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A CHARACTERIZATION OF THE TOPOLOGY OF COMPACT CONVERGENCE ON $C(X)$

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The function space of all continuous real-valued functions on a realcompact topological space X is denoted by $C(X)$. It is shown that a topology τ on $C(X)$ is a topology of uniform convergence on a collection of compact subsets of X if and only if (*) $C_\tau(X)$ is a locally m-convex algebra and a topological vector lattice. Thus, the topology of compact convergence on $C(X)$ is characterized as the finest topology satisfying (*). It is also established that if $C_\tau(X)$ is an A-convex algebra (a generalization of locally m-convex) and a topological vector lattice, then each closed (algebra) ideal in $C_\tau(X)$ consists of all functions vanishing on a fixed subset of X. Some consequences for convergence structures are investigated.

Introduction. Throughout this paper, X will denote a realcompact topological space and $C(X)$ the algebra and lattice of all real-valued continuous functions on X under the pointwise defined operations. After preliminary remarks in §1, we describe (Theorem 1) closed (algebra) ideals in $C(X)$ endowed with a topology τ making $C_\tau(X)$ an A-convex algebra (a generalization of locally m-convex introduced in [4]) and a topological vector lattice. As a corollary, we state sufficient conditions for τ so that an ideal in $C_\tau(X)$ is closed if and only if it consists of all functions vanishing on a subset of X. Then, in Theorem 3, we characterize topologies on $C(X)$, which are topologies of uniform convergence on a collection of compact subsets of X. In particular, the corollary of Theorem 3 provides a characterization of the topology of compact convergence on $C(X)$. We conclude the note (§3), by discussing generalizations applicable to convergence structures on $C(X)$.

1. Definitions and preliminary results.

Since our major concern is the algebra $C(X)$, we restrict our definitions to commutative algebras over the reals.

DEFINITION 1. Given a commutative R-algebra $\mathscr{A}$, an absolutely convex subset $S \subset \mathscr{A}$ is said to be *m-convex* (respectively, *A-convex*) if $S \cdot S = \{fg : f, g \in S\}$ is contained in S (respectively, $fS = \{fg : g \in S\}$ is absorbed by S for each f in S). Now $(\mathscr{A}, \tau)$, the algebra $\mathscr{A}$ together with a convergence structure τ (see [1]) is said to be an *m-convex* (respectively, *A-convex*) *convergence algebra* if τ is a convergence vector space structure (see [1]) and for every filter θ convergent to zero (in $(\mathscr{A}, \tau)$), there exists a coarser filter Φ convergent to zero with a basis consisting of m-convex (respectively, A-convex) sets.

It is evident that if τ is a topology, these definitions coincide with the concepts of a locally m-convex algebra (respectively, an A-convex algebra as defined in [4]). Since every m-convex set is A-convex, every m-convex convergence algebra is also an A-convex convergence algebra.

DEFINITION 2. In a vector lattice L, a subset S of L is said to be *solid*, if $f \subset S$ whenever $|f| \leqq |g|$ $(f \in L)$ and $g \in S$ (e.g., see [8], p. 35). Given a convergence vector space structure τ on L with the property that for every filter θ convergent to zero, there exists a coarser filter Φ convergent to zero, where Φ has a basis of solid sets, we call (L, τ) a *convergence vector lattice*.

Clearly, a convergence vector lattice that is also a topological space is a topological vector lattice. Further, one can readily verify that in every convergence vector lattice, the lattice operations are continuous.

The algebra $C(X)$ is a lattice with respect to the order induced by the cone of nonnegative functions. Thus for $f \in C(X)$, the function $|f|$ may be characterized by $|f|(x) = |f(x)|$ for every $x \in X$. The symbols "$\vee$" and "$\wedge$" will denote the lattice operations of "sup" "inf", respectively. In addition, we will use the notation "1" to represent the function of constant value 1.

