PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 2, 1974

MONOTONE MAPPINGS OF A TWO-DISK ONTO ITSELF WHICH FIX THE DISK'S BOUNDARY CAN BE CANONICALLY APPROXIMATED BY HOMEOMORPHISMS

WILLIAM E. HAVER

The theorem stated in the title is proven. As a corollary it is shown that the space of all such monotone mappings is an absolute retract.

1. Introduction. Let D^n denote the standard n-ball of radius one in E^n and $H(D^n)$ the space of all homeomorphisms of D^n onto itself which equal the identity on the boundary of D^n. Let $\overline{H(D^n)}$ denote the space of all mappings of D^n onto itself which can be approximated arbitrarily closely by elements of $H(D^n)$. Under the supremum topology, $H(D^n)$ and $\overline{H(D^n)}$ are separable metric spaces; $\overline{H(D^n)}$ is complete under the supremum metric. It is known that $\overline{H(D^n)}$ is locally contractible [7], $\overline{H(D^n)} \times l_2 \approx \overline{H(D^n)}$ [4], $\overline{H(D^n)}$ is homogeneous [7], and that $\overline{H(D^1)} \approx l_2$ [3]. In this paper we shall be concerned with the case $n = 2$, and to simplify notation we shall write D for D^2. It is well-known (cf. [8]) that $\overline{H(D)}$ is the space of all monotone mappings of D onto itself which equal the identity when restricted to the boundary of D.

We shall show [Theorem 1] that the elements of $\overline{H(D)}$ can be "canonically approximated" by elements of $H(D)$ and [Theorem 2] that $\overline{H(D)}$ is an absolute retract. The work of this paper depends heavily on W. K. Mason's paper, "The space of all self-homeomorphisms of a two-cell which fix the cell's boundary is an absolute retract", [9]. The crux of Mason's paper is the definition of a basis for $H(D)$ which can be shown to possess some particularly nice properties. We shall review the definition of this basis in the following section and then define a basis for $\overline{H(D)}$. Familiarity will be assumed with the notation and basic definitions of [9].

2. Mason's basis for $H(D)$. Consider D to be a rectangle in R^2 with horizontal and vertical sides. A grating, P, on D consists of a finite number of spanning segments (crosscuts) across D, parallel to its sides, with the same number of horizontal and vertical crosscuts. Let $P_1, P_2, \cdots$ be a sequence of gratings on D such that (a) the mesh of P_i approaches 0 as i increases and (b) if l is a crosscut of P_i and $j \geq i$, then l is a crosscut of P_j.

Let $\mathscr{H}$ be the collection of all polyhedral disks H contained in D

such that Bd (H) is the union of a vertical segment in the left side of Bd (D), a vertical segment in the right side of Bd (D), a polygonal spanning arc of D, H^T, that is contained in the closure of the same component of $H(D) - H$ as the top of Bd (D), and a polygonal spanning arc of D, H^B, that is contained in the closure of the same component of $H(D) - H$ as the bottom of Bd (D). Let $\mathscr{V}$ be the collection of all polyhedral disks V contained in D such that Bd (V) is the union of a horizontal segment in the top of Bd (D), a horizontal segment in the bottom of Bd (D), a polygonal spanning arc of D, V^L, that is contained in the closure of the same component of $H(D) - V$ as the left side of Bd (D) and a polygonal spanning arc of D, V^R, that is contained in the closure of the same component of $H(D) - V$ as the right side of Bd (D).

Let P_j be a grating from the sequence $P_1, P_2, \cdots$. Let $\{l_1, \cdots, l_n\}$ be the set of horizontal crosscuts of P_j and $\{m_1, \cdots, m_n\}$ the set of vertical crosscuts. Let $\{H_1, \cdots, H_n\} \subset \mathscr{H}$ satisfy $H_i \cap H_j = \emptyset$ if $i \neq j$ and $\{V_1, \cdots, V_n\} \subset \mathscr{V}$ satisfy $V_i \cap V_j = \emptyset$ if $i \neq j$. Then define

$$\begin{aligned} &O(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n) \\ &\quad = \{f \in H(D) \mid f(l_i) \subset H_i - \{\mathrm{Cl}\,(D - H_i)\} \text{ and} \\ &\qquad f(m_i) \subset V_i - \{\mathrm{Cl}\,(D - V_i)\} \text{ for } 1 \leqq i \leqq n\} . \end{aligned}$$

Then the basis for $H(D)$, which Mason denotes HVT, is the collection of all such open sets.

