PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 1, 1974

PHRAGMÉN-LINDELÖF TYPE THEOREMS FOR A SYSTEM OF NONHOMOGENEOUS EQUATIONS

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Hyperbolic systems of N equations are considered

(1) $\partial u/\partial t = P(D)u + f(x, t)$, with u in R^N and (x, t) in $R^n \times R^1$,

where $D = i\partial/\partial x$. For suitable source functions $f(x, t)$ there are solutions satisfying the boundedness condition

(2) $|u(x, t)| \leqq C \exp \{a|x|^\gamma + b|t|^\theta\}$, $0 \leqq \theta < 1$, $0 \leqq \gamma < p$,

where p is the conjugate of $2p_0$, with p_0 the reduced order of the matrix $P(\xi)$. Furthermore, the solutions are polynomials in t if the initial states $u(x, 0)$ grow at infinity like polynomials. However, these solutions are not unique; a requirement of a certain type of $u(x, 0)$ at infinity is needed. The one-dimensional classical Phragmén-Lindelöf theorem and some results of Shilov for homogeneous systems are instances of this. It is the purpose here to supply a general (necessary and for some cases sufficient) condition for uniqueness. Preliminary to that a necessary condition is found on $f(x, t)$ so that (1) admits solutions that are polynomials in t.

The one-dimensional classical theorem of Phragmén-Lindelöf can be stated as follows: If u is a soultion of the Cauchy-Riemann equation $\partial u/\partial t = iu/\partial x$ on $R^1 \times R^1$ fulfilling the boundedness condition (2) with $\gamma = \theta < 1$ and

$$|u(x, 0)| \leqq C(1 + |x|)^\nu, \quad \nu \geqq 0, \tag{3}$$

then u is a polynomial in (x, t). Therefore, u is identically zero provided condition (3) is replaced by the decay condition on the initial state

$$u(x, 0) = \theta(|x|^{-d}), \quad d \geqq 0, \quad \text{when} \quad x \to \infty. \tag{4}$$

Shilov [4], [5] or [6] has improved this theorem for a system of N partial differential equation (1) with $f = 0$ under the boundedness conditions (2) and (3). If the eigenvalues of $P(\xi)$ are real for each real vector $\xi \in R^n$, then u has the expression

$$u(x, t) = \sum_0^r U_k(x)t^k, \qquad r = 2[(n + \nu)/2] + N + 1 \tag{5}$$

where the functions U_k are solutions of the systems

$$P(D)U_{k-1} = kU_k, \qquad k = 0, \cdots, r-1,$$
$$P(D)U_r = 0 \quad \text{on} \quad R^n.$$

Both of the above Phragmén-Lindelöf theorems consider systems of homogeneous equations only. For nonhomogeneous equations, a necessary condition on $f(x, t)$ is obtained to permit the solutions to be polynomials in t (i.e., Lemma 1.1).

In Shilov's result, if condition (3) is replaced by condition (4), it cannot be concluded that u is the trivial (hence, unique) solution. This leaves open a uniqueness problem which is solved by applying the results of the symmetrization of distributions (cf. Chen [1], [2], and [3]), to obtain a uniqueness condition. Moreover, if this uniqueness condition is not assumed, a smooth nontrivial solution is constructed.

The remainder of the paper is organized as follows: First the Shilov result is extended to nonhomogeneous equations. Next, some consequences of symmetrization of distributions are recalled. In § 3, the above uniqueness problem is treated.

The author wants to express his gratitude to Professor Avner Friedman for reminding him of the Shilov result [4], and to Professor Joe R. Foote for his discussion on equation (1.2) and the conclusion in Remark 1.3.

An extension of the Shilov result will now be given. The convention is adopted that giving a vector a certain property means that each component has that property.

Let $P(\xi) = (P_{ij}(\xi))$ be an $N \times N$ matrix with polynomial entries of reduced order p_0 (cf. [5]), and let $P(\xi)$ have real eigenvalues for each $\xi \in R^n$. Then, it is clear that system (1) is hyperbolic. To obtain the Shilov result of solutions being polynomial in t, consider first a necessary condition on the source function $f(x, t)$.