DEFINITION 3. Let $C_\tau(X)$ denote $C(X)$ together with the convergence structure τ. Now, the space $C_\tau(X)$ will be called an *m-convex* (respectively, *A-convex*) *convergence lattice* if it is both an m-convex (respectively, A-convex) convergence algebra and a convergence vector lattice with respect to the natural order defined above. If, in addition, $C_\tau(X)$ is a topological space, we will substitute the word "topological" for "convergence".

In discussing A-convex convergence lattices, the following two technical results will prove important.

LEMMA 1. *Let $C_\tau(X)$ be an A-convex convergence lattice. For every filter θ convergent to zero in $C_\tau(X)$, there exists a coarser filter Φ convergent to zero, where Φ has a basis consisting of solid, A-convex sets. If, in addition τ is a topology, members of a basis for Φ can be chosen closed, A-convex and solid.*

Proof. Let θ be a filter convergent to zero in $C_\tau(X)$. Since the Fréchet filter $\mathscr{F}$ generated by $\left(\frac{1}{n}1\right)_{n \in N}$ converges to zero, the assumption implies that there exists a filter ψ coarser than $\mathscr{F} \cap \theta$ and having a basis of solid sets. In turn, ψ is finer than a filter Ω having a basis $\mathscr{B}$ consisting of A-convex sets. For each $B \in \mathscr{B}$, we define B' to be the collection of all functions $f \in B$ with the property that if $|g| \leqq |f|$ $(g \in C(X))$ then $g \in B$. Clearly, B' is solid and since B contains a solid set S in ψ, we have $B' \supset S$ (i.e., $B' \in \psi$). Moreover, we will show that B' is an A-convex set. First, to verify that B' is absolutely convex, let f, g be in B' and k any function in $C(X)$ with $|k| \leqq |\lambda f + \beta g|$, where $(|\lambda| + |\beta|) \leqq 1$. We need only verify that $k \in B$. Now the function

$$h = (k \wedge |\lambda||f|) \vee (-|\lambda||f|)$$

has the property that $|h| \leqq |\lambda||f|$ and $|k - h| \leqq |\beta||g|$. Since f and g are elements of B', both h/λ and $(k - h)/\beta$ are in B. The absolute convexity of B implies k is in B. Now, to show that B' is A-convex, let g be any member of B'. Since Ω is contained in $\mathscr{F}$, the function 1 is absorbed by B' and hence, $|g| + 1$ is absorbed by B'. We know that B' is contained in the A-convex set B, which implies that for some $\lambda > 0$,

$$(*)(|g| + 1)\cdot B' \subset \alpha B \text{ for every } \alpha \geqq \lambda\ .$$

We wish to demonstrate that $gB' \subset \alpha B'$ for $\alpha \geqq \lambda$. Given $k \in C(X)$ with $|k| \leqq |gf|$ and f an element of B', it suffices to show that $k \in \lambda B$. Clearly, $|k|/(|g| + 1)$ is less than or equal to $|f|$, and thus $k/(|g| + 1)$ is in B'. It follows from (*) that $(|g| + 1)k/(|g| + 1)$ is in λB as desired. It is evident that $(B_1 \cap B_2)' \subset (B_1' \cap B_2')$ for B_1, $B_2 \in \mathscr{B}$, and thus $\{B' : B \in \mathscr{B}\}$ is a basis for a filter Φ convergent to zero since $\Phi \supset \Omega$. We conclude that Φ has the required properties. Finally, if $C_\tau(X)$ is a topological space, it can be easily verified that the closures of neighborhoods B' remain A-convex and solid.

Any function $f \in C(X)$ can be regarded as a continuous function from X into $\dot{R}$, the one point compactification of the reals. Thus, the function f can be extended (uniquely) to a function $\bar{f}$ from βX, the Stone-Čech compactification of X, into $\dot{R}$. If f is an element of $C^0(X)$, the collection of all bounded elements in $C(X)$, then $\bar{f}$ can, of course, be regarded as a real-valued function. We will use the concept of a support set as introduced by Nachbin in [7]. Let V be a convex subset of $C^0(X)$ (respectively, $C(X)$). A compact subset $G \subset \beta X$ with the property that f is an element of V whenever $\bar{f}$ vanishes on G and $f \in C^0(X)$ (respectively, $C(X)$), is called a *support set* for V. The following result for a convex subset of $C(X)$ is due to Nachbin. (See [7].) Its extension to $C^0(X)$ is easily obtained from Nachbin's proof.

