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NORMED KÖTHE SPACES AS INTERMEDIATE SPACES OF L_1 AND L_∞

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Let (Δ, Σ, μ) be a totally σ-finite measure space and let $M(\Delta)$ be the set of all complex-valued μ-measurable functions on Δ. This paper is concerned with determining whether certain classes of normed Köthe spaces (Banach function spaces) are intermediate spaces of $L_1 = L_1(\mu)$ and $L_\infty = L_\infty(\mu)$. It is proven that $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are associate Orlicz spaces and that for every nontrivial Young's function Φ there is an equivalent Young's function Φ_1 such that the Orlicz space $L_{M\Phi_1}$ is an intermediate space of L_1 and L_∞. The notion of a universal Köthe space is presented and it is proven that if Λ is a universal Köthe space then $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$. Furthermore, if Λ is normed, in particular $\Lambda = L_\rho$, then there is an equivalent universally rearrangement invariant norm ρ_1 for which L_{ρ_1} is an intermediate space of L_1 and L_∞.

1. Introduction. Let X_1 and X_2 be two Banach spaces contained in a linear Hausdorff space Y such that the injection of $X_i (i = 1, 2)$ into Y is continuous. Denote the norm of X_i by $\|\cdot\|_i$. The space $X_1 \cap X_2$ is the set of all elements which are in both X_1 and X_2, and the space $X_1 + X_2$ is the set of all $f \in Y$ of the form $f = f_1 + f_2$ with $f_1 \in X_1$ and $f_2 \in X_2$. The spaces $X_1 \cap X_2$ and $X_1 + X_2$ are Banach spaces under the norms $\|f\|_{X_1 \cap X_2} = \max\{\|f\|_1, \|f\|_2\}$ and $\|f\|_{X_1+X_2} = \inf\{\|f_1\|_1 + \|f_2\|_2 : f = f_1 + f_2, f_i \in X_i\}$ (see [1, p. 165, Prop. 3.2.1]). A Banach space $X \subset Y$ satisfying $X_1 \cap X_2 \subset X \subset X_1 + X_2$ and $\|f\|_{X_1+X_2} \leqq \|f\|_X \leqq \|f\|_{X_1 \cap X_2}$ is called an *intermediate space* of X_1 and X_2.

Much work has been done on intermediate spaces and the related topic of interpolation theory. (See [1], [2], [12].) In particular, it has been shown that the Lebesgue spaces L_p and the Lorentz spaces L_{pq} ([6] and [7]) are intermediate spaces of L_1 and L_∞. In this paper we investigate what other classes of normed Köthe spaces are intermediate spaces of L_1 and L_∞. In §7 we introduce the notion of a universal Köthe space, which we prove to be equivalent to Luxemburg's notion of a universally rearrangement invariant Köthe space [9]. We have been able to show that if Λ is a universal Köthe space, then $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$. Furthermore, if Λ is normed, in particular $\Lambda = L_\rho$, then there is an equivalent norm ρ_1 which is universally rearrangement invariant and L_{ρ_1} is an intermediate space of L_1 and L_∞.

Section 2 contains preliminaries and §3 deals with Orlicz spaces. We show that $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are Orlicz spaces and prove that they are associate Orlicz spaces. It is shown that for any nontrivial

Young's function Π, there is an equivalent Young's function Π_1 such that $L_{M\Pi_1}$ is an intermediate space of L_1 and L_∞. This means that $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are the smallest and the largest Orlicz spaces, respectively. Section 4 deals with the monotonic rearrangement of a measurable function. Sections 5 and 6 deal with universal and universally rearrangement invariant function norms.

2. **Preliminaries.** Let (Δ, Σ, μ) be a σ-finite measure space where Δ is a point set, Σ is a σ-algebra of measurable sets, and μ is a totally σ-finite measure. Let M^+ be the set of all nonnegative μ-measurable functions on Δ. We allow that a function can assume the value $+\infty$ at some or all points $x \in \Delta$.

A mapping ρ on M^+ to the extended reals is called a *function norm* if ρ satisfies the following conditions for all f and g in M^+:

(i) $\rho(f) \geqq 0$ and $\rho(f) = 0$ if and only if $f = 0$ a.e. (almost everywhere).

(ii) $\rho(af) = a\rho(f)$ for $a \geqq 0$.

(iii) $\rho(f + g) \leqq \rho(f) + \rho(g)$.

(iv) $f(x) \leqq g(x)$ a.e. implies $\rho(f) \leqq \rho(g)$.

In addition, we assume that ρ satisfies:

(v) (*Fatou property*) $f_0, f_1, \cdots \in M^+$ and $f_n \uparrow f_0$ (pointwise a.e.) implies $\rho(f_n) \uparrow \rho(f_0)$.

(vi) (Saturated) there are no sets $E \in \Sigma$ such that $\rho(\chi_B) = \infty$ for every measurable $B \subset E$ with $\mu(B) > 0$ (χ_B is the characteristic function for the set B).

The domain of definition of ρ is extended to $M = M(\Delta, \mu)$, the set of all complex-valued, μ-measurable functions on Δ, by defining $\rho(f) = \rho(|f|)$ for $f \in M$. We denote by $L_\rho = L_\rho(\Delta, \Sigma, \mu)$ the set of all $f \in M$ satisfying $\rho(f) < \infty$. If we assume μ-almost equal functions are identified in the usual way, the spaces L_ρ are complete normed linear spaces. Such spaces are commonly called *normed Köthe spaces* or *Banach function spaces.* (For theory of normed Köthe spaces see [10].) Examples of normed Köthe spaces are Orlicz spaces, the spaces of Ellis and Halperin [3], and the Lorentz spaces [6, 7].

The *associate norm* ρ' of any function norm ρ is defined by

$$\rho'(f) = \sup\left\{\int_\Delta |fg|d\mu \colon \rho(g) \leqq 1\right\} .$$

The *associate space,* denoted $(L_\rho)'$ or $L_{\rho'}$, is defined to be $L_{\rho'} = \{f \in M \colon \rho'(f) < \infty\}$. The associate norm ρ' has the Fatou property (even if ρ did not) and hence is a normed Köthe space. (For the details see [10].)

Let (Δ, Σ, μ) be as outlined earlier, and let Δ_n be a fixed increasing sequence of sets of finite measure whose union is Δ. Let $\Omega =$

$\left\{f: \int |f\chi_{\Delta_n}| d\mu < \infty \text{ for all } n\right\}$ be the space of locally integrable function on Δ. For any subset $\Gamma \subset \Omega$ we define the *Köthe space* $\Lambda(\Gamma)$ associated with Γ to be $\Lambda = \Lambda(\Gamma) = \left\{f \in \Omega: \int_\Delta |fg| d\mu < \infty \text{ for all } g \in \Gamma\right\}$. The *associate Köthe space* Λ' is defined to be $\Lambda' = \Lambda(\Lambda(\Gamma)) = \left\{g \in \Omega: \int_\Delta |gf| d\mu < \infty \text{ for all } f \in \Lambda(\Gamma)\right\}$. Notice that our normed Köthe space L_ρ is also a Köthe space (since ρ is assumed to saturated).