LEMMA 1.1. *Suppose that $u(x, t)$ is a solution of system* (1) *and is a polynomial in t:*

$$u(x, t) = \sum_1^r u_j(x)t^j \quad on \quad R^n \times R^1 . \tag{1.1}$$

Then, f is a solution of the system

$$(P^r(D) + P^{r-1}(D)\partial + \cdots + P(D)\partial^{r-1} + \partial^r)f(x, t)\,|_{t=0} = 0 \text{ on } R^n, \tag{1.2}$$

with $\partial = \partial/\partial t$.

Proof. The substitution of (1.1) into (1) yields

$$\Sigma j u_j(x)t^{j-1} = \Sigma P(D)u_j(x)t^j + f(x, t) , \qquad j = 1, \cdots, r .$$

For $t = 0$, then $u_1(x) = f(x, 0)$. Differentiating both sides of the r relations implies that for $i = 2, \cdots, r + 1$,

$$\begin{aligned}&\Sigma j \cdots (j - i + 1)u_j(x)t^{j-i}\\ &= \Sigma k \cdots (k - i + 2)P(D)u_k(x)t^{k-i+1} + \partial^{i-1}f(x, t) ,\\ &\qquad\qquad j = i, \cdots, r; \quad k = i - 1, \cdots, r .\end{aligned}$$

With $t = 0$ and comparing coefficients,

$$\begin{aligned}&i!\, u_i(x) = (i - 1)!\, P(D)u_{i-1}(x) + \partial^{i-1}f(x, 0) , \quad i = 2, \cdots, r ;\\ &r!\, P(D)u_r(x) + \partial^r f(x, 0) = 0 .\end{aligned}$$

Substitution of the $r - 1$ relations into the last relation yields (1.2), completing the proof.

REMARK 1.2. Since a solution of system (1) can be written as sum of a particular solution and a solution of the corresponding homogeneous system, and since each particular solution vanishes identically at $t = 0$, the solutions considered in the lemma are selected in the form (1.1). If $f(x, t) = \sum_0^r F_j(x)t^j$ is a polynomial in t, the coefficient functions satisfy the system

$$P^r(D)F_0 + \cdots + j!\, P^{r-j}(D)F_j + \cdots + F_r = 0 . \tag{1.3}$$

In particular, $f = 0$ is the Shilov case. More generally, for any $F_r \in C_0^\infty(R^n)$, the system

$$P^r(D)u = -F_r \quad \text{on} \quad R^n$$

has a distributional solution. System (1.3) always has either $C_0^\infty(R^n)$-solutions or C^∞-solutions with polynomial growth in x at infinity. For their construction, refer to Chen [1] or [2].

REMARK 1.3. Instead of system (1.2), consider a system with a stronger restriction:

$$(P^r(D) + P^{r-1}(D)\partial + \cdots + P(D)\partial^{r-1} + \partial^r)f(x, t) = 0 \quad \text{on} \quad R^n \times R^1 . \tag{1.4}$$

Operating on this vector equation with $P(D) - \partial$ reduces the problem to consideration of solutions for

$$(P^{r+1}(D) - \partial^{r+1})u = 0 .$$

This equation is again of Shilov type, but for a larger system of $N + r$ equations.

Let $P(\xi)$ be an $N \times N$ matrix with real eigenvalues and with reduced order p_0; let f be a solution of (1.2) such that

$$|f(x, t)| \leqq C_0(t)(1 + |x|)^\mu \quad \text{on } R^n , \quad \mu \geqq 0 ,$$

with constant μ independent of t, with both $C_0(t)$ and μ independent of x, and with $r < r_p$, where r_p is the smallest integer not less than

$\mu + N + n$. Then, the solutions of system (1) have the following properties:

THEOREM 1.4. *Assume that* u *satisfies system* (1) *and the boundedness conditions* (2) *and* (3). *Then,* u *has the representation*

$$u(x, t) = u_h(x, t) + u_p(x, t) . \tag{1.5}$$

The function u_h *is a solution of the homogeneous system corresponding to* (1) *in the form*

$$u_h(x, t) = \sum_0^{r_h} U_k(x) t^k , \tag{1.6}$$

where the coefficient matrices satisfy

$$P(D) U_{k-1} = k U_k , \quad k = 1, \cdots, r_h , \quad P(D) U_{r_h} = 0 \quad \text{on } R^n , \tag{1.7}$$

and where r_h *is the smallest even integer not less than* $\nu + n + N$. *The function* u_p *is a solution of* (1) *of the form*

$$u_p(x, t) = \sum_1^{r_p+1} V_k(x) t^k , \tag{1.8}$$

such that

$$\begin{cases} V_1 = f(\cdot, 0); P(D) V_k = (k+1) V_{k+1} - \partial^k f(\cdot, 0)/k! , & k = 1, \cdots, r_p , \\ P(D) V_k = -\partial^k f(\cdot, 0)/k! , \quad k = r_p + 1 \quad \text{on} \quad R^n , \end{cases} \tag{1.9}$$

where $\partial^k f(x, 0)$ *is* $\partial^k f(x, t)/\partial t^k$ *evaluated at* $t = 0$.