LEMMA 2. *If V is a convex subset of $C^0(X)$ (respectively, $C(X)$) such that $V \supset \{f: |f| \leqq \delta 1\}$ for some $\delta > 0$, then $G(V)$, the intersection of all support sets for V, is again a support set for V. Further, given any support set G for V, if $f \in C^0(X)$ (respectively, $f \in C(X)$) and*

$$\|f\|_G = \sup \{|\bar{f}(x)|: x \in G\}$$

is less than $\delta/2$, then f is an element of V.

2. Topological algebras. We first prove the following proposition for an A-convex topological lattice $C_\tau(X)$ to facilitate the study of closed ideals. The symbol $C_k^0(X)$ will denote the algebra of all bounded elements in $C(X)$ endowed with the topology of convergence.

PROPOSITION 1. *If $C_\tau(X)$ is an A-convex topological lattice, then the inclusion map from $C_k^0(X)$ into $C_\tau(X)$ is continuous.*

Proof. In view of Lemma 1, we can assume that the neighborhood filter of zero in $C_\tau(X)$ has a basis, $\mathscr{B}$, consisting of solid A-convex sets. Given any element U in $\mathscr{B}$, we will show that U contains a neighborhood of zero in $C_k^0(X)$. Clearly, $U^0 = U \cap C^0(X)$ is a solid A-convex subset of $C^0(X)$. Since U is absorbing, U^0 contains $\{f \in C^0(X): |f| < \delta 1\}$ for some $\delta > 0$. Lemma 2 states that

$$U^0 \supset \left\{f \in C^0(X): \|f\|_{G(U^0)} < \frac{\delta}{2}\right\}$$

where $G(U^0)$ is the smallest support set for U^0. Thus, it is sufficient to show that $G(U^0)$ is a subset of X. Let p be an arbitrary point in $\beta X \backslash X$. Since X is realcompact, we can choose a function $f \in C(X)$ such that $\bar{f}(p) = \infty$ (for example, the function $1/|f|$ on page 119 in [5]). For convenience, we can assume f is greater than zero and an element of U, since U is absorbing (divide by an appropriate constant). Now U is also A-convex so that $fU \subset \alpha U$ for some $\alpha > 1$. Letting $k = f/\alpha$, one can verify that k^n is in U for every natural number n. We will establish that

$$G = \{x \in \beta X: \bar{k}(x) \leqq 2\}$$

is actually a support set for U^0. To this end let h be in $C^0(X)$ with $\bar{h}$ vanishing on G. There exists a positive integer m so that $|h(x)| < 2^m$ for every $x \in X$, and thus $|h| \leqq k^m$. Since U is solid, h is in both U and U^0. Hence, G is a support set for U^0 disjoint from p, and it follows from Lemma 2 that $G(U^0)$ is contained in X as desired.

The term *ideal* will always mean a proper algebra ideal. In $C(X)$ (respectively, in $C^0(X)$), an ideal is said to be *full* if it consists of all functions in $C(X)$ (respectively, in $C^0(X)$) vanishing on a nonempty subset of X. In particular, for $N \subset X$, let $I(N)$ denote the full ideal in $C(X)$ of all functions vanishing on N.

THEOREM 1. *In an A-convex topological lattice $C_\tau(X)$, every closed ideal is full.*

Proof. Let J be a closed ideal in $C_\tau(X)$. Proposition 1 implies that $J^0 = J \cap C^0(X)$ is a closed ideal in $C_k^0(X)$. Since an ideal is closed in $C_k^0(X)$ if and only if it is full (e.g., this follows from Lemma 3 and Proposition 3 in [2], and the fact that the topology of compact convergence is coarser than the continuous convergence structure), we can write $J^0 = I(N) \cap C^0(X)$, where $N \subset X$. We will show that J is actually the full ideal $I(N)$. Given any function $f \in C(X)$, we have

$$((-1 \vee f) \wedge 1) = uf \ ,$$

where u is a unit (an invertible function) in $C(X)$ (see [5], p. 21). Now if $f \in J$, then uf is an element of J^0, which implies that uf and hence f vanish on N, i.e., $J \subset I(N)$. Conversely, if $f \in I(N)$, we have $uf \in J^0$, and finally, $u^{-1}uf = f$ must be an element of J. Hence, J is the full ideal $I(N)$.