Endow the space $M(\Delta, \mu)$ with the topology of convergence in measure on sets of finite measure. Then M becomes a linear Hausdorff space and the injection of L_ρ into M is continuous. Thus we have established the framework necessary to consider L_ρ as an intermediate space of L_1 and L_∞.

Let $\mu(\Delta) < \infty$. Then $L_\infty = L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty = L_1$ if and only if $\rho(\chi_\Delta) < \infty$ and $\rho'(\chi_\Delta) < \infty$. Furthermore, there is an equivalent norm which makes this embedding norm-reducing (Theorem 6.4). For this reason, *we will proceed under the assumption that* $\mu(\Delta) = \infty$.

Finally, we given a representation of the $L_1 + L_\infty$ norm which we will denote by $\|\cdot\|_+$.

THEOREM 2.1. *Let $f \in L_1 + L_\infty$ and let $s = \sup\{t: \mu\{|f| \geqq t\} \geqq 1\}$. Then*

$$\|f\|_+ = s + \int_{\{|f|>s\}} (|f| - s) d\mu .$$

A proof can be derived from Butzer and Berens [1, pp. 185–186].

3. Orlicz spaces as intermediate spaces.

For basic Orlicz space theory, the reader is referred to [5], [8], or [15].

Let $\Phi: [0, \infty) \to [0, \infty)$ and $\Psi: [0, \infty) \to [0, \infty)$ be complementary Young's functions. Hence Φ and Ψ are increasing, absolutely continuous on the sets where they are finite, and convex. Let

$$\|f\|_{M\Phi} = \inf\left\{k > 0: \int_\Delta \Phi(|f|/k) d\mu \leqq 1\right\} .$$

The *Orlicz space* $L_{M\Phi}$ is the set of all complex-valued, μ-measurable functions satisfying $\|f\|_{M\Phi} < \infty$. Hence the Orlicz space $L_{M\Phi}$ is a normed Köthe space and, as such, it satisfies the properties stated in §2. In particular we can form the associate norm, denoted $\|\cdot\|_\Psi$,

$$\|f\|_\Psi = \sup\left\{\int_\Delta |fg| d\mu: \|g\|_{M\Phi} \leqq 1\right\} ,$$

and the associate space $L_\Psi = \{g: \|g\|_\Psi < \infty\}$.

We will denote the $L_1 \cap L_\infty$ norm by $\|\cdot\|_\cap$.

THEOREM 3.1. (a) *If Π is a (nontrivial) Young's function, then $L_1 \cap L_\infty \subset L_{M\Pi}$.* (b) *$L_1 \cap L_\infty$ is an Orlicz space. In particular there is a Young's function Ψ such that $\|f\|_\cap = \|f\|_{M\Psi}$ for all $f \in M$.*

Proof. Consider the Orlicz space given by $\Psi(u) = u$ for $0 \leqq u \leqq 1$ and $\Psi(u) = \infty$ for $1 < u$.

From Theorem 3.1 we see that $L_1 \cap L_\infty$ is the smallest Orlicz space.

Let Ψ be as defined in the proof of Theorem 3.1. Let Φ be the complementary Young's function of Ψ. One can check that $\Phi(u) = 0$ for $0 \leqq u \leqq 1$ and $\Phi(u) = u - 1$ for $1 \leqq u$.

LEMMA 3.2. *$L_{M\Phi}$, $(L_1 \cap L_\infty)'$, and $L_1 + L_\infty$ all consist of the same functions.*

It is not true that $\|\cdot\|_+ = \|\cdot\|_{M\Phi}$. For example let (Δ, Σ, μ) be $[0, \infty)$ with Lebesgue measure and let $f = 10\chi_{(0,1/2]} + 5\chi_{[1,3]}$. Then $\|f\|_{M\Phi} \leqq 5$ but $\|f\|_+ = 15/2$. However, the following is true.

THEOREM 3.3. (a) *For any $f \in L_1 + L_\infty$, we have $\|f\|_\Phi = \|f\|_+$.* (b) *$L_1 + L_\infty$ is an Orlicz space; in particular $(L_1 + L_\infty, \|\cdot\|_+) = (L_\Phi, \|\cdot\|_\Phi)$.*

Proof. Let $f \in L_1 + L_\infty$ and $g \in L_{M\Psi} = L_1 \cap L_\infty$. Then by Theorem 2.1 we get $\int |f|(g/\|g\|_\cap)d\mu \leqq \|f\|_+$. Hence

$$\|f\|_\Phi = \sup\left\{\int |f(g/\|g\|_\cap)|\,d\mu \colon g \in L_{M\Psi}\right\} \leqq \|f\|_+ .$$

To show the reverse inequality let $f \in L_1$ with $f \geqq 0$ and $s = \sup\{t \colon \mu\{f \geqq t\} \geqq 1\}$. Furthermore assume that f is a simple function (i.e., f is a linear combination of characteristic functions of sets of finite measure). Because f is simple, one can show that $\mu\{f > s\} \leqq 1$, $\mu\{f \geqq s\} \geqq 1$, and $\mu\{f = s\} \neq 0$. Now define $\alpha\colon \Delta \to [0, \infty)$ by $\alpha(x) = 1$ if $x \in \{f > s\}$, $\alpha(x) = (1 - \mu\{f > s\})/\mu\{f = s\}$ if $x \in \{f = s\}$ and $\alpha(x) = 0$ otherwise. Then $\|\alpha\|_\cap = 1$ and

$$\int |f\alpha|\,d\mu = s + \int_{\{f>s\}} (f - s)d\mu = \|f\|_+ .$$

Therefore, $\|f\|_+ = \int |f\alpha|\,d\mu \leqq \|f\|_\Phi$ by Hölders inequality [8, p. 7] and we have shown the equality for any simple function. Since both $\|\cdot\|_+$ and $\|\cdot\|_\Phi$ have the Fatou property, it is an easy matter to extend the result to an arbitrary $f \in L_1 + L_\infty$.

Combining Theorem 3.1 and Theorem 3.3, we can say $L_{M\Pi} \subset (L_1 \cap L_\infty)' = L_1 + L_\infty$ for any Young's function Π. Hence $L_1 + L_\infty$ is the largest Orlicz space and we have

$$L_1 \cap L_\infty \subset L_{M\Pi} \subset L_1 + L_\infty .$$

An element $B \in \Sigma$ is called an *atom* if $A \in \Sigma$ and $A \subset B$ implies $\mu(A) = 0$ or $\mu(A) = \mu(B)$. If we restrict ourselves to the case that (Δ, Σ, μ) is nonatomic (i.e., has no atoms), then G. G. Gould [4] and Luxemburg and Zaanen [11] have obtained some results similar to ours. If μ has no atoms, then define the function norm $\|\cdot\|_G$ as

$$\|f\|_G = \sup\left\{\int_E |f|\, d\mu : \mu(E) = 1\right\} .$$

It was shown by Luxemburg and Zaanen and by Gould that for $f \in L_1 + L_\infty$, $\|f\|_G = \|f\|_+$. This is also mentioned by Butzer and Berens [1, p. 183]. Luxemburg and Zaanen have shown that the associate space of $(L_1 + L_\infty, \|\cdot\|_G)$ is the space $(L_1 \cap L_\infty, \|\cdot\|_\cap)$. One might hope that for each $f \in L_1 + L_\infty$ there exists a set E_f such that $\mu(E_f) = 1$ and $\|f\|_+ = \|f\|_G = \int_{E_f} |f|\, d\mu$. This is true for simple function, but it is not true for general functions as is shown by the following example.

