof.

Two other methods for obtaining dendritic compactifications of dendritic spaces may be found in the literature. In [2] Ward proves, by embedding in a Tychonoff cube, that every locally connected dendritic space satisfying a certain convexity condition has a dendritic compactification. In [1] Proizvolov proves, by considering maximal collections of closed connected sets having the finite intersection property, that every locally peripherally compact dendritic space has a dendritic compactification.

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Received April 28, 1972 and in revised form July 13, 1972.

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