REMARK 1.5. For more detailed discussion about the conditions imposed here, the reader is referred to [4], [5] or [6]. Since the idea and arguments of the following proof are similar to those of Shilov [4] or [5], the proof here is given rigorously only on the principal points of difference. For convenience, the notation in [5] is used here.

Proof. Let $g(\xi, t)$, $v(\xi, t)$ and $v_0(\xi)$ be the Fourier transforms in x only of $f(x, t)$, $u(x, t)$, and $u(x, 0)$, respectively. Consider system (1) on the space W_q^q, $r < q < (2p_0)'$, with $(2p_0)'$ as the conjugate of $2p_0$. Then, the Fourier transform of system (1) with respect to x is a system of differential equations in t:

$$\partial v/\partial t = P(\xi) v + g(\xi, t) \quad \text{in} \quad R^n \times R^1 . \tag{1.10}$$

With use of the initial condition $v_0(\xi)$, the system admits the unique solution in the space $[W_{q'}^{q'}]'$ (cf. Chap. 7, [4]), given by

$$v(\xi, t) = \exp\{tP(\xi)\} v_0(\xi) + \int_0^t \exp\{(t - \tau) P(\xi)\} g(\xi, \tau) d\tau . \tag{1.11}$$

The inverse Fourier transform of (1.11) is the expression (1.5), where u_h and u_p are the inverse Fourier transforms of the first and last terms, respectively, of (1.11). Furthermore,

$$u_p(x, t) = \int_0^t G(x, t - \tau) * f(x, \tau) d\tau \tag{1.12}$$

with convolution in x, where $G(x, t)$ is the inverse Fourier transform of $\exp\{tP(\xi)\}$ in ξ.

Since u_h is the solution on W_q^q of the homogeneous part of system (1) with the initial condition $u_h(x, 0) = u(x, 0)$, it satisfies the required conditions in the proof of the Shilov result [5]. Hence, u_h is in the form (1.6) and (1.7).

To prove (1.8) the Shilov technique [5] for proving (1.6) can be extended as follows; i.e., the method for treating the homogeneous part can be applied successfully to the nonhomogeneous part. Thus, the function

$$F(t, \tau; \phi) = (g(\cdot, t), \exp\{tP^*(\cdot)\}\psi(\cdot))$$

satisfies the arguments in pp. 83–86 for each $\tau \in R^1$ and for each $\phi \in W_q^q$, where ψ is the Fourier transform of ϕ. This implies that $F(t - \tau_0, \tau; \phi)$ is a polynomial in $t - \tau_0$ of degree not higher than r_p and independent of τ. Following the methods of pp. 86–89, $G(x, t - \tau_0) * f(x, \tau)$ is a polynomial in $t - \tau_0$ independent of τ. Hence, $u_p(x, t)$ is a polynomial in t in the form (1.8) with the summation running from 0 through $r_p + 1$, where the order of differentiability of the coefficients $V_k(x)$ is the same as that of $u_p(x, t)$ in x for each t. But, by (1.11), $u_p(x, t)$ vanishes at $t = 0$. Therefore, $V_0 = 0$. To prove (1.9), the same arguments for the proof of Lemma 1.1 and the condition (1.2) yield the assertion for all V_k, $k = 1, \cdots, r_p + 1$. This completes the proof of the theorem.

REMARK 1.6. The particular solution u_p of system (1) can be represented explicitly in terms of derivatives of $f(x, t)$ by using relations (1.8) and (1.9); i.e.,

$$\begin{aligned} u_p(x, t) = \Sigma(t^k/k!)[&P^{k-1}(D)\partial^0 + P^{k-2}(D)\partial + \cdots \\ &+ P(D)\partial^{k-2} + \partial^{k-1}]f(x, 0)\,, \qquad k = 1, \cdots, r_p + 1\,. \end{aligned}$$

2. A Liouville-type theorem for systems of equations. The results of Chen [1] and [2] are extended to certain systems of convolution equations from single convolution equations. First those proofs need to be modified both to make them complete and to make possible the desired extension.