Let (Δ, Σ, μ) be $[0, \infty)$ with Lebesgue measure and let $f(t) = (1 - 1/t)\chi_{[1,\infty)}$. Using Theorem 2.1 $\|f\|_G = \|f\|_+ = 1$. For any $E \subset [0, \infty)$ such that $\mu(E) = 1$ it follows that $\int_E |f|\, dt < 1 = \|f\|_+$.

Let us return to the question of whether all Orlicz spaces are intermediate spaces of L_1 and L_∞. It is easy to see that there are many spaces whose embeddings are not norm-reducing (e.g. $L_{M2\Psi}$, where $L_{M\Psi} = L_1 \cap L_\infty$). But we prove the following.

THEOREM 3.4. *Every Orlicz space $L_{M\Pi}$ has an equivalent Orlicz norm $\|\cdot\|_{M\Pi_1}$ for which it becomes an intermediate space of L_1 and L_∞.*

Proof. Let Ψ and Φ denote the Young's functions for $L_1 \cap L_\infty$ and $L_1 + L_\infty$, respectively. Let Π be a nontrivial Young's function. It may happen that there exists $u_0 (u < u_0 < \infty)$ such that $\Pi(u) = 0$ for $u \leqq u_0$ and $\Pi(u) = \infty$ for $u > u_0$. In this case $L_{M\Pi} = L_\infty$ as sets, so $\|\cdot\|_{M\Pi}$ is equivalent with the L_∞ norm. In all other cases, there is a $u_0 > 0$ such that $0 < \Pi(u_0) < \infty$. Now define Π_2 and Π_1 by $\Pi_2(u) = \Pi(u_0 u)/\Pi(u_0)$ for $u \geqq 0$ and $\Pi_1(u) = \Pi_2(u)$ for $0 \leqq u \leqq 1$ and $\Pi_1(u) = 2\Pi_2(u) - 1$ for $1 \leqq u$. Notice that Π_2 is continuous, convex, $\Pi_2(u) \geqq 0$ for all u, $\Pi_2(0) = 0$, and $\Pi_2(1) = 1$. This means that Π_1 is continuous, convex, $\Pi_1(u) \geqq 0$ for all u, $\Pi_1(0) = 0$ all and $\Pi_1(1) = 1$.

Thus Π_1 is a Young's function [8, p. 38, Remark (1)].

Because Π_2 is convex and $\Pi_2(1) = 1$, we have $\Pi_2(u) \geqq u$ for $u \geqq 1$; so $\Pi_1(u) \geqq 2u - 1$ for $u \geqq 1$. Therefore, $2\Phi(u) = 2u - 2 \leqq \Pi_1(u) \leqq \infty = \Psi(u)$ for $u \geqq 1$. Now for $0 \leqq u \leqq 1$, we have

$$2\Phi(u) = 0 \leqq \Pi_1(u) = \Pi(uu_0)/\Pi(u_0) \\ \leqq \frac{u\Pi(u_0)}{\Pi(u_0)} = u = \Psi(u) .$$

Hence for all $u \geqq 0$, $2\Phi(u) \leqq \Pi_1(u) \leqq \Psi(u)$. This means that

$$||f||_+ = ||f||_\Phi \leqq 2\,||f||_{M\Phi} \leqq ||f||_{M\Pi_1} \leqq ||f||_{M\Psi} = ||f||_\cap .$$

Next we will show that $L_{M\Pi}$ and $L_{M\Pi_1}$ consist of the same functions which means that $||\cdot||_{M\Pi}$ and $||\cdot||_{M\Pi_1}$ are equivalent. First notice that $\Pi_2(u) \leqq \Pi_1(u) \leqq 2\Pi_2(u)$ for all $u \geqq 0$. From which it follows that $\int \Pi(|f|/k)d\mu < \infty$ if and only if $\int \Pi_1(|f|/k)d\mu < \infty$. Therefore, $f \in L_{M\Pi}$ if and only if $f \in L_{M\Pi_1}$.

What about the space L_Π? Let Ω be the complementary Young's function for Π. Let Ω_1 be given by Theorem 3.4. Then the associate norm of $||\cdot||_{M\Omega_1}$ denoted by $||\cdot||_{\Pi_2}$ will make L_Π an intermediate space of L_1 and L_∞.

4. Monotonic rearrangement.

Let $f \in M(\Delta, \mu)$, then the *monotonic rearrangement* of f is the function f^*: $[0, \infty) \to [0, \infty]$ defined by

$$f^*(t) = \inf \{y \geqq 0: \mu\{|f(x)| > y\} \leqq t\} .$$

Let f and g belong to $M(\Delta, \mu)$. Then f and g are called *equimeasurable* whenever $\mu\{|f(x)| > r\} = \mu\{|g(x)| > r\}$ for all $r \geqq 0$. If f and g are equimeasurable we write $f \sim g$. Notice that $f \sim g$ if and only if $f^* = g^*$. Since $\mu\{|f(x)| > r\} = m\{f^*(t) > r\}$ for all r, we will say that f and f^* are equimeasurable even though they are defined on different measure spaces. Hence f^* is the unique, nonnegative, monotonic nonincreasing, right-continuous function on $[0, \infty)$ which is equimeasurable with f. For properties of the montonic rearrangement refer to [9] and [14].

The following lemma, whose proof is straightforward, has several important consequences.

LEMMA 4.1. *Let Π be any Young's function and let f be μ-measurable. Then* $\int_\Delta \Pi(|f|)d\mu = \int_0^\infty \Pi(f^*)dt$.

COROLLARY 4.2. *Let Π be a Young's function and let f and g belong to $M(\mu)$.*

(i) $\|f\|_{M\Pi} = \|f^*\|_{M\Pi}$.
(ii) *If* $f \sim g$, *then* $\|f\|_{M\Pi} = \|g\|_{M\Pi}$.
(iii) *If* $f \in L_1 \cap L_\infty$ *and* $g \sim f$, *then* $g \in L_1 \cap L_\infty$.
(iv) $\|f\|_+ = \|f^*\|_{L_1([0,\infty)) \cap L_\infty([0,\infty))}$.

Now we are able to quickly prove a result which is stated by Butzer and Berens [1, p. 184, Prop. 3.3.7].

