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A COMPACT SET THAT IS LOCALLY HOLOMORPHICALLY CONVEX BUT NOT HOLOMORPHICALLY CONVEX

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It is shown that a certain simple imbedding T of the ordinary two-dimensional torus in C^2 contains a polynomially convex compact T-neighborhood of each of its points, but T is not holomorphically convex in even the weakest presently accepted sense. This example illustrates some of the limitations of a theory of lower dimensional sets in C^n. In particular, it shows the difficulty of developing a theory based on local information.

In the following K will denote a compact set in C^n, $\mathscr{C}(K)$ the Banach algebra of continuous functions on K, and $\mathscr{O}(K)$ the algebra of functions holomorphic on some C^n neighborhood of K. Also, let $A(K)$ denote the Banach subalgebra of $\mathscr{C}(K)$ obtained by taking the closure of the image of $\mathscr{O}(K)$ in $\mathscr{C}(K)$. A compact set K is said to be *holomorphically convex* if K and the spectrum of $A(K)$ are homeomorphic under the natural map. In [5] a notion of the "envelope of holomorphy", for K a compact subset of C^n, was introduced; there it was proved, in particular, that K is equal to its envelope if and only if K is holomorphically convex. The Cartan Theorems A and B for open holomorphically convex sets in C^n admit analogues for compact holomorphically convex sets in C^n (see [5]). One might conjecture that the E. E. Levi problem for open sets in C^n admits a compact analogue. That is, one might conjecture that if K is locally holomorphically convex (i.e., for each point $z \in K$ there exists a compact neighborhood N of z in K such that N is holomorphically convex) then K is holomorphically convex. The example presented below shows that this is not the case.

If "holomorphic approximation" holds on a compact set $K \subset C^n$ (i.e., $\mathscr{O}(K)$ is dense in $\mathscr{C}(K)$) then the spectrum of $A(K) = \mathscr{C}(K)$ is of course homeomorphic to K so that K is holomorphically convex according to the above definition. Even if a compact set K has the property that "local holomorphic approximation" holds (i.e., for each point $z \in K$ there exists a compact neighborhood N of z in K such that $\mathscr{O}(N)$ is dense in $\mathscr{C}(N)$) the set K need not be (globally) holomorphically convex because of the example presented below. In particular, this provides an example of a compact set in C^n where local holomorphic approximation holds but global holomorphic approximation does not hold; as distinguished from the well-known fact that if K is a compact subset of the complex line C and local holomorphic approximation holds then it is true that global holomorphic approximation holds (see for example [2]).

In [4] a notion of a "totally real set in C^n" was introduced in order to better understand the properties of R^n in C^n which are crucial for the development of Sato's theory of hyperfunctions. Sato's basic theory [12] was shown to hold with R^n replaced by a totally real set. In the definition of a compact totally real subset K of C^n there are two local requirements which heuristically ensure that K has no (locally) "complex structure of dimension $\geqq 1$" (see [4], Definition 3.4 and the Remark 1 afterward). The example presented below shows that the local information contained in the assertion that K is a totally real set (which is more than just local holomorphic convexity but less than local holomorphic approximation) is not sufficient to ensure that K is holomorphically convex. In particular, in the duality result, Corollary 3.10 of [4], the hypothesis that K be holomorphically convex is necessary.

We would like to acknowledge that R. O. Wells has independently verified that the example given here is not holomorphically convex.

The example is very simple. It is just the two-dimensional torus T imbedded in C^2 as $T = \{z\colon (|z_1| - 3)^2 + x_2^2 = 1, y_2 = 0\}$. In fact, (a) the envelope of holomorphy of T is the set

$$\tilde{T} = \{z\colon (|z_1| - 3)^2 + x_2^2 \leqq 1, y_2 = 0\}$$

obtained by filling up T in $C \times R \times \{0\}$; but (b) each point a of T has a compact T-neighborhood N on which the polynomials $C[z_1, z_2]$ are dense in the Banach space $\mathscr{C}(N)$ of continuous functions on N. Of course this implies in particular that each compact subset of N is polynomially and hence holomorphically convex.

