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AN IDENTITY FOR MATRIX FUNCTIONS

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Let $G \subset S_n$. Let η be a character on G. For $A = (a_{ij})$ an n-square matrix, define

$$d_\eta^G(A) = \sum_{g \in G} \eta(g) \prod_{t=1}^{n} a_{tg(t)} .$$

A general identity for idempotents in group algebras is proved. A very special example of the consequences is this: If χ is a linear character on G and H a normal subgroup of G, then $[G: H]d_\chi^H(A) = \sum \eta(1)d_\eta^G(A)$, where the summation is over those irreducible characters η of G whose restriction to H contain χ as a component.

1. Introduction. In this note we prove two theorems involving idempotents in group algebras. The first is a very general identity between the central idempotents of a group algebra and similar items involving only elements of certain conjugate subgroups.

For nonnormal subgroups, Theorem 1 is not especially pretty. The second theorem gives a more appealing inequality for the nonnormal case.

Applications of these results to matrix functions improve and unify earlier results of Williamson [10], Merris [7], Freese [4], and Merris and Watkins [8].

2. Relations in the group algebra. Let H be a subgroup of the finite group G. Let χ be a (not necessarily irreducible) complex character of H. By KG we mean the complex group algebra of G. Define

$$t(H, \chi) = \frac{\chi(1)}{o(H)} \sum_{h \in H} \chi(h)h \in KG ,$$

where $o(H)$ is the order of H. Denote by $I(H)$ the set of irreducible complex characters on H. We know that $\{t(G, \eta): \eta \in I(G)\}$ is a set of mutually annihilating idempotents which spans $Z(KG)$, the center of KG [1, p. 83], [2, pp. 233-236]. (Moreover, $t(G, \eta)$ generates the simple two sided ideal in KG to which η corresponds, and $\sum_{\eta \in I(G)} t(G, \eta) = 1 \in G$.)

THEOREM 1. *Let*

$$L = \{g \in N_G(H): \chi(g^{-1}hg) = \chi(h): \forall h \in H\} .$$

Let $g_1, \cdots, g_r$ be right coset representatives for L in G. Let χ_i be *the character on* $g_i^{-1}Hg_i$ *defined by* $\chi_i(g_i^{-1}hg_i) = \chi(h)$ *for all* $h \in H$. *Then*

$$(1) \qquad \frac{1}{r}\sum_{i=1}^{r} t(g_i^{-1}Hg_i, \chi_i) = \sum_{\eta \in I(G)} \frac{\chi(1)(\chi, \eta)_H}{\eta(1)} t(G, \eta) ,$$

where $(\chi, \eta)_H$ *is the inner product,*

$$(\chi, \eta)_H = \frac{1}{o(H)} \sum_{h \in H} \chi(h)\eta(h^{-1}) .$$

Proof. Let

$$t = \sum_{i=1}^{r} t(g_i^{-1}Hg_i, \chi_i) .$$

We first show that $t \in Z(KG)$. Let $g \in G$. Then

$$(2) \qquad g^{-1}tg = \frac{\chi(1)}{o(H)} \sum_{i=1}^{r} \sum_{h=H} \chi(h)g^{-1}g_i^{-1}hg_ig .$$

Now, $Lg_i \to Lg_ig$ is a permutation of $\{Lg_i : 1 \leqq i \leqq r\}$. Say $Lg_ig = Lg_{\sigma(i)}$. Thus, there exists $l_i \in L$ such that $g_ig = l_ig_{\sigma(i)}$, $1 \leqq i \leqq r$. Equation (2) becomes

$$\begin{aligned} g^{-1}tg &= \frac{\chi(1)}{o(H)} \sum_{i=1}^{r} \sum_{h \in H} \chi(h)g_{\sigma(i)}^{-1}l_i^{-1}hl_ig_{\sigma(i)} \\ &= \frac{\chi(1)}{o(H)} \sum_{i=1}^{r} \sum_{h \in H} \chi(l_i^{-1}hl_i)g_{\sigma(i)}^{-1}l_i^{-1}hl_ig_{\sigma(i)} \\ &= \frac{\chi(1)}{o(H)} \sum_{i=1}^{r} \sum_{h \in H} \chi(h)g_{\sigma(i)}^{-1}hg_{\sigma(i)} \\ &= t . \end{aligned}$$

It follows that $t \in Z(KG)$. From our previous remarks, there exist complex numbers $\mathscr{A}_\eta$, $\eta \in I(G)$, such that

$$t = \sum_{\eta \in I(G)} \mathscr{A}_\eta t(G, \eta) .$$

Since the $t(G, \eta)$ are annihilating idempotents, it follows that

$$(3) \qquad tt(G, \eta) = \mathscr{A}_\eta t(G, \eta) .$$

We now view t and $t(G, \eta)$ as linear operators on KG. For example, to obtain $t(g)$, just multiply g on the left by t, i.e., $t(g) = tg$. Then, from (3),

$$\mathscr{A}_\eta = \frac{\operatorname{tr}(tt(G, \eta))}{\operatorname{tr}(t(G, \eta))} .$$