3. **A Basis for $\overline{H(D)}$.** In this section we define a basis, β, for $\overline{H(D)}$ and demonstrate that it possesses some nice properties. Let P_j, $\{H_1, \cdots, H_n\}$ and $\{V_1, \cdots, V_n\}$ be as in the definition of HVT. The basis, β, will consist of all sets of the following form:

$$\begin{aligned} &B(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n) \\ &\quad = \{f \in \overline{H(D)} \mid f^{-1}(l_i) \subset H_i - \{\mathrm{Cl}\,(D - H_i)\} \text{ and} \\ &\qquad f^{-1}(m_i) \subset V_i - \{\mathrm{Cl}\,(D - V_i)\}, \text{ for } 1 \leqq i \leqq n\} . \end{aligned}$$

We note that $f \in B(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n) \cap H(D)$ if and only if $f^{-1} \in O(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n)$. To see that the elements of β are open subsets of $\overline{H(D)}$, let f be an arbitrary element of $B(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n)$. Then let

$$\varepsilon = \min_{1 \leqq i \leqq n} \{d(f(H_i^T \cup H_i^B), \bigcup_{j=1}^{n} l_j), d(f(V_i^L \cup V_i^R), \bigcup_{j=1}^{n} m_j)\} .$$

Let g be an arbitrary element of $\overline{H(D)}$ satisfying $d(f, g) < \varepsilon$. Suppose that $g \notin B(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n)$. Then, without loss of generality, we can assume that there exists an integer i such that $g^{-1}(l_i)$

is not contained in $H_i - \mathrm{Cl}(D - H_i)$. Since (i) $g^{-1}(l_i) \cap (\mathrm{Bd}\, D) = f^{-1}(l_i) \cap (\mathrm{Bd}\, D) \subset H_i - \mathrm{Cl}(D - H_i)$, (ii) $g^{-1}(l_i)$ is a connected set and (iii) $H_i^T \cup H_i^B$ separates $H_i - \mathrm{Cl}(D - H_i)$ from $D - H_i$, there is an $x \in H_i^T \cup H_i^B$ such that $g(x) \in l_i$. But this implies that $d(f(x), g(x)) \geqq d(f(H_i^T \cup H_i^B), l_i) \geqq \varepsilon$ and hence $d(f, g) \geqq \varepsilon$.

LEMMA 1. *β is a basis for $\overline{H(D)}$.*

Proof. Let $f \in \overline{H(D)}$ and $\varepsilon > 0$ be given. We wish to find $B \in \beta$ such that $f \in B$ and $d(f, g) < \varepsilon$ for all $x \in B$. Pick a grating, P_j, such that diam $|st(x, P_j)| < \varepsilon$ for every $x \in D$. Let l_i be the ith crosscut from the top of D. Choose H_1^T to be a polygonal spanning arc of D with one endpoint in each side of D that separates the top of D from $f^{-1}(l_1)$. Choose H_1^B to be a polygonal spanning arc of D that separates $f^{-1}(l_1)$ from $f^{-1}(l_2)$. The polyhedral disk H_1 is thus uniquely defined. Assume inductively that disks $H_1, \cdots, H_{i-1}$ have been defined in such a manner that $H_j \cap H_k = \varnothing$ if $1 \leqq j \leqq i-1$, $1 \leqq k \leqq i-1$, and $j \neq k$ and that for each j, $1 \leqq j \leqq i-1$, H_j^T separates H_{j-1}^B from $f^{-1}(l_j)$ and H_j^B separates $f^{-1}(l_j)$ from $f^{-1}(l_{j+1})$. Choose H_i^T to be a polygonal spanning arc of D that separates H_{i-1}^B from $f^{-1}(l_i)$. Finally choose H_i^B to be a pylygonal spanning arc of D that separates $f^{-1}(l_i)$ from $f^{-1}(l_{i+1})$ (or from the bottom of D if $i = n$). We have thus uniquely defined H_i in such a way that the inductive hypothesis is satisfied. Define $V_1, \cdots, V_n$ in a similar manner. Now, by construction $f \in B(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n)$ and if $g \in B(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n)$ then $d(f, g) < \varepsilon$.