COROLLARY. *Let $C_\tau(X)$ be an A-convex topological lattice such that τ restricted to the bounded functions is finer than the topology of pointwise convergence. An ideal in $C_\tau(X)$ is closed if and only if it is full.*

Proof. In view of the previous theorem, we need only verify that $I(N)$, for $N \subset X$, is closed in $C_\tau(X)$. Assume that θ is a filter in $C(X)$ convergent to g with a basis in $I(N)$. We define $t(f)$ to be $(-1 \vee f) \wedge 1$ for $f \in C(X)$. Since the lattice operations are continuous, the image filter $t(\theta)$ converges to $g' = ((-1 \vee g) \wedge 1)$. Furthermore, by assumption $t(\theta) \cap C^0(X)$ converges pointwise to g', hence, g itself vanishes on N. Therefore, $I(N)$ is a closed ideal in $C_\tau(X)$.

A seminorm s on $C(X)$ will be called a *supremum seminorm,* if, for any $f \in C(X)$,

$$s(f) = \sup \{|f(x)| : x \in K\} \ ,$$

where K is a compact subset of X. We say that a topology τ on $C(X)$ is a *topology of k-convergence* if τ is generated by a collection of supremum seminorms, i.e., τ is a topology of uniform convergence on a collection of compact subsets of X.

Before characterizing the topologies of k-convergence, we prove the following lemma.

LEMMA 3. *Let $C_\tau(X)$ be an A-convex topological lattice. Either of the following conditions implies that the sequence $(|f| \wedge n1)_{n\in N}$ converges to $|f|$ for each $f \in C(X)$:*

(1) *$C^0(X)$ is dense in $C_\tau(X)$.*

(2) *Inversion is continuous on the set of invertible elements.*

Proof. Given that (1) is satisfied and f is any member of $C(X)$, there exists a filter θ convergent to $|f|$ with a trace on $C^0(X)$. This implies that $\theta - |f|$ is finer than a filter ψ convergent to zero with a basis $\mathscr{B}$ of solid sets. Now for any $B \in \mathscr{B}$, there exists an $A \in \theta$ so that B contains $A - |f|$. In particular, $B \supset (g - |f|)$, where g is an element of $A \cap C^0(X)$. If n_0 is a natural number greater than the supremum of $|g|$ on X, it follows that

$$|(|f| \wedge n1) - |f|| \leqq |g - |f||$$

for every $n \geqq n_0$. Since B is solid, $((|f| \wedge n1) - |f|)$ is an element of B for every $n \geqq n_0$ and we conclude that $(|f| \wedge n1)_{n\in N}$ converges to $|f|$.

Given that (2) is satisfied, we will show that $C^0(X)$ is dense in $C_\tau(X)$, i.e., (1) is satisfied. Let f be a member of $C(X)$ and U any solid neighborhood of zero in $C_\tau(X)$. Now there exists a solid neighborhood V of zero with $V + V$ contained in U. Proposition 1 implies that there is a $\delta > 0$ so that $((|f| \vee \delta 1) - |f|)$ is in V. Since inversion is continuous, we can choose a neighborhood W of zero with the property that if g is a unit and $(g^{-1} - (|f| \vee \delta 1)^{-1}) \in W$, then $(g - (|f| \vee \delta 1))$ is an element of V. It is evident that for an appropriate choice of $m > \delta$,

$$[((|f| \vee \delta 1) \wedge m1)^{-1} - (|f| \vee \delta 1)^{-1}] \in W$$

and thus $[((|f| \vee \delta 1) \wedge m1) - |f|]$ is in $V + V$. Since $(V + V) \subset U$ and

$$|((-m1 \vee f) \wedge m1) - f| \leqq |((|f| \vee \delta 1) \wedge m1) - |f||,$$

we conclude that $C^0(X)$ is dense in $C_\tau(X)$.