THEOREM 4.3. *Let* $f \in M(\mu)$, *then* $\|f\|_+ = \int_0^1 f^*(t)dt$.

Proof. From Corollary 4.2, we know that $\|f\|_+ = \|f^*\|_+$. So we will show that $\|f^*\|_+ = \int_0^1 f^*(t)dt$. Since f^* is a monotonic decreasing function, we know that $\{f^* > s_{f^*}\} \subset [0, 1) \subset \{f^* \geqq s_{f^*}\}$. So by Theorem 2.1

$$\|f\|_+ = s_{f^*} + \int_0^1 f^* dt - \int_0^1 s_{f^*} dt = \int_0^1 f^* dt .$$

This representation of $\|\cdot\|_+$ allows us to make the following statement about general Köthe spaces.

COROLLARY 4.4. *Let* Λ *be a Köthe space and let* Λ^* *be the set of all monotonic rearrangements of functions in* Λ *and let* Λ' *be the Köthe dual of* Λ. *Then the following are equivalent*:
(i) $L_1(\mu) \cap L_\infty(\mu) \subset \Lambda \subset L_1(\mu) + L_\infty(\mu)$.
(ii) $(\Lambda^* \cup \Lambda'^*) \subset L_1(m) + L_\infty(m)$.
(iii) $\int_0^1 f^*(t)dt < \infty$ *for all* $f \in (\Lambda \cup \Lambda')$.
(iv) $\int_0^r f^*(t)dt < \infty$ *for all* $f \in (\Lambda \cup \Lambda')$ *for any* $r > 0$.

5. Rearrangement invariant Köthe spaces.

DEFINITION 5.1. A Köthe space Λ is called *rearrangement invariant* if $f \in \Lambda$ and g equimeasurable with f implies $g \in \Lambda$.

(ii) A function norm ρ is called *rearrangement invariant* if $f \in L_\rho$ and g equimeasurable with f implies $\rho(f) = \rho(g)$.

Notice that if ρ is a rearrangement invariant function norm, then L_ρ is a rearrangement invariant Köthe space. However, a normed Köthe space may be rearrangement invariant but not norm rearrangement invariant. Most of the well-known examples of normed Köthe spaces are rearrangement invariant. Included are the L_p spaces $(1 \leqq p \leqq \infty)$, Orlicz spaces and Lorentz spaces L_{pq}. Furthermore, given any Young's function Π and any $f \in M(\mu)$ we have that $\|f\|_{M\Pi} = \|f^*\|_{M\Pi}$ (Corollary 4.2).

DEFINITION 5.2. A function norm λ defined on $M([0, \infty), m)$ is called *universal* if for each totally σ-finite measure space (Δ, Σ, μ) the functional ρ defined on $M(\Delta, \mu)$ by $\rho(f) = \lambda(f^*)$ is a function norm. In this case we say that *ρ is induced by λ*.

Not every function norm on $M([0, \infty), m)$ is universal. Consider λ defined on $M([0, \infty), m)$ by $\lambda(f) = \|f\chi_{[0,1)}\|_1 + \|f\chi_{[1,\infty)}\|_\infty$. Let (S, ν) be a totally σ-finite measure space with sets A, B, and C such that $\nu(A) = 1/4$, $\nu(B) = 1/2$, and $\nu(C) = 3/4$. Let $f = 5\chi_B + 3\chi_A$ and $g = 4\chi_C$. Then $\rho(f) + \rho(g) = 25/4 < 17/2 = \rho(f + g)$ which means ρ is not a function norm. Therefore, λ is not universal.

Next we state a theorem that was proven by Silverman [14] and that has proven very useful for us.

LEMMA 5.3. (Silverman). *If (Δ, μ) has no atoms and if $f, g \in M(\mu)$, then $\int_0^\infty f^*g^*dt = \infty$ if and only if $\int_\Delta |f'g|\,d\mu = \infty$ for some $f' \sim f$.*

The theory of rearrangement invariant function norms has received some attention, most notably from Luxemburg [9]. However, each time the setting has been somewhat more restrictive than ours. Hence several cases of Lemma 5.4 and Theorem 5.5 are known. See [9] and [13].

LEMMA 5.4. *If (Δ, Σ, μ) is nonatomic, then for any $f, g \in M(\mu)$ we have $\int_0^\infty f^*g^*\,dt = \sup\left\{\int_\Delta |fg'|\,d\mu \colon g' \sim g\right\}$.*

Proof. Because of Lemma 5.3 we can assume that $\int_0^\infty f^*g^*\,dt < \infty$. Further, without loss of generality we may assume that $f, g \in M^+(\mu)$. Let $\varphi = \sum_{i=1}^{m+1} a_i\chi_{A_i}$ be a simple function in $M^+(\mu)$ where $a_1 > a_2 > \cdots > a_m > a_{m+1} = 0$ and $A_{m+1} = \Delta\backslash(\bigcup_{i=1}^m A_i)$. Let $g \in M^+(\mu)$ be arbitrary. Then $g^* \in M^+([0, \infty))$, so for each pair of integers $\langle n, k\rangle$ such that $0 \leqq k \leqq 2^{2n}$ let

$$E_{n,k} = \{t \in [0, \infty)\colon k/2^n < g^*(t) \leqq (k + 1)/2^n\}$$

and

$$E_{n,2^{2n}+1} = [0, \infty)\Big\backslash\left(\bigcup_{k=0}^{2^{2n}} E_{n,k}\right).$$

Set

$$\psi_n = \sum_{k=0}^{2^{2n}} (k/2^n)\chi_{E_{n,k}}.$$

Then $\{\psi_n\}_{n=1}^\infty$ is as a sequence of simple functions such that $\psi_n^* \uparrow g^*$. Notice that for a fixed n_0 the sets $\{E_{n_0,k}\}_{k=0}^{2^{2n_0}}$ are disjoint sets and each $E_{n_0,k}$ is the disjoint union of a finite number of sets $\{E_{n_0+1,j}\}_{j\in F_{n_0,k}}$. Hence, since (Δ, μ) has no atoms, by induction we can define the sets $\tilde{E}_{n,k}$ in Δ such that

(1) $\tilde{E}_{n_0,k_1} \cap \tilde{E}_{n_0,k_2}$ is empty for $k_1 \neq k_2$.

(2) $\mu(\tilde{E}_{n,k}) = m(E_{n,k})$.

(3) $\mu(A_i \cap \tilde{E}_{n,k}) = m(A_i^* \cap E_{n,k})$.

(4) $\mu(\tilde{E}_{n_1,k_1} \cap \tilde{E}_{n_2,k_2}) = m(E_{n_1,k_1} \cap E_{n_2,k_2})$.