For a temperate distribution w, let $\tilde{w}(\xi)$ denote its Fourier transform. In the convolution equation $T*u = 0$, if the Fourier transform $\tilde{T}(\xi)$ of the "convolutor" T is complex-valued for real variable ξ, as shown below the equation can be rewritten into two equations with convolutors having real-valued Fourier transforms. Each of these two equations can be treated as if the convolutor T had the real-valued Fourier transform $\tilde{T}(\xi)$. Indeed, let Q be the inverse Fourier transform of the real part of $\tilde{T}(\xi)$; wherever needed, Q could represent the imaginary part through a similar argument. Let $\tilde{v}$ be the sum of the squares of the real part and the imaginary part of $\tilde{u}$, and supp $\tilde{v}$ be the support of $\tilde{v}$. Then the Fourier transform of the convolution equation $T*u = 0$ implies that supp $\tilde{v}$ is contained in the null space $\mathscr{N}(\tilde{Q})$:

$$\mathscr{N}(\tilde{Q}) = \{\xi \in R^n : \tilde{S}(\xi) = 0\} ,$$

and the inverse Fourier transform v of $\tilde{v}$ satisfies the convolution equation $Q*v = 0$. Therefore, it suffices to consider the equation $T*u = 0$ with real-valued $\tilde{T}(\xi)$.

Consider next the system of N convolution equations:

$$S*u = f \quad \text{on} \quad R^n . \tag{2.1}$$

Let $S = (S_{ij})$, called a convolutor, be an $N \times N$ matrix with finite distributions $S_{ij} \in \mathscr{E}'(R^n)$ as entries, $\det \tilde{S}$ be the determinant of its Fourier transform $\tilde{S}$, and $\det S$ be the inverse Fourier transform of $\det \tilde{S}$. It is well-known (cf. Friedman [4]) that $\det \tilde{S}$ is an entire function of finite exponential type and $\det S$ is the determinant of S in the distributional sense. For a finite distribution T such that $\mathscr{N}(\tilde{T})$ is nonempty and regular (i.e., the gradient of $\tilde{T}$ does not vanish on $\mathscr{N}(\tilde{T})$), let j be the minimum number of nonzero principal curvatures of $\mathscr{N}(\tilde{T})$. Denote by C_j the class of such finite distributions. In the rest of the section, let the convolutor S satisfy the following assumptions:

(I) *S is nonsingular*; i.e., $\det S \neq 0$.

(II) *If $\tilde{T}$ is an irreducible factor of* $\det \tilde{S}$ *such that $\mathscr{N}(\tilde{T}) \neq \emptyset$, then $T \in C_j$.*

(III) *Let k be the smallest number of the indices j such that C_j contains a factor of* $\det S$ *which is not in C_{j+1}.*

THEOREM 2.1. *Let the source function f be smooth and compactly supported. Let the continuous vector-valued function u be a solution of system* (2.1) *and satisfy the decay property at infinity*:

$$u(x) = o(|x|^{-d}) \text{ with } d > 0 \text{ for } k = 0;\ d \geqq n - 1 - k/2 \text{ for } k > 0 . \tag{2.2}$$

Then, u is a smooth function with compact support. Furthermore, $u = 0$ provided $f = 0$.

COROLLARY 2.2. *Let $P(\xi)$ be an $N \times N$ matrix with polynomial entries $P_{ij}(\xi)$, $\xi \in R^n$, such that $P(D)\delta$ fulfills the assumptions on S. Then, if the continuous function u is a solution, in the distributional sense, of the system*

$$P(D)u = f, \qquad (f \in \mathscr{D}(R^n)),$$

and satisfies the decay property (2.2), *u is smooth and compactly supported.*

This corollary is an immediate consequence of Theorem 2.1.