Let ρ be the character of the regular representation of G. Then

$$\operatorname{tr}(t(G, \eta)) = \frac{\eta(1)}{o(G)} \sum_{g \in G} \eta(g)\rho(g) = \eta(1)^2 .$$

Similarly, compute

$$\begin{aligned}
\mathscr{A}_\eta &= \frac{1}{\eta(1)^2} \operatorname{tr}(tt(G, \eta)) \\
&= \frac{\chi(1)}{\eta(1)o(H)o(G)} \operatorname{tr}\Big(\sum_{i=1}^{r} \sum_{h \in H} \sum_{g \in G} \chi(h)\eta(g) g_i^{-1} h g_i g\Big) \\
&= \frac{\chi(1)}{\eta(1)o(H)o(G)} \sum_{i=1}^{r} \sum_{h \in H} \sum_{g \in G} \chi(h)\eta(g_i^{-1}h^{-1}g_i g)\rho(g) \\
&= \frac{\chi(1)}{\eta(1)o(H)} \sum_{i=1}^{r} \sum_{h \in H} \chi(h)\eta(h^{-1}) \\
&= \frac{\chi(1)r}{\eta(1)} (\chi, \eta)_H .
\end{aligned}$$

(We have used the fact that $\operatorname{tr}(g) = \rho(g) = 0$ if $g \neq 1$, and $\rho(1) = o(G)$.)

REMARKS. Suppose H is normal in G. Let ξ be a character of G and let $\chi = \xi \mid H$. Then $L = G$, $r = 1$, and (1) becomes

$$(4) \qquad t(H, \chi) = \chi(1) \sum_{\eta \in I(G)} \frac{(\chi, \eta)_H}{\eta(1)} t(G, \eta) .$$

If ξ happens to be an irreducible character of G, then

$$\chi = e(\lambda_1 + \cdots + \lambda_m) ,$$

where $\lambda_1 = \lambda$ is an irreducible component of χ and $\lambda_2, \cdots, \lambda_m$ are the inequivalent conjugates of λ. The integer e is the index of ramification of ξ with respect to H [3, p. 53]. In this case, (4) becomes

$$(5) \qquad t(H, \chi) = \frac{\chi(1)}{\lambda(1)} e \sum t(G, \eta) ,$$

where the summation is over those irreducible characters, η, of G whose restriction to H contains λ as a component. Expression (5) reveals that $t(H, \chi)$ is essentially idempotent when $H \triangle G$ and $\chi \in I(G)$. (This can, of course, be verified directly.)

If A and B are positive semidefinite hermitian operators on an n dimensional complex inner product space, then $A \geqq B$ means that $A - B$ is positive semidefinite.

THEOREM 2. *Let H be a subgroup of G. Let χ be an irreducible character on H. Then*

$$(6) \qquad t(H, \chi) \geqq \sum t(G, \eta) ,$$

where the summation is over those irreducible characters of G whose restriction to H is a multiple of χ.

Proof. With respect to the inner product on KG which makes G an o.n. basis, $t(H, \chi)$ and $t(G, \eta)$ are hermitian. Since they are idempotent, they are orthogonal projections and hence are positive semidefinite. It is proved in [7, Corollary 2.4] that $t(H, \chi) \geqq t(G, \eta)$ if and only if $\eta \mid H$ is a multiple of χ. Since $t(G, \eta) \in Z(KG)$, and since any two of them annihilate each other, the result follows.

3. **Applications to matrix functions.** We now assume that G is a subgroup of S_n, the symmetric group of degree n. We still take χ to be a character on the subgroup H of G. Let r be a fixed but arbitrary integer in $[1, n]$. For the generic n-square complex matrix $A = (a_{ij})$, define

$$\delta_\chi^H(A) = \frac{\chi(1)}{o(H)} \sum_{g \in H} \chi(g) E_r(a_{1g(1)}, \cdots, a_{ng(n)}) ,$$

where E_r is the rth elementary symmetric function.

When $r = n$, δ_χ^H is $\chi(1)/o(H)$ times the generalized matrix function, d_χ^H, of Schur [4], [5], and [9]. (If $H = S_n$ and χ is the alternating character, $d_\chi^H =$ determinant. If $H = \{1\}$ and $\chi = 1$, $d_\chi^H = h$, the product of the main diagonal elements. If $H = S_n$ and $\chi = 1$, $d_\chi^H =$ permanent.)

When $r = 1$, δ_χ^H is $\chi(1)/o(H)$ times the generalized trace function, t_χ^H [6]. (If $H = \{1\}$ and $\chi = 1$, $t_\chi^H =$ trace. If $H = \langle(1, 2, \cdots, n)\rangle$ and $\chi = 1$, $t_\chi^H(A)$ is the sum of the entries of A.)

The case for general r has also been discussed. See, for example, [7].

COROLLARY 1. *Let H be a normal subgroup of G. Let ξ be an irreducible character of G and $\chi = \xi \mid H$. Then*

$$\delta_\chi^H(A) = \frac{\chi(1)e}{\lambda(1)} \sum \delta_\eta^G(A) , \tag{7}$$

where the notation and summation are as they were in (5).