Next we define the simple functions $\tilde{\psi}_n: \Delta \to [0, \infty)$ by

$$\tilde{\psi}_n = \sum_{k=0}^{2^{2n}} (k/2^n)\chi_{\tilde{E}_{n,k}} .$$

Because of the properties of the sets $\{\tilde{E}_{n,k}\}$, one can show that ψ_n and $\tilde{\psi}_n$ are equimeasurable for all n and that $\{\tilde{\psi}_n(x)\}_{n=1}^\infty$ is an increasing sequence for each $x \in \Delta$. Also $\int_\Delta \varphi\tilde{\psi}_n d\mu = \int_0^\infty \varphi^*\psi_n dt$ since $\mu(A_i \cap \tilde{E}_{n,k}) = m(A_i^* \cap E_{n,k})$. Let $\tilde{g}(x) = \lim_{n\to\infty} \tilde{\psi}_n(x)$. Then $\tilde{g}^* = \lim_n \tilde{\psi}_n^* = \lim_n \psi_n^* = g^*$, so $\tilde{g}$ and g are equimeasurable and $\int_\Delta \varphi\tilde{g}d\mu = \int_0^\infty \varphi^* g^* dt$.

Hence the equation is true for arbitrary g and simple functions φ. The extension to arbitrary functions follows easily.

The next result was also stated by Luxemburg [9]. A proof follows from Lemma 5.4.

THEOREM 5.5. *Let (Δ, μ) be a nonatomic measure space and let ρ be a function norm defined on $M(\mu)$.*

(i) *If ρ is rearrangement invariant, then ρ' is rearrangement invariant.*

(ii) *ρ is rearrangement invariant if and only if*

$$\rho(f) = \sup\left\{\int_0^\infty f^* g^* dt : \rho'(g) \leqq 1\right\} .$$

A partition $P = \{E_j\}_{j=1}^n$ in Δ is defined to a finite disjoint collection of sets of positive measure. Define the average function of $f \in M(\mu)$ with respect to P to be

$$f_P = \sum_{j=1}^n \left(\int_{E_j} f d\mu / \mu(E_j)\right)\chi_{E_j} .$$

A function norm ρ defined on $M(\mu)$ is said to satisfy *Property* (J) if for each partition P and any $f \in L_\rho$, we have $\rho(f_P) \leqq \rho(f)$. This is similar to the levelling length property introduced by Ellis and Halperin [3].

Let R be the set of all nonnegative, monotonic nonincreasing, right-continuous functions defined on $[0, \infty)$. Then the monotonic rearrangement of any measurable function belonging to $M(\mu)$ is contained in R. Also $g^* = g$ for any $g \in R$.

The next result is stated in terms of the levelling length property by Luxemburg ([9, p. 132]).

THEOREM 5.6. *Let (Δ, μ) be non-atomic and let ρ be a rearrangement invariant function norm on $M(\mu)$. Then ρ has property (J).*

Proof. Let $f \in M^+(\mu)$ and let $P = \{E_j\}_{j=1}^n$ be a partition in Δ. Let $b_j = \left(\int_{E_j} f d\mu / \mu(E_j)\right)$. Renumber the E_j, if necessary, so that $b_1 \geqq b_2 \geqq \cdots \geqq b_n$. Set $E_{n+1} = \Delta \backslash \bigcup_{j=1}^n E_j$ and $b_{n+1} = 0$; hence

$$f_P^* = \sum_{j=1}^{n+1} b_j \chi E_j^*$$

where

$$E_j^* = [y_{j-1}, y_j) = \left[\sum_{l=1}^{j-1} \mu(E_l), \sum_{l=1}^{j} \mu(E_l)\right)$$

with the understanding that $y_0 = 0$ and $y_{n+1} = \infty$.

Define the function h: $[0, \infty) \to [0, \infty)$ by

$$h(t) = \sum_{j=1}^{n} (f\chi_{E_j})^*(t - y_{j-1})\chi_{E_j^*}(t) .$$

The collection $P' = \{E_j^*\}_{j=1}^n$ is a partition in $[0, \infty)$, and

$$h_{P'} = \sum_{j=1}^{n} \frac{\int_0^{\mu(E_j)} (f\chi_{E_j})^*(t)dt}{m(E_j^*)} \chi_{E_j^*} = \sum_{j=1}^{n} \frac{\int_{E_j} f d\mu}{\mu(E_j)} \chi_{E_j^*} = f_P^* .$$

For each x such that $y_{j-1} \leqq x \leqq y_j$ we know that

$$(1) \qquad \int_{y_{j-1}}^{x} h(t)dt \geqq \int_{y_{j-1}}^{x} h_{P'}(t)dt = \int_{y_{j-1}}^{x} f_P^*(t)dt$$

since h is nondecreasing on E_j^*. Let $\varphi = \sum_{i=1}^{m+1} a_i \chi_{A_i} (a_1 > a_2 > \cdots > a_m > a_{m+1} = 0, A_{m+1} = [0, \infty) \backslash \bigcup_{i=1}^m A_i)$ be a simple function in R (the set of monotonic rearrangements). Then by Hardy's theorem (Luxemburg [9, p. 34]) we have

$$\int_{E_j^*} h\varphi dt = \int_{E_j^*} f_P^* \varphi dt .$$

For $1 \leqq j \leqq n+1$, set $\varphi_j = \varphi \chi_{E_j^*}$. Since h and φ are nonincreasing on E_j^* we know that $(h\chi_{E_j^*})^*(t) = h(t + y_{j-1})$ and $\varphi_j^*(t) = \varphi(t + y_{j-1})$. Hence

$$\int_0^\infty (f\chi_{E_j})^*\varphi_j^* dt = \int_0^\infty (h\chi_{E_j^*})^*\varphi_j^* dt = \int_{E_j^*} h\varphi dt .$$

Because (Δ, μ) is nonatomic, for each $j = 1, 2, \cdots, n+1$ we can define a function $\tilde{\varphi}_j: E_j \to [0, \infty)$ which is equimeasurable with φ_j. Since φ_j is simple, we have seen in the proof of Lemma 5.4 that there exist functions $\tilde{f}_j: E_j \to [0, \infty) (1 \leqq j \leqq n+1)$ such that $\tilde{f}_j$ is equimeasurable with $f\chi_{E_j}$ and $\int_{E_j} \tilde{f}_j \tilde{\varphi}_j d\mu = \int_0^\infty (f\chi_{E_j})^*(\varphi_j)^* dt$. Let

$$\tilde{\varphi} = \sum_{j=1}^{n+1} \tilde{\varphi}_j \chi_{E_j} \quad \text{and} \quad f_1 = \sum_{j=1}^{n+1} \tilde{f}_j \chi_{E_j} .$$

Then f_1 is equimeasurable with f and

$$\int_\Delta f_1 \tilde{\varphi} d\mu \geqq \sum_{j=1}^{n} \int_0^\infty (f\chi_{E_j})^*\varphi_j^* dt \geqq \sum_{j=1}^{n} \int_{E_j^*} f_P^* \varphi dt = \int_0^\infty f_P^* \varphi dt .$$

Hence

$$\int_0^\infty f^*\varphi dt = \sup \left\{ \int_\Delta |f_1 \varphi'| d\mu : \varphi' \sim \varphi \right\} \geqq \int_\Delta f_1 \tilde{\varphi} d\mu \geqq \int_0^\infty f_P^* \varphi dt .$$

Now let $g \in R$ be arbitrary, then there exists a sequence of simple functions φ_k such that $\varphi_k \uparrow g$ a.e. on $[0, \infty)$. Then φ_k can be chosen to lie in R for each k. Since ρ is rearrangement invariant

$$\rho(f_P) = \sup \left\{ \lim \int_0^\infty f_P^* \varphi_n dt : \varphi_n \uparrow g \quad \text{and} \quad \rho'(g) \leqq 1 \right\}$$

$$\leqq \sup \left\{ \lim \int_0^\infty f^* \varphi_n dt : \varphi_n \uparrow g \quad \text{and} \quad \rho'(g) \leqq 1 \right\} = \rho(f) .$$

Therefore ρ has property (J).