LEMMA 2. *Let $B_1, \cdots, B_j$ be elements of β. Then $B = \bigcap_{k=1}^j B_k$ is an element of β.*

Proof. Assume $B \neq \varnothing$ and that P_k is the grating associated with B_k, $1 \leqq k \leqq j$. Hence for any k, $1 \leqq k \leqq j$, every crosscut of P_k is a crosscut of P_j. Let l_1 be the first horizontal crosscut of P_j. For each k, $1 \leqq k \leqq j$, let $H_{1,k}$ be the element of $\mathscr{H}$ associated with l_1 and B_k (if there is one). Let H_1 be the component of $D - (\bigcup_{k=1}^j H_{1,k}^T \cup \bigcup_{k=1}^j H_{1,k}^B)$ that contains $f^{-1}(l_1)$. Define in an analogous manner $H_2, \cdots, H_n$ and $V_1, \cdots, V_n$. It is clear that $B(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n) = \bigcap_{k=1}^j B_k$. The elements of $\{H_1, \cdots, H_n\}$ are pairwise disjoint since each H_i is contained in $H_{i,n}$ and the elements of $\{H_{1,j}, \cdots, H_{n,j}\}$ are pairwise disjoint.

Lemma 3 will follow as a corollary to the following theorem of Mason. (The proof of this theorem constitutes the bulk of [9].)

THEOREM (Mason). *Let U be an element of HVT and K a finite* *dimensional compact subset of U. Then there is an embedding ψ of the cone over K into U such that $\psi(f, 0) = f$, for all $f \in K$.*

LEMMA 3. *Let $B = B(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n)$ be an element of β and K a finite dimensional compact subset of $B \cap H(D)$. Then there is an embedding λ of the cone over K into $B \cap H(D)$ such that $\psi(f, 0) = f$, for all $f \in K$.*

Proof. Since $H(D)$ is a topological group, the function $G: H(D) \to H(D)$ defined by $G(f) = f^{-1}$ is a homeomorphism. Therefore, by the note following the definition of β, $G(K)$ is a finite dimensional compact subset of $U(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n)$. Hence by Mason's theorem there is an embedding ψ of the cone over $G(K)$ into $U(P_j; H_1, \cdots, H_n; V_1, \cdots, V_n)$ such that $\psi(f, 0) = f$, for all $f \in G(K)$. Define $\lambda: K \times I \to B \cap H(D)$ by $\lambda(k, t) = G^{-1}(\psi(G(k), t))$.

4. The main results. The following theorem shows that the elements of $\overline{H(D)}$ can be canonically approximated by elements of $H(D)$.

THEOREM 1. *Let α be an open cover of $\overline{H(D)}$. Then there exists a locally finite polyhedron, $\mathscr{P}$, and maps $b: \overline{H(D)} \to \mathscr{P}$, $\psi: \mathscr{P} \to H(D)$, and $\theta: \overline{H(D)} \times I \to \overline{H(D)}$ such that*

(a) *for each $f \in \overline{H(D)}$, there is an element, U_f, of α such that $\theta(f, t) \in U_f$, for each $t \in I$,*

(b) *$\theta(f, 1) = f$, for each $f \in \overline{H(D)}$,*

(c) *$\theta(f, 0) = \psi b(f)$, for each $f \in \overline{H(D)}$,*

(d) *$\theta(f, t) \in H(D)$ for each $f \in \overline{H(D)}$ and $t \in [0, 1)$.*

Proof. Let α' be an open barycentric refinement α (i.e., if $f \in \overline{H(D)}$, $|st(f, \alpha')|$ is contained in some element of α). For each positive integer, k, let α_k be an open cover of $\overline{H(D)}$ such that

(i) α_k is a refinement of α',

(ii) if $V \in \alpha_k$, diam $V < 1/k$,

(iii) if $V \in \alpha_k$, then $V \in \beta$.