The proof of (a) rests on the observation that T has a basis for its neighborhood system consisting of the Hartogs domains

$$U_\varepsilon = \{z\colon |(|z_1| - 3)^2 + x_2^2 - 1| < \varepsilon, |y_2| < \varepsilon\}, \varepsilon > 0$$

(which are clearly circled in z_1 for each fixed z_2), and on the proposition below, which asserts that the envelope of holomorphy of U_ε is

$$\tilde{U}_\varepsilon = \{z\colon (|z_1| - 3)^2 + x_2^2 < 1 + \varepsilon, |y_2| < \varepsilon\}, \frac{1}{2} \geqq \varepsilon > 0 .$$

This shows that any function holomorphic in a neighborhood of T has a holomorphic extension to a neighborhood of $\tilde{T}$. Moreover, since each $\tilde{U}_\varepsilon$ is holomorphically convex, so is $\tilde{T} = \bigcap_{\varepsilon>0} \tilde{U}_\varepsilon$. Thus $\tilde{T}$ is the envelope of holomorphy of T (see [5] for the precise definition of envelope of holomorphy of T).

PROPOSITION. *$\tilde{U}_\varepsilon$ is the envelope of holomorphy of U_ε for $0 < \varepsilon \leqq 1/2$.*

Proof. The open set $\tilde{U}_\varepsilon$ is a domain of holomorphy because it is pseudoconvex (see [3] or [8]). The fact that the functions $z \to (|z_1| - 3)^2 + x_2^2$ and $z \to |y_2|^2$ are plurisubharmonic on $\tilde{U}_\varepsilon$, $\varepsilon \leqq 1/2$, implies that $\tilde{U}_\varepsilon$ is pseudoconvex by [8] Theorem 2.6.7 (iii).

Each function f holomorphic on U_ε has a holomorphic extension to $\tilde{U}_\varepsilon$. For this it suffices to see that the Hartogs-Laurent expansion (see [13] page 130) for f on U_ε,

$$f(z) = \sum_{n=-\infty}^{\infty} f_n(z_2) z_1^n \,, \tag{1}$$

is normally convergent on $\tilde{U}_\varepsilon$, for then its sum will extend f as asserted. Here the coefficients f_n are holomorphic on $\{z_2: x_2^2 < 1 + \varepsilon, |y_2| < \varepsilon\}$. From the normal convergence of (1) on U_ε it follows that

$$\sum_{n=-\infty}^{\infty} \sup \{|f_n(z_2) z_1^n| : z \in K_\delta\} < \infty \,, \tag{2}$$

where $0 \leqq \delta < \varepsilon$ and $K_\delta = \{z: (|z_1| - 3)^2 + x_2^2 = 1 + \delta, |y_2| \leqq \delta\}$ (a product of a torus in $\boldsymbol{C} \times \boldsymbol{R}$ and a closed interval in $\boldsymbol{R}$). Now the maximum principle applied (for fixed z_2) to $z_1 \to f_n(z_2) z_1^n$ shows that the suprema in (2) become no larger if extended over

$$\tilde{K}_\delta = \{z: (|z_1| - 3)^2 + x_2^2 \leqq 1 + \delta, |y_2| \leqq \delta\}.$$

Thus (2) holds with K_δ replaced by $\tilde{K}_\delta$, and since any compact subset of $\tilde{U}_\varepsilon$ is contained in the interior of some $\tilde{K}_\delta$, the normal convergence of (1) on $\tilde{U}_\varepsilon$ is proved. Thus $\tilde{U}_\varepsilon$ is the envelope of holomorphy of U_ε.

PROPOSITION. *Each point a of T has a compact neighborhood N in T such that $\boldsymbol{C}[z_1, z_2]$ is dense in $\mathscr{C}(N)$.*

Proof. Two cases will be distinguished.

(1) The point a is not on one of the top or bottom circles $|z_1| = 3$, $z_2 = \pm 1 + 0i$. Then a is a totally real point of T (i.e., the ordinary real-linear tangent space T_a to T at a is not complex-linear). The proposition is known for this case (see [11], [9] or [6]) but a simple direct proof can be based on the real-analyticity of T. It will be shown that a has an open neighborhood U such that $U \cap T$ is mapped into $\boldsymbol{R}^2$ by a biholomorphic map $\psi = (\psi_1, \psi_2): U \to \boldsymbol{C}^2$. Then if N is any compact subset of $T \cap U$, the ordinary Weierstrass Theorem implies that $\boldsymbol{C}[w_1, w_2]$ is uniformly dense in $\mathscr{C}(\psi(N))$. Since ψ is invertible, the polynomial combinations of ψ_1, ψ_2 are dense in $\mathscr{C}(N)$.