THEOREM 2. *Let $C_\tau(X)$ be an A-convex topological lattice. The topology τ is coarser than the topology of compact convergence if:*

(1) *$C^0(X)$ is dense in $C_\tau(X)$. Or,*

(2) *Inversion is continuous on the set of invertible elements.*

Proof. Let U be any neighborhood of zero in $C_\tau(X)$, an A-convex topological lattice. In view of Lemma 1, we may assume U is A-convex, solid and closed. Now, Lemma 2 and the fact that U is absorbing, implies that U contains $\{f \in C(X): \|f\|_{G(U)} < \delta/2\}$, where $\delta > 0$ and $G(U)$ is the smallest support set for U. To establish the result, we will show (by an argument similar to that in the proof of Proposition 1) that $G(U)$ is contained in X assuming condition (1) or (2) is satisfied. For p any point in $\beta X \backslash X$, choose a positive function f in U with $f(p) = \infty$. We can suppose $f^n \in U$ for every $n \in N$ (otherwise, pick an appropriate scalar multiple of f), and we claim

$$G = \{x \in \beta X: \bar{f}(x) \leqq 2\}$$

is a support set for U. If $h \in C(X)$ with $\bar{h}(G) = \{0\}$, then $(|h| \wedge n1) \leqq f^n$ for every $n \in N$, and hence, $(|h| \wedge n1)$ is in U. It follows from the supposition and the previous lemma that h itself is in U as desired.

Given that $C_\tau(X)$ is an m-convex topological lattice, we know that inversion is continuous (e.g., see [6], Proposition 2.8), and thus, the stipulation of (1) or (2) in the previous theorem can be omitted. Furthermore, we can now prove the following:

THEOREM 3. *$C_\tau(X)$ is an m-convex topological lattice, if and only if τ is a topology of k-convergence.*

Proof. Let $C_\tau(X)$ be an m-convex topological lattice. An m-convex closed neighborhood M of zero in $C_\tau(X)$ contains a solid closed neighborhood of zero, call it N. The proof of Lemma 1 established that the set M' consisting of all $f \in M$ with the property that $|g| \leqq |f|$ implies that $g \in M$ is an absolutely convex set containing N. We claim that M' is also m-convex. Given $|k| \leqq |f| \cdot |g|$ for $f, g \in M'$, we need only verify that k is in M. For any solid neighborhood U of zero, there exists a solid neighborhood $\tilde{U}$ of zero with $|g| \cdot \tilde{U} \subset U$. Now, Proposition 1 implies that $\tilde{U}$ contains $\{f \in C^0(X): |f| \leqq \delta 1\}$ for some $\delta > 0$. We have $|k|(|f| + \delta 1)^{-1} \leqq |g|$ since

$$|k| \leqq |f||g| \leqq (|f| + \delta 1)|g| .$$

By writing

$$\begin{aligned} k &= [k \cdot (|f| + \delta 1)^{-1}] \cdot (|f| + \delta 1) \\ &= [k \cdot (|f| + \delta 1)^{-1}] \cdot |f| + [k(|f| + \delta 1)^{-1}] \cdot \delta 1 , \end{aligned}$$

it follows that $k \in (M \cdot M + U)$ and thus $k \in (M + U)$. Since U was arbitrary and M is closed, we conclude that k is indeed in M. One may verify that the closure of M' is still m-convex and solid, and hence, we may choose as a basis for the neighborhoods of zero in $C_\tau(X)$ a collection $\mathscr{M}$ of closed, m-convex, solid sets. For $V \in \mathscr{M}$, as in the proof of Theorem 2, we have

