PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

A PROOF OF THE FINITE GENERATION OF INVARIANTS OF A NORMAL SUBGROUP

JOHN BRENDAN SULLIVAN

A fundamental theorem in the development of the quotient theory of an affine algebraic group G shows that the coordinate functions invariant under a normal subgroup form a finitely generated algebra. We show that this theorem follows from the finite field generation of the quotient field of the algebra of invariant coordinate functions in the connected case.

Notation. Let K be an algebraically closed field and A an integral domain Hopf algebra over K. Denote by $[A]$ the field of fractions of A, and by $G(A)$ the group of K-algebra morphisms from A to K. There is the natural left action (translation) of $G(A)$ on A, denoted by $(\cdot)$.

For the sake of completeness, we include a proof of a known proposition for fields:

PROPOSITION 0. *$K \subset L \subset E$ fields. If E is finitely generated as a field over K, then L is finitely generated as a field over K.*

Proof. Let $x_1, \cdots, x_t$ be a transcendence basis for L over K. We will see that L is a finite extension of $\ell = K(x_1, \cdots, x_t)$. Let $y_1, \cdots, y_n$ be a transcendence basis for E over ℓ; since E is finitely generated over K, E is a finite extension of $\ell(y_1, \cdots, y_n)$. Since $\ell(y_1, \cdots, y_n)$ is purely transcendental over ℓ and L is algebraic over ℓ, $\ell(y_1, \cdots, y_n)$ and L are linearly disjoint over ℓ. Therefore, the dimension of E over $\ell(y_1, \cdots, y_n)$ is at least as large as the dimension of L over ℓ. So L is finite over ℓ and L is finitely generated over K.

PROPOSITION 1. *Let $A_1 \subset A$ be domain Hopf algebras. Then $[A_1] \cap A = A_1$.*

Proof. Let $f = a/b$ be an element of $[A_1] \cap A$, where $a, b \in A_1$. For the purpose of demonstrating this proposition, we may suppose that A is generated as a Hopf algebra by a, b, and f and that A_1 is generated by a and b. Let M be the K-linear span of $G(A)\cdot f$; M is a finite-dimensional $G(A)$-invariant subspace of $[A_1] \cap A$. Let $I \subset A_1$ be the ideal $\{c \in A_1 \mid cM \subset A_1\}$; since M is finite-dimensional, $I \neq (0)$. If $I = A_1$, then $f \in A_1$, as was to be shown. Otherwise, from $g \cdot M = M$ and $g \cdot A_1 = A_1$ for $g \in G(A)$, it follows that $g \cdot I = I$; so, I is a

$G(A)$-module. Moreover, since $I \subset A_1$ and $G(A)$ separates the points of A_1, we have $G(A_1) \cdot I = I$. Since $I \neq A_1$, by the Nullstellensatz there is an element x of $G(A_1)$ which vanishes on I. Therefore, $0 = x(I) = x(G(A_1) \cdot I) = (x \cdot G(A_1))(I) = G(A_1)(I)$. By the Nullstellensatz, $I = (0)$. This contradicts $I \neq (0)$.

PROPOSITION 2. *If $A_1 \subset A$ are domain Hopf algebras and $[A_1] = [A]$, then $A_1 = A$.*

Proof. $A_1 = [A_1] \cap A = A$.

THEOREM. *For $A \subset B$ domain Hopf algebras, if B is a finitely generated K-algebra, then A is a finitely generated K-algebra.*

Proof. Since B is finitely generated as a K-algebra, $[B]$ is finitely generated as a field over K. By Proposition 0, $[A]$ is finitely generated as a field over K. Let $a_1, \cdots, a_n$ be field generators for $[A]$ where $a_i \in A$. Let A_1 be the sub-Hopf algebra of A generated by $a_1, \cdots, a_n$; so, $[A_1] = [A]$. By Proposition 2, $A_1 = A$. Therefore, A is finitely generated as a K-algebra.

For $G(B)$ a connected affine algebraic group with domain Hopf algebra B of coordinate functions and H a normal subgroup, the subalgebra B^H of B of H-invariant elements is a sub-Hopf algebra; since B is finitely generated, so is B^H by the theorem.

From this point, the extension to the nonconnected case is given in [1, p. 38].

REFERENCE

1. G. Hochschild, *Introduction to Affine Algebraic Groups*, Holden-Day, San Francisco, 1971.

Received February 2, 1973.

UNIVERSITY OF CALIFORNIA, BERKELEY