Proof of Theorem 2.1. Let $({}^{co}S_{ji}(\xi))^{\sim}$ be the matrix formed by the cofactors of $(\tilde{S}_{ij}(\xi))$ and let $({}^{co}S)$ be its inverse Fourier transform. Let $\Delta = (\Delta_{ij})$ be the $N \times N$ matrix with entries $\Delta_{ij} = 0$ if $i \neq j$, and $\Delta_{ij} = \delta$, $i = 1, 2, \cdots, N$, where δ is the Dirac-measure in R^n. Then each element ${}^{co}S_{ij}$ is in $\mathscr{E}'(R^n)$ and

$$({}^{co}S) * S = S * ({}^{co}S) = (\det S) * \Delta \,. \tag{2.3}$$

Hence, equations (2.1) and (2.3) imply that for each $j = 1, \cdots, N$, u_j is a solution of the single convolution equation

$$(\det S) * u_j = g_j\,, \tag{2.4}$$

where the functions g_j are smooth and compactly supported:

$$g_j = \Sigma({}^{co}S_{ji}) * f_i\,, \qquad i = 1, \cdots, n\,. \tag{2.5}$$

By the assertions in [2] for $k > 0$ or in [3] for $k = 0$, it follows that u_j, $j = 1, \cdots, n$, are smooth functions with compact supports and vanish if $g_j = 0$. But the vanishing of f implies $g_j = 0$, $j = 1, \cdots, N$. This completes the proof of the theorem.

The next theorem shows that condition (2.2) in both Theorem 2.1 and Corollary 2.2 is crucial for the cases with $k = 0, n$; or with $d \leqq k/2$ when $0 < k < n - 1$. However, the problem is still open for the case with $k/2 < d < n - 1 - k/2$ when $0 < k < n - 1$.

Let the $N \times N$ matrix T be a finite distribution with $\det T$ in C_k but not in C_{k+1} for some $k > 0$. For any natural number r, let $\tilde{S}$ be $\tilde{T}^r$ and S its inverse Fourier transform. Then, S is the r-fold convolution of matrix T, denoted by $[T]_*^r$.

THEOREM 2.3. *For each integer μ, $0 \leqq \mu < r$, there is a nontrivial solution u of the system of convolution equations*

$$S * u = 0 \quad in \quad R^n\,, \tag{2.6}$$

such that u can be extended to an entire vector-valued function of finite exponential type and

(2.7) $$u(x) = o(|x|^{\mu-k/2}) \text{ but } u(x) \neq o(|x|^{\mu-k/2}) \text{ at infinity} .$$

Proof. For a vector $b = (b_1, \cdots, b_n)$, let

$$G_b = \{\xi \in R^n : |\xi_j| \leqq b_j,\ j = 1, \cdots, N\} .$$

Let b_j be so large that $G_b \cap \mathscr{N}(\det \tilde{T}) \neq \emptyset$, that there is a point ξ_0 in $G_{b-\varepsilon} \cap \mathscr{N}(\det \tilde{T})$, $0 < \varepsilon < b$, where at least k of the $n-1$ principal curvatures of $\mathscr{N}(\det \tilde{T})$ are different from zero. For each integer $h \geqq 0$, define a Dirac-measure $\delta_b^{(h)}(\det \tilde{T})$ in R^n as follows:

(2.8) $$(\delta_b^{(h)}(\det \tilde{T}), \phi) = (\delta^{(h)}, \phi^{\rho(b)}),\ \phi \in \mathscr{D}(R^n) ,$$

where $\phi^{\rho(b)}$ is the function used in the definition of the symmetrization $\phi^{\sigma(b)}$ of ϕ with respect to $\det \tilde{T}$ as defined in [1] or [2].

Let $\omega = (1, \cdots, 1) \in R^n$ and u_μ be the inverse Fourier transform of the product $({}^{co}S)^{\sim} \cdot (\delta_b^{(h)}(\det \tilde{T})\omega)$. By definition of u_μ, it suffices to prove that u_μ satisfies (2.6) and (2.7). Since the assertion of Theorem 2.1 in [2] is still true if χ_s is replaced by $({}^{co}S)^{\sim}\chi_s\omega$, the function u_μ has property (2.7). Now let ϕ be the Fourier transform of ψ in $\mathscr{D}(R^n)$. Then $(\det \tilde{T})^r\phi$ is a smooth function with rapid decay and

$$\begin{aligned}(S^* u_\mu, \psi) &= ((\det \tilde{T})^r \delta_b^{(\mu)}(\det \tilde{T}), \phi) \\ &= (\delta^{(\mu)}(q), q^r \phi^{\rho(b)}(q)) = (-1)^\mu D_q^\mu [q^r \phi^{\rho(b)}(q)]\,|_{q=0} = 0 ,\end{aligned}$$

where $D_q = \partial/\partial q$. Hence,

$$S * u_\mu = 0 , \qquad \mu = 0, \cdots, r-1 .$$

This concludes the proof of the theorem.