We next define an open cover, η, of $\overline{H(D)} \times [0, 1)$.

$$\text{Let } \eta = \{V \times [0, 1/2) \mid V \in \alpha_1\} \cup \bigcup_{k=2}^{\infty} \left\{ V \times \left(\frac{2^k - 3}{2^k}, \frac{2^k - 1}{2^k}\right) \middle| V \in \alpha_k \right\}.$$

Let γ be a countable refinement of η such that

(a) if $h \in \gamma$, $st(h, r)$ is a finite set,

(b) if $h \in \gamma$, then there is an element of η, $V \times J$, such that $|st(h, \gamma)| \subset V \times J$.

Let $\mathscr{P}$ be the nerve of γ and $B: \overline{H(D)} \times [0, 1) \to \mathscr{P}$ be the standard barycentric map. Order the element of γ, and for each $h_i \in \gamma$, let $V_i \times J_i$ be an element of η such that $|st(h_i, \gamma)| \subset V_i \times J_i$. Note that $V_i \in \beta$.

We shall define a map $\psi: \mathscr{P} \to H(D)$ by induction on the skeletons of $\mathscr{P}$. For each vertex (h_i) of $\mathscr{P}$, let $\psi^0((h_i))$ be an arbitrary element of $H(D)$ intersected with the projection of h_i onto $\overline{H(D)}$.

Now assume that for $m = 1, 2, \cdots, n$ we have defined $\psi^m: \mathscr{P}^m \to H(D)$ such that ψ^m extends ψ^{m-1} and for each simplex $\sigma^m = (h_{\lambda_0}, \cdots, h_{\lambda_m})$:

(a) $\psi^m(\sigma^m)$ is finite dimensional,

(b) $\psi^m(\sigma^m) \subset H(D) \cap \{V_i \mid h_i \subset \bigcap_{j=0}^m st(h_{\lambda_j}, \gamma)\}$.

Now let $\sigma^{n+1} = \langle h_0, \cdots, h_{n+1}\rangle$ be any simplex of $\mathscr{P}^{n+1}$. Let

$$U = \bigcap \{V_i \mid h_i \subset \bigcap_{j=0}^{n+1} st(h_j, \gamma)\} .$$

Since each V_i is an element of β, by Lemma 2, $U \in \beta$. By the inductive hypothesis the image under ψ^n of the boundary of σ^{n+1} is a finite dimensional compact subset of $U \cap H(D)$, denoted K. By Lemma 3 there is an embedding

$$\lambda: \subset (K) \to U \cap H(D)$$

such that $\lambda(f, 0) = f$ for all $f \in K$. We consider σ^{n+1} to be the cone over its boundary, and so for $(x, t) \in \sigma^{n+1}$, let $\psi^{n+1}(x, t) = \lambda(\psi^n(x), t)$.

Extending over each $n + 1$ simplex in this manner gives $\psi^{n+1}: \mathscr{P}^{n+1} \to H(D)$ and completes the induction. Hence $\lim_{n\to\infty} \psi^n = \psi: \mathscr{P} \to H(D)$ is continuous by the continuity of each ψ^n and the local finiteness of $\mathscr{P}$.

Let $b: \overline{H(D)} \to \mathscr{P}$ be defined by $b(f) = B((f, 0))$.

We next define the homotopy $\theta: \overline{H(D)} \times I \to \overline{H(D)}$ in the following manner:

$$\theta(f, t) = \begin{cases} \psi(B(f, t)), & t \neq 1 \\ f, t = 1 . \end{cases}$$

Conditions (b), (c), and (d) are obviously satisfied. We show simultaneously that θ is continuous and that for each $f \in \overline{H(D)}$ there is an element U_f of α such that for each $t \in I$, $\theta(f, t) \in U_f$.

Suppose that $(f, t) \in \overline{H(D)} \times [0, 1)$ and that $(2^k - 3)/2^k < t < (2^k - 1)/2^k$. Let h_0 be any element of γ which contains (f, t). By the definition of ψ, $\psi B(f, t) \in V_0$. But $V_0 \in \alpha_{k-1} \cup \alpha_k \cup \alpha_{k+1}$ and therefore the diameter of V_0 is less than $1/(k - 1)$ which implies that $d(\psi B(f, t), f) < 1/(k - 1)$ and thereby that θ is continuous. Since each α_k is a refinement of α', there exists an element of α', $U_{(f,t)}$, such that

$\{f\} \cup \{\psi B(f, t)\} \subset V_0 \subset U_{(f,t)}$. Since α' is a barycentric refinement of α, there is some element, U_f, of α such that $\bigcup_{t \in [0,1)} U_{(f,t)} \subset U_f$ and hence $\theta(f, t) \in U_f$, for each $t \in I$.