$$V \supset \{f \in C(X): \|f\|_{G(V)} \leqq \eta\}$$

where $\eta > 0$ and $G(V) \subset X$. We will show that

$$V \subset \{f \in C(X): \|f\|_{G(V)} \leqq 3\} .$$

Assume to the contrary that there exists an $f \in V$ with $f(p) > 3$ for $p \in G(V)$. Since V is solid, we can assume f is bounded and non-negative. We will establish that

$$H = \{x \in \beta X: \bar{f}(x) \leqq 2\}$$

is a support set for V, which will contradict the fact that $G(V)$ is minimal (as $H \cap G(V)$ is a support set). If $g \in C(X)$ with $\bar{g}(H) = \{0\}$, then $(|g| \wedge n1) \leqq f^n$ for every $n \in N$. The fact that V is m-convex implies that each $(|g| \wedge n1)$ is in V, and since V is closed and solid, we conclude by an application of Lemma 3 that g is in V. Thus, we have proved that $C_\tau(X)$ carries a topology of k-convergence. The converse is immediate.

COROLLARY. *The topology of compact convergence on $C(X)$ is the finest among all topologies τ making $C_\tau(X)$ an m-convex topological lattice.*

3. Consequences for convergence spaces. Here, we will provide sufficient conditions for closed ideals in A-convex convergence lattices to be full.

In studying these structures that are not necessarily topological, we will utilize properties of the Marinescu space $C_{I'}(X)$ introduced in [3]. A set $Z \subset \beta X$ is said to be a *zero-set* if $Z = \{p \in \beta X: f(p) = 0\}$ for some function f in $C(\beta X)$. For any zero-set $Z \subset \beta X \backslash X$, the algebra $C(\beta X \backslash Z)$ can be identified with a subalgebra of $C(X)$; namely, restriction of the functions to X. The space $C_{I'}(X)$ is the inductive limit in the category of convergence spaces of the family

$$(*) \qquad \{C_k(\beta X \backslash Z): Z \subset \beta X \backslash X \text{ is a zero-set}\}$$

together with the order defined by inclusion (k again denotes the topology of compact convergence). Actually, $C_{I'}(X)$ can be regarded as $C(X)$ endowed with the finest convergence structure making all the inclusion maps from members of (*) into $C(X)$ continuous. Furthermore, it is easily verified that $C^0_{I'}(X)$, the bounded functions together with the convergence structure inherited from $C_{I'}(X)$, coincides with the inductive limit of the family of all $C^0_k(\beta X \backslash Z)$ for Z a zero-set contained in $\beta X \backslash X$.

In analogy to Proposition 1, we prove:

PROPOSITION 2. *If $C_\tau(X)$ is an A-convex convergence lattice, then the inclusion map from $C^0_{I'}(X)$ into $C_\tau(X)$ is continuous.*

Proof. Because $C^0_{I'}(X)$ is itself an inductive limit, it is sufficient to show that the inclusion map

$$i: C^0_k(\beta X \backslash Z) \longrightarrow C_\tau(X)$$

is continuous for each zero-set $Z \subset \beta X \backslash X$. Let $\mathscr{U}$ denote the image filter under i of the neighborhood filter at zero in $C^0_k(\beta X \backslash Z)$. The subset Z can be considered a zero-set of a function $\bar{f}$, where f is a positive function in $C^0(X)$. Since f is invertible, we set $g = f^{-1}$ and note that $\mathscr{F}_1$ and $\mathscr{F}_2$, the Fréchet filters generated by $\left(\frac{1}{n} 1\right)_{n \in N}$ and $\left(\frac{1}{n} g\right)_{n \in N}$, both converge to zero in $C_\tau(X)$. Thus, we can find a filter θ coarser than both $\mathscr{F}_1$ and $\mathscr{F}_2$, where θ converges to zero and has a basis $\mathscr{B}$ of solid, A-convex sets. To complete the proof, we wish to establish that $\mathscr{U}$ contains θ. Given any $B \in \mathscr{B}$, clearly, $B \cap C^0(X)$ satisfies the conditions of Lemma 2, and thus, it is sufficient to show that some support set for $B \cap C^0(X)$ is contained in $\beta X \backslash Z$. Since g is absorbed by B, proceeding as in the proof of Proposition 1, we can choose a scalar multiple of g, call it k, with the property that $k^n \in B$ for every $n \in N$. Since $\bar{g}(Z) = \{\infty\}$, it is clear that the set

$$G = \{x \in \beta X : \bar{k}(X) \leqq 2\}$$

is contained in $\beta X \backslash Z$. To establish that G is actually a support set, let h be a bounded function in $C(X)$ with $\bar{h}(G) = \{0\}$. It follows that $|h| \leqq k^m$ for some $m \in N$, and thus, h itself is in B, since B is solid.