We will give an example at the end of this section to show that a universal function norm does not necessarily have property (J).

Let Γ be any nontrivial subset of R. Define the functional $F = F_\Gamma$ on $M(\Delta, \mu)$ by $F(f) = \sup \left\{ \int_0^\infty f^* h dt : h \in \Gamma \right\}$. Then F is a function norm with the Fatou property.

THEOREM 5.7. (a) *If λ is a rearrangement function norm on $M([0, \infty))$, then λ is universal.*

(b) *Let ρ be a function norm defined on $M(\Delta, \mu)$ which is induced by a universal function norm λ. Then for each $f \in M(\Delta, \mu)$ we have $\rho'(f) = \sup \left\{ \int_0^\infty f^* h dt : h \in R \text{ and } \lambda(h) \leqq 1 \right\}$.*

(c) *If λ is rearrangement invariant on $M([0, \infty))$, then λ' is universal; moreover, if $\rho(f) = \lambda(f^*)$, then $\rho'(f) = \lambda'(f^*)$.*

Proof. To prove (a) let $\Gamma = \{g^* : \lambda'(g) \leqq 1\}$. Then for $f \in M([0,$

∞)) we have $F_\Gamma(f) = \lambda(f)$ which means λ is universal.

In the proof of (b) we may assume that λ is rearrangement invariant and by Theorem 5.6 λ has property (J).

It is not hard to see that

$$\rho'(f) \leqq \sup\left\{\int_0^\infty f^* h dt \colon h \in R \quad \text{and} \quad \lambda(h) \leqq 1\right\}.$$

Now we will show the reverse inequality for simple functions. Assume $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ is a simple function in $M^+(\Delta, \mu)$ where $a_1 > a_2 > \cdots > a_n > 0$ and the A_i are mutually disjoint. Then $\varphi^* = \sum_{i=1}^n a_i \chi_{A_i^*}$ where $m(A_i^*) = \mu(A_i)$. Let $g \in R$ and define $\tilde{g}\colon \Delta \to [0, \infty)$ by

$$\tilde{g} = \sum_{i=1}^n \left(\int_{A_i^*} g dt / m(A_i^*)\right)\chi_{A_i}.$$

Then $\tilde{g}^* = g_P$ where P is the partition $\{A_i^*\}_{i=1}^n$ in $[0, \infty)$. So if $\lambda(g) \leqq 1$, by property (J), $\rho(\tilde{g}) = \lambda(\tilde{g}^*) = \lambda(g_P) \leqq \lambda(g) \leqq 1$. Also

$$\int_\Delta \varphi \tilde{g} d\mu = \int_0^\infty \varphi^* g dt$$

which means

$$\sup\left\{\int_0^\infty \varphi^* g dt \colon g \in R,\ \lambda(h) \leqq 1\right\} \leqq \sup\left\{\int_\Delta \varphi h d\mu \colon h \in M(\Delta, \mu),\ \rho(h) \leqq 1\right\}$$
$$= \rho'(\varphi).$$

Therefore, (b) is true for every simple function in $M(\Delta, \mu)$ and the extension to arbitrary functions follows from the Fatou property.

We conclude this section with the following example. Let $\mathscr{I} = \{I_i\}_{i=1}^\infty$ be the partition of $[0, \infty)$ with $I_i = [i - 1, i)$. For any $f \in M^+([0, \infty))$ define $f_{\mathscr{I}}$ to be the average function $f_{\mathscr{I}} = \sum_{i=1}^\infty \left(\int_{I_i} f dt\right)\chi_{I_i}$. Some of the properties of $f_{\mathscr{I}}$ are

(i) $f_{\mathscr{I}} = 0$ if and only if $f = 0$ a.e. on $[0, \infty)$.

(ii) $(af_{\mathscr{I}}) = a(f_{\mathscr{I}})$.

(iii) $(f + g)_{\mathscr{I}} = f_{\mathscr{I}} + g_{\mathscr{I}}$.

(iv) If $f_n \uparrow f$, then $(f_n)_{\mathscr{I}} \uparrow f_{\mathscr{I}}$.

Define the functional λ_0 on $M^+([0, \infty))$ by $\lambda_0(f) = \|f_{\mathscr{I}}\|_\infty$. Then λ_0 is a function norm with the Fatou property.

λ_0 *is universal.* Notice that λ_0 is universal if and only if $(\lambda_0)_m(f) = \lambda_0(f^*)$ is a function norm. For any $f \in M([0, \infty))$, $f^* \in R$ which means that $\int_{I_1} f^* dt \geqq \int_{I_i} f^* dt$ for all $i = 1, 2, \cdots$. Hence $(\lambda_0)_m(f) = \int_{I_1} f^* dt = \int_0^1 f^* dt = \|f\|_{L_1 + L_\infty}$. Therefore, $(\lambda_0)_m$ is a function norm which makes λ_0 universal.

λ_0 *is not rearrangement invariant* and in fact L_{λ_0} is not even rearrangement invariant. Let $f = \sum_{i=1}^{\infty} i\chi_{[i,i+1/i)}$. Then

$$\lambda_0(f) = \sup\left\{\int_{I_i} f\,dt\right\}_{i=1}^{\infty} = 1 .$$

Let $\{A_i\}_{i=1}^{\infty}$ be the subsets of $[0, \infty)$ defined by $A_i = [\sum_{k=1}^{i-1} 1/k, \sum_{k=1}^{i} 1/k)$. Define $f_1 = \sum_{i=1}^{\infty} i\chi_{A_i}$. Then f and f_1 are equimeasurable but $\lambda_0(f_1) = \infty$. Hence L_{λ_0} is not rearrangement invariant.

λ_0 *does not have property* (J). Let $P = \{[1/2, 2)\}$ and let $\varphi = 6\chi_{[1/2,1)} + 4\chi_{[1,2]}$. Then $\varphi_P = (14/3)\chi_{[1/2,2)}$ and $\lambda_0(\varphi_P) = 14/3$. But $\lambda_0(\varphi) = 4$. Thus $\lambda_0(\varphi) < \lambda_0(\varphi_P)$ which means λ_0 does not have property (J).