REMARKS. If $H = \{1\}$, then (7) becomes

$$E_r(a_{11}, \cdots, a_{nn}) = \sum_{\eta \in I(G)} \delta_\eta^G(A) . \tag{8}$$

When $r = n$, (8) was obtained by Freese [4, eq. (8)].

If

$$H = G_\xi = \{g \in G: |\xi(g)| = \xi(1)\} ,$$

and $r = n$, equation (7) was obtained in [8].

When $\xi | H$ remains irreducible, (7) is a significant generalization of a result of Williamson [10, Theorem 1]. He showed that if $\xi(1) = 1$, then

$$d_\xi^G(A) \leqq [G: H] d_\xi^H(A) \tag{9}$$

for all positive semidefinite hermitian A. Equation (7) shows what has been discarded to obtain (9). In a similar way, equation (7) improves some of the results in [7].

A more general corollary could be obtained using the full power of Theorem 1. For simplicity, we use only (5).

Proof of Corollary 1. We begin by obtaining an alternate expression for δ_χ^G. First, some notation: Let $\{P(g): g \in S_n\}$ be the standard representation of S_n by n-square permutation matrices, i.e., the i, j element of $P(g)$ is $\delta_{ig(j)}$. Let $P_r(g)$ be the rth Kronecker power of $P(g)$. Let $D_{r,n}$ be the set of one-to-one functions from $\{1, 2, \cdots, r\}$ into $\{1, 2, \cdots, n\}$. (In particular, $D_{n,n} = S_n$.) Let $Q_{r,n}$ be the subset of $D_{r,n}$ of order preserving one-to-one functions, i.e., $Q_{r,n}$ is the set

$$\{\alpha = (\alpha(1), \cdots, \alpha(r)): 1 \leqq \alpha(1) < \alpha(2) < \cdots < \alpha(r) \leqq n\} .$$

Now, observe

$$\begin{aligned}
r! \frac{o(H)}{\chi(1)} \delta_\chi^H(A) &= r! \sum_{g \in H} \chi(g) \sum_{\alpha \in Q_{rn}} \prod_{i=1}^r a_{\alpha(i) g\alpha(i)} \\
&= \sum_{g \in H} \chi(g) \sum_{\alpha \in D_{rn}} \prod_{i=1}^r a_{\alpha(i) g\alpha(i)} \\
&= \sum_{g \in H} \chi(g) \sum_{\alpha, \beta \in D_{rn}} \prod_{i=1}^r (\delta_{\beta(i) g\alpha(i)} a_{\alpha(i)\beta(i)}) \\
&= \sum_{g \in H} \chi(g) \sum_{\alpha, \beta \in D_{rn}} \prod_{i=1}^r (P(g)_{\beta(i)\alpha(i)} a_{\alpha(i)\beta(i)}) \\
&= \sum_{g \in H} \chi(g) \sum_{\alpha, \beta \in D_{rn}} (P_r(g)_{\beta\alpha} K_r(A)_{\alpha\beta}) \\
&= \operatorname{trace} (C_r'(H, \chi) K_r'(A)) , \qquad (10)
\end{aligned}$$

where $K_r(A)$ is the rth Kronecker power of A,

$$C_r(H, \chi) = \sum_{g \in H} \chi(g) P_r(g) ,$$

and primes indicate the principal submatrices corresponding to $D_{r,n}$.

Now, $g \rightarrow P_r(g)$ is a representation of S_n. Thus, we may extend it linearly to a homomorphism, $\bar{P}_r$ of KS_n. To complete the proof, apply $\bar{P}_r$ to equation (5), restrict the resulting equation to the principal submatrices corresponding to $D_{r,n}$, and exploit the linearity of the expression (10).

A similar corollary is available for Theorem 2. Since the proof is very similar, we will omit it.

COROLLARY 2. *We have*

$$\delta_{\chi}^{H}(A) \geqq \sum \delta_{\gamma}^{G}(A)$$

for all positive semidefinite hermitian A. The notation and summation are as they were in Theorem 2.

REFERENCES

1. Hermann Boener, *Representations of Groups*, American Elsevier, New York, 1970.
2. Charles W. Curtis and Irving Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962.
3. Walter Feit, *Characters of Finite Groups*, Benjamin, New York, 1967.
4. Ralph Freese, *Inequalities for generalized matrix functions based on arbitrary characters*, preprint.
5. Marvin Marcus and Henryk Minc, *Generalized matrix functions*, Trans. Amer. Math. Soc., **116** (1965), 316-329.
6. Russell Merris, *Trace functions I*, Duke Math. J., **38** (1971), 527-530.
7. ———, *A dominance theorem for partitioned hermitian matrices*, Trans. Amer. Math. Soc., **164** (1972), 341-352.
8. Russell Merris and William Watkins, *Character induced subgroups*, J. Research Natl. Bur. Stds. 77B (1973), 93-99.
9. I. Schur, *Über endliche Gruppen und Hermitesche Formen*, Math. Z., **1** (1918), 184-207.
10. S. G. Williamson, *On a class of combinatorial inequalities*, J. Combinatorial Theory, **6** (1969), 359-369.

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