The following result is an immediate corollary of Theorem 1 and a theorem of Hanner [5] which states that a metric space X is an ANR if given an arbitrary cover, α, of X there exists a locally finite polyhedron $\mathscr{P}$, maps $b: X \to \mathscr{P}$, $\psi: \mathscr{P} \to X$, and $\theta: X \times I \to X$ such that $\theta(x, 0) = \psi b(x)$ for all $x \in X$, $\theta(x, 1) = x$ for all $x \in X$ and for each $x \in X$ there is an element U of α such that $\theta(x, t) \in U$ for all $t \in [0, 1]$.

THEOREM 2. *$\overline{H(D)}$ is an absolute retract.*

Proof. By the preceding comments, $\overline{H(D)}$ is an ANR. But $\overline{H(D)}$ is contractible by the Alexander isotopy [1] applied to $\overline{H(D)}$. The theorem follows since every contractible absolute neighborhood retract is an absolute retract.

5. **Applications.** (a) The author has shown [6] that $\overline{H(M)}$, the space of all mappings of a compact manifold onto itself which can be approximated arbitrarily closely by homeomorphisms, is weakly locally contractible. Theorem 1 can be used [7] to show that for any compact 2-manifold, M^2, $\overline{H(M^2)}$ is locally contractible.

(b) A problem of current interest is whether $H(D)$ is homeomorphic to l_2; it can easily by shown using a result of Anderson [2] that if $\overline{H(D)}$ is homeomorphic to l_2, then $H(D)$ is homeomorphic to l_2. Perhaps the results of this paper and the fact that $\overline{H(D)}$ is complete under the usual metric will be helpful in showing that $\overline{H(D)}$ is homeomorphic to l_2.

(c) L. C. Siebenmann [10] has asked whether the inclusion map $i: H(M) \to \overline{H(M)}$ is a homotopy equivalence. Theorem 1 provides an affirmative answer to the question for the special case $i: H(D) \to \overline{H(D)}$.

Added in proof. Recent work of Torunczyk ("Absolute retracts as factors of normed linear spaces," Fund. Math., to appear) implies that since $\overline{H(D)}$ is an AR and $H(D) \times l_2 \approx \overline{H(D)}$, $\overline{H(D)}$ is in fact homeomorphic to l_2.

REFERENCES

1. J. W. Alexander, *On the deformation of an n-cell,* Proc. Nat. Acad. Sci., U.S.A., **9** (1923), 406-407.

2. R. D. Anderson, *Strongly negligible sets in Frechet manifolds,* Bull. Amer. Math. Soc., **75** (1969), 64-67.

3. R. Geoghegan, *On spaces of homeomorphisms, embeddings, and functions I,* Topology, **11** (1972), 159-177.

4. R. Geoghegan and D. W. Henderson, *Stable function spaces*, to appear.
5. O. Hanner, *Some Theorems on Absolute Neighborhood Retracts*, Arkiv Fur Matematik, **1** (1951), 389–408.
6. W. Haver, *Homeomorphisms and UV^∞ maps*, Proc. of SUNY Bingh. Conf. on Monotone Mappings and Open Mappings, (1970), 112–121.
7. ———, *The Closure of the Space of Homeomorphisms on a Manifold*, Trans. Amer. Math. Soc., to appear.
8. J. Hocking, *Approximations to monotone mappings on noncompact two-dimensional manifolds*, Duke Math. J., **21** (1954), 639–651.
9. W. K. Mason, *The space of all self-homeomorphisms of a 2-cell which fix the cell's boundary is an absolute retract*, Trans. Amer. Math. Soc., **161** (1971), 185–206.
10. L. C. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology, **11** (1972), 271–294.

Received September 6, 1972. Research partially supported by NSF Grant GP. 33872.

UNIVERSITY OF TENNESSEE