λ_0' *is not universal*. One can show that $\lambda_0'(g) = \sum_{i=1}^{\infty} \|g\chi_{I_i}\|_\infty$. Let $f = 3\chi_{[0,1)}$ and $g = 2\chi_{[1/2,3/2)}$. Then $(\lambda_0')_m(f) + (\lambda_0')_m(g) = 5 < 7 = (\lambda_0')_m(f + g)$ which means λ_0' is not universal.

6. **Universally rearrangement invariant function norms.** If (Δ, Σ, μ) is a σ-finite measure space, then Δ can be written as the union of a sequence of disjoint sets $\Delta_0, e_1, e_2, \cdots$ belonging to Σ such that Δ_0 is atom free and each e_i is an atom of finite measure. Let $\{B_i\}_{i=1}^{\infty}$ be a collection of disjoint intervals on the positive real axis such that $B_i = [a_i, b_i]$ and $b_i - a_i = \mu(e_i) (i = 1, 2, \cdots)$. Set $\Delta_1 = \Delta_0 \cup (\bigcup_{i=1}^{\infty} B_i)$ and let $(\Delta_1, \Sigma_1, \mu_1)$ be the direct sum of the measure space $(\Delta_0, \Sigma \cap \Delta_0, \mu)$ and the spaces $(B_i, m) (i = 1, 2 \cdots)$. Then $(\Delta_1, \Sigma_1, \mu_1)$ is a nonatomic σ-finite measure space with $\mu_1(\Delta_1) = \mu(\Delta) = \infty$. Furthermore, $M(\Delta, \Sigma, \mu)$ can be identified with a subset of $M(\Delta_1, \Sigma_1, \mu_1)$, in particular the set of all functions which are constant on the intervals B_i. We will say that (Δ, Σ, μ) is *embedded* in $(\Delta_1, \Sigma_1, \mu_1)$.

The next definition is due to Luxemburg [9, p. 98].

DEFINITION 6.1. Let (Δ, Σ, μ) be embedded in $(\Delta_1, \Sigma_1, \mu_1)$. Define the transformation $T_\mu: M(\Delta_1, \mu_1) \to M(\Delta, \mu)$ by

$$T_\mu(f) = f\chi_{\Delta_0} + \sum_{i=1}^{\infty}\left(\int_{B_i} f\,dt/m(B_i)\right)\chi_{e_i} .$$

A function norm ρ on $M(\Delta, \Sigma, \mu)$ is said to be *universally rearrange-invariant* whenever $\rho(T_\mu f_1) \geqq \rho(f)$ for all $f \in M^+(\Delta, \mu)$ and all $f_1 \in M(\Delta_1, \mu_1)$ satisfying $f_1 \sim f$.

Notice that if (Δ, μ) is non-atomic, then ρ is universally rearrangement invariant if and only if ρ is rearrangement invariant.

Lemma 6.2 relates the subjects of the previous section to the concept of universally rearrangement invariant (compare [9, p. 121, Theorem 12.2]).

LEMMA 6.2. (a) *Let ρ be a function norm defined on $M(\Delta, \mu)$.*

Then the following are equivalent:

(i) ρ *is induced by a universal function norm.*

(ii) ρ *is universally rearrangement invariant.*

(iii) $\rho(f) = \sup\left\{\int_0^\infty f^*g^*dt \colon \rho'(g) \leqq 1\right\}$ *for all* $f \in M^+(\Delta, \mu)$.

(b) *If* ρ *is universally rearrangement invariant, then* ρ' *is universally rearrangement invariant.*

We are now able to show that the function norms induced by a universal function norm behave very much like the Orlicz norms with respect to $L_1 \cap L_\infty$ and $L_1 + L_\infty$. We will need to use a result of Silverman [14, p. 230].

THEOREM 6.3. (Silverman). *Let* (Δ, μ) *be nonatomic and let* Λ *be a Köthe space in* $M(\Delta, \mu)$. *If* Λ *is rearrangement invariant then* $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$.

THEOREM 6.4. *Let* ρ *be a universally rearrangement invariant function norm defined on* $M(\Delta, \mu)$. *Then*

(a) $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$.

(b) *there is an equivalent universally rearrangement invariant function norm* ρ_1 *such that* L_{ρ_1} *is an intermediate space of* L_1 *and* L_∞.

Proof. To prove (a) notice that since ρ is universally rearrangement invariant, there exists a rearrangement invariant function norm λ defined on $M([0, \infty))$ such that $\rho(f) = \lambda(f^*)$. λ' is rearrangement invariant so by Theorem 6.3 we have $L_1 \cap L_\infty \subset L_\lambda$, $L_{\lambda'} \subset L_1 + L_\infty$. Hence $\|f\|_{L_1+L_\infty} = \int_0^1 f^*dt < \infty$ for all $f \in (L_\rho \cup L_{\rho'})$. So by Corollary 4.4 we know $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$.

To prove (b) let $\Gamma = \{g \colon \rho'(g) \leqq 1\}$ be the unit ball for $L_{\rho'}$ and let $B_\cap = \{g \colon \|g\|_\cap \leqq 1\}$ and $B_+ = \{g \colon \|g\|_+ \leqq 1\}$ be the unit balls for $L_1 \cap L_\infty$ and $L_1 + L_\infty$, respectively. ρ' is universally rearrangement invariant which means $L_1 \cap L_\infty \subset L_{\rho'} \subset L_1 + L_\infty$. Hence there is a constant a such that $(1/a)\rho' \leqq \|\cdot\|_\cap$, i.e., $B_\cap \subset a\Gamma$. Now set $\Gamma_1 = a\Gamma \cap B_+$ and define ρ_1 by $\rho_1(f) = \sup\left\{\int_0^\infty f^*g^*dt \colon g \in \Gamma_1\right\}$. Lemma 6.2 says that ρ_1 is universally rearrangement invariant. Because $B_\cap \subset \Gamma_1 \subset B_+$ we have $\|\cdot\|_+ \leqq \rho_1 \leqq \|\cdot\|_\cap$.

Now we will show that ρ_1 and ρ are equivalent. Notice that $a\rho(f) = \sup\left\{\int_0^\infty f^*g^*dt \colon g \in a\Gamma\right\}$. Hence $\rho_1 \leqq a\rho$ because $\Gamma_1 \subset a\Gamma$. Since $L_{\rho'} \subset L_1 + L_\infty$, there is a constant b_1 such that $1/b_1\|\cdot\|_+ \leqq \rho'$ (we may choose b_1, such that $b_1 > 1/a$). So $\Gamma \subset b_1B_+$ and thus $a\Gamma \subset ab_1B_+$. Let $b = ab_1$, then $b\Gamma_1 = b(a\Gamma \cap B_+) = ba\Gamma \cap bB_+$. Notice that $a\Gamma \subset b\Gamma_1$ which means that $(a/b)\Gamma \subset \Gamma_1$ or $(a/b)\rho \leqq \rho_1$. Hence ρ and ρ_1 are equivalent.