A completely regular topological space Y will be called *z-realcompact*, if every compact subset of $\beta Y \backslash Y$ is contained in a zero-set Z, where $Z \subset \beta Y \backslash Y$.

It is evident that any z-realcompact space is realcompact, and on the other hand, every locally compact σ-compact (Hausdorff) topological space is z-realcompact (see [3], p. 445 and [5], p. 115).

For a z-realcompact space X, it can be established by modifying the proof of Theorem 2 in [3] that an ideal in $C^0_{I'}(X)$ is closed, if and only if it is full. (In modifying the proof, the set K would be replaced by a zero-set in $\beta X \backslash X$ containing K.) Now, given a closed ideal J in an A-convex convergence lattice $C_\tau(X)$, where X is z-realcompact, Proposition 2 implies that $J \cap C^0(X)$ is closed in $C^0_{I'}(X)$. Thus, $J \cap C^0(X)$ is a full ideal, and arguing as in the proof of Theorem 1, we conclude that J itself is a full ideal. To summarize, we state:

THEOREM 4. *Let X be a z-realcompact space and $C_\tau(X)$ an A-convex convergence lattice. Every closed ideal in $C_\tau(X)$ is full.*

The argument in the proof of the corollary to Theorem 1 is valid for convergence spaces, and thus, we have:

COROLLARY. *Let X be a z-realcompact topological space and $C_\tau(X)$ an A-convex convergence lattice. If τ restricted to $C^0(X)$ is finer than the topology of pointwise convergence, then an ideal in $C_\tau(X)$ is closed if and only if it is full.*

It is clear that $C_{I'}(X)$ and in fact any inductive limit (in the category of convergence spaces) of m-convex (respectively, A-convex) topological lattices are m-convex (respectively, A-convex) convergence lattices. The following proposition provides examples of m-convex (and hence, A-convex) convergence lattices which cannot be realized as inductive limits of topological vector spaces.

For a completely regular topological space Y, let $C_c(Y)$ denote the algebra $C(Y)$ endowed with the continuous convergence structure (see [1]). In [3] (Theorem 8), it is shown that $C_c(Y)$ is not in general an inductive limit of topological vector spaces, while we can prove:

PROPOSITION 3. *Let Y be a completely regular topological space. $C_c(Y)$ is an m-convex convergence lattice.*

Proof. A filter θ converges to zero in $C_c(Y)$ if and only if for every $p \in Y$ and $n \in N$, there exists a neighborhood U of p and a B in θ such that

$$B \cdot U \subset \left[\frac{-1}{n}, \frac{1}{n}\right]$$

(i.e., $|f(x)| \leqq 1/n$ for every $f \in B$ and $x \in U$). Given Φ convergent to zero in $C_c(Y)$, we associate to every point $y \in Y$ and $n \in N$ a neighborhood $U_{(y,n)}$ of y and a B in Φ so that

$$B \cdot (U_{(y,n)}) \subset \left[\frac{-1}{n}, \frac{1}{n}\right].$$

Now, we define

$$T_{(y,n)} = \left\{f \in C(Y): f(U_{(y,n)}) \subset \left[\frac{-1}{n}, \frac{1}{n}\right]\right\}.$$

It is clear that all the sets $T_{(y,n)}$ for $y \in Y$ and $n \in N$ generate a filter ψ coarser than Φ and convergent to zero in $C_c(Y)$. Furthermore, it is easy to check that finite intersections of sets $T_{(y,n)}$ are also m-convex and solid, which completes the proof.

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