If U is taken from the beginning as a polycylinder, then ψ_1 and ψ_2 are approximable on N by polynomials in z_1, z_2, which proves the proposition in case (1).

The map ψ will be found by constructing its inverse. Note that there is an open neighborhood V of 0 in $\boldsymbol{R}^2$ and a real-analytic map $\phi: V \to T$ such that $\phi(0) = a$ and $d_0\phi(\boldsymbol{R}^2) = T_a$. Here $d_0 f$ denotes the Fréchet derivative of f at 0. Then there is an open set $\tilde{V}$ in $\boldsymbol{C}^2$ such that $\tilde{V} \cap \boldsymbol{R}^2 = V$ and a holomorphic map $\tilde{\phi}: \tilde{V} \to \boldsymbol{C}^2$ such that $\tilde{\phi} \mid V = \phi$. Clearly, $d_0\tilde{\phi}(\boldsymbol{R}^2) = T_a$. Moreover, $T_a \cap iT_a = \{0\}$, so $\boldsymbol{C}^2 = T_a + iT_a = d_0\tilde{\phi}(\boldsymbol{R}^2) + id_0\tilde{\phi}(\boldsymbol{R}^2) = d_0\tilde{\phi}(\boldsymbol{R}^2 + i\boldsymbol{R}^2) = d_0\tilde{\phi}(\boldsymbol{C}^2)$. Thus $d_0\tilde{\phi}$ is invertible, so $\tilde{\phi}$ has a holomorphic inverse ψ near 0 by the inverse function theorem.

(2) $|a_1| = 3, a_2 = \pm 1 + i0$. Then there is a closed disk $D = \{z_1: |z_1 - a_1| \leqq \varepsilon\}$ on which the graph of $g(z_1) = (\operatorname{sign} a_2)\sqrt{1 - (|z_1| - 3)^2}$ defines a compact set $N = \{(z_1, g(z_1)): |z_1 - a_1| \leqq \varepsilon\} \subset T$. Clearly, N is a T-neighborhood of a. Moreover, the level curves of g, as arcs of radii > 1, do not disconnect $\boldsymbol{C}$ and have no interior points. Therefore, by Mergelyan's Theorem [10] (§5, Theorem 1.5), the polynomial combinations of z_1 and g are dense in $\mathscr{C}(D)$. The proposition is proved by transporting this property to N via the homeomorphism $z_1 \to (z_1, g(z_1))$.

There is a result (going back to Grauert [3]) of a positive nature which enables one to conclude from local information that a compact subset K of $\boldsymbol{C}^n$ is holomorphically convex. Briefly, the method is as follows (cf. [11], [9] or [6]). Suppose that in some $\boldsymbol{C}^n$ neighborhood U_a of each point $a \in K$ there exists a $\boldsymbol{C}^2$ nonnegative strictly plurisubharmonic function φ such that $K \cap U_a$ equals $\{z \in U_a: \varphi(z) = 0\}$. By using a partition of unity one can construct a nonnegative strictly plurisubharmonic function φ in a neighborhood U of K such that $K = \{z \in U: \varphi(z) = 0\}$. Then for sufficiently small $\varepsilon > 0$, each of the sets $W_\varepsilon = \{z \in U: \varphi(z) < \varepsilon\}$ is a Stein open neighborhood of K and K is $\mathscr{O}(W_\varepsilon)$-convex. Hence K is holomorphically convex. The use of this result is limited by the fact (see [7]) that sets K which satisfy the local condition described above must be (locally) contained in a $\mathscr{C}^1$ submanifold of $\boldsymbol{C}^n$ all of whose points are totally real.

On the other hand, this technique is extended in [1], where such a function φ (which is only required to be plurisubharmonic-not strictly) is constructed in a neighborhood of a point on a two-manifold where its tangent space is complex linear but whose second-order behavior is sufficiently "hyperbolic" (in a precise sense given in [1]). This result is delimited by the above example T, which (in the same sense of [1]) exhibits a kind of "parabolic" behavior at such points.

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