7. Universal and universally rearrangement invariant Köthe spaces. The concepts of the previous sections of this paper can be generalized to the general Köthe spaces.

DEFINITION 7.1. A Köthe space $\Lambda(\Gamma)$ is called *universal* if

$$\Lambda = \left\{ f \in M(\Delta, \mu) : \int_0^\infty f^* g^* dt < \infty \quad \text{for all} \quad g \in \Gamma \right\} .$$

Hence the functions in a universal Köthe space are characterized by the action of their monotonic rearrangements as was the case of a normed Köthe space induced by a universal function norm.

The following concept is due to Luxemburg [9].

DEFINITION 7.2. A Köthe space $\Lambda = \Lambda(\Gamma)$ defined on $M(\Delta, \mu)$ is said to be *universally rearrangement invariant* whenever $f \in \Lambda$ implies $T_\mu f_1 \in \Lambda$ for all $f_1 \in M(\Delta_1, \mu_1)$ satisfying $f_1 \sim f$.

Observe that if (Δ, μ) is nonatomic then Λ is universally rearrangement invariant if and only if Λ is rearrangement invariant.

LEMMA 7.3. *Let $\Lambda(\Gamma)$ be a Köthe space.*

(a) *Λ is universal if and only if Λ is universally rearrangement invariant.*

(b) *If Λ is universal, then Λ' is also universal.*

Proof. Assume $\Lambda(\Gamma)$ is universal. Let $f \in \Lambda$, $f_1 \in (\Delta_1)$, and $f_1 \sim f$. Then for any $g \in \Gamma$ we have $\int_\Delta T_\mu f_1 g d\mu = \int_\Delta f_1 g d\mu \leqq \int_0^\infty f^* g^* dt < \infty$. Therefore, Λ is universally rearrangement invariant.

Next assume that Λ is universally rearrangement invariant. Let $\Pi = \left\{ f : \int_0^\infty f^* g^* dt < \infty \text{ for all } g \in \Gamma \right\}$. Easily $\Pi \subset \Lambda$. Suppose $f \in \Lambda$ but $f \notin \Pi$. This means that $\int_0^\infty f^* g_0^* dt = \infty$ for some $g_0 \in \Gamma$. By Lemma 5.3 we know that there exists an $f_1 \in M(\Delta_1)$ such that $\int_{\Delta_1} f_1 g_0 d\mu_1 = \infty$ and $f_1 \sim f$. But $\int_\Delta T_\mu f_1 g_0 d\mu = \int_{\Delta_1} f_1 g_0 d\mu_1 = \infty$ which contradicts the fact that Λ is universally rearrangement invariant. Therefore, $\Pi = \Lambda$ and Λ is universal.

The next result is an extension of Theorem 6.3.

THEOREM 7.4. *If $\Lambda(\Gamma)$ is a universal Köthe space in $M(\Delta, \mu)$, then $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$.*

Proof. In $[0, \infty)$ let $I_n = [0, n)$ and let $\Omega([0, \infty))$ be the locally integrable functions in $M([0, \infty))$ with respect to $\{I_n\}_{n=1}^{\infty}$. Let $\Gamma^* = \{g^*: g \in \Gamma\}$ and $\Gamma_1 = \{h \in \Omega([0, \infty)): h^* \in \Gamma^*\}$. Form the Köthe space $\Lambda_1 = \Lambda(\Gamma_1)$ in $M([0, \infty))$. If $f \in \Lambda_1$ and $g \in \Gamma_1$, then $\int_0^\infty fg'dt < \infty$ for all $g' \sim g_1$. Hence $\int_0^\infty f^*g^*dt < \infty$ and therefore

$$\Lambda_1 = \left\{f \in \Omega([0, \infty)): \int_0^\infty f^*h^*dt < \infty \quad \text{for all} \quad h \in \Gamma_1\right\}$$

which means Λ_1 is rearrangement invariant. So $L_1([0, \infty)) \cap L_\infty([0, \infty)) \subset \Lambda_1 \subset L_1([0, \infty)) + L_\infty([0, \infty))$. This means that $(\Lambda^* \cup \Lambda'^*) \subset L_1([0, \infty)) + L_\infty([0, \infty))$. Hence by Corollary 4.4 $L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty$.

Returning to normed Köthe spaces we are now able to prove

THEOREM 7.5. *If L_ρ is a universal Köthe space, then there is a norm ρ_1 such that ρ and ρ_1 are equivalent and ρ_1 is universally rearrangement invariant.*

Proof. Define ρ_1 by $\rho_1(f) = \sup\left\{\int_0^\infty f^*g^*dt: \rho'(g) \leqq 1\right\}$. Easily ρ_1 is universally rearrangement invariant. In order to show that ρ_1 and ρ are equivalent, we will show that $L_{\rho_1} = L_\rho$. It is easy to show that $L_{\rho_1} \subset L_\rho$. On the other hand, suppose $f \in L_\rho$ and $f \notin L_{\rho_1}$. There is a sequence of functions $\{g_n\} \subset L_{\rho'}$ such that $g_n \geqq 0$, $\rho'(g_n) \leqq 1$, and $\int_0^\infty f^*g^*dt > n^3$. Let $h_k = \sum_{n=1}^{k} g_n/n^2$ and $h = \sum_{n=1}^{\infty} g_n/n^2$. Then $\rho'(h) \leqq \liminf \sum_{n=1}^{k} 1/n^2\rho'(g_n) \leqq \pi^2/6$. Since all the g_n are nonnegative we know that $h_k \geqq g_k$ for each k, which means $\int_0^\infty f^*h^*dt \geqq \int_0^\infty f^*g_k^*dt > k^3$ for all $k = 1, 2, \cdots$. Therefore $\int_0^\infty f^*h^*dt = \infty$. But as before this contradicts the fact that L_ρ is universal. Therefore, $L_{\rho_1} = L_\rho$ and we have completed the proof.

Theorem 7.5 was also given by Luxemburg [9] for his restricted case.

Combining Theorem 7.4, Theorem 7.5, and Theorem 6.4(b) we have

THEOREM 7.6. *If Λ is a universal Köthe space, then*

$$L_1 \cap L_\infty \subset \Lambda \subset L_1 + L_\infty .$$

Furthermore, if Λ is normed, i.e., *$\Lambda = L_\rho$, then there exists an equivalent universally rearrangement invariant norm ρ_1 such that $\|\cdot\|_+ \leqq \rho_1 \leqq \|\cdot\|_\cap$.*

We conclude with an example that shows that $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$ does not necessarily imply that L_ρ is universal. Let (Δ, μ) be $(-\infty, \infty)$ with Lebesgue measure and let

$$\rho(f) = \|f\chi_{(-\infty,0)}\|_\infty + \|f\chi_{[0,\infty)}\|_1 .$$

Clearly $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$ but L_ρ is not universal.

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