As a consequence of this theorem and the proof of Theorem 2.1, there is

COROLLARY 2.4. *For the same convolutor S, if the nonhomogeneous system*

(2.9) $$S * u = f , \qquad (f \in \mathscr{D}(R^n))$$

has a distributional solution, then the system has smooth solutions with compact support.

3. Applications. As an applications of Theorems 2.1 and 2.3, a Phragmén-Lindelöf theorem of the Liouville type is presented. More precisely, condition (3) is replaced by the decay property (2.2) in the consideration of Theorem 1.4. This property permits proof that the solution of the generalized Phragmén-Lindelöf theorem is unique.

Denote by $K(k, p_0, N)$ the class of all $N \times N$ matrices $P(\xi)$ of reduced order p_0, with polynomial entries $P_{ij}(\xi)$, $\xi \in R^n$, satisfying assumptions (I), (II), (III) in the last section, and satisfying also

(IV) *the eigenvalues of* $P(\xi)$ *are real for each real vector* $\xi \in R^n$.

For P in $K(k, p_0, N)$ with $k > 0$, consider the hyperbolic system of N nonhomogeneous equations:

$$\partial u/\partial t = P(D)u + f(x, t) \quad \text{on} \quad R^n \times R^1 . \tag{3.1}$$

The source function $f(x, t)$ is a solution of system (1.2) with $r \leqq r_p + 1$, and satisfies the boundedness condition

$$|f(x, t)| \leqq C_0(t)(1 + |x|)^\mu \quad \text{on} \quad R^n . \tag{3.2}$$

The constant μ is independent of t and x; $C_0(t)$ is independent of x; and for each $h = 1, \cdots, r$,

$$\partial^h f(\cdot, 0) \equiv \partial^h f(\cdot, 0)\partial t^h \tag{3.3}$$

is smooth and compactly supported.

THEOREM 3.1. *Assume that the continuous function u is a solution of system* (3.1) *and fulfills the boundedness condition* (2) *and, as* $x \to \infty$,

$$\begin{aligned} &u(x, 0) = o(|x|^{-d}), \\ &\qquad \textit{with } d \geqq n - 1 - k/2 \textit{ if } k > 0 \textit{ and } d > 0 \textit{ if } k = 0 . \end{aligned} \tag{3.4}$$

Then, $u(x, t)$ is a polynomial in t of degree r in the form

$$u(x, t) = \Sigma V_h(x) t^h , \qquad h = 1, \cdots, r , \tag{3.5}$$

where the V_h's are the solutions of the system

$$\begin{cases} P(D)V_h = (h + 1)V_{k+1} - \partial^h f(\cdot, 0)/h! , \quad h = 1, \cdots, r - 1 ; \\ P(D)V_r = -\partial^r f(\cdot, 0)/r! \quad \textit{on} \quad R^n. \end{cases} \tag{3.6}$$

If, with the same constant d in (3.4), *u satisfies*

$$\partial u(x, 0) = o(|x|^{-d}) \quad \textit{as} \quad x \to \infty , \tag{3.7}$$

then there is at most one solution $u(x, t)$ and this solution is smooth in (x, t) and has representation (3.5), *where the V_h are compactly supported.*

Proof. Using condition (3.4) and equation (1.5),

$$U_0(x) = u_h(x, 0) + u_p(x, 0) = u(x, 0) = o(|x|^{-d}) , \quad \text{as} \quad x \to \infty .$$

By system (1.7), $U_0(x)$ is a solution of the system of differential equations

$$[P(D)]^{\alpha} U_0 = 0 \quad \text{on} \quad R^n \quad \text{with} \quad \alpha = r_n + 1 .$$

The assertion of Corollary 2.2 yields $U_0 = 0$. System (1.7) then implies that $U_j = 0$ and thus $u_h = 0$. Therefore, $u = u_p$, where the representation is given by (1.8) and (1.9). This proves (3.5).

Condition (3.7) and differentiation with respect to t on both sides of relation (3.6) lead to the conclusion

$$V_1(x) = o(|x|^{-d}) \quad \text{as} \quad x \to \infty . \tag{3.8}$$

Using this result with system (3.6) implies that V_1 is a solution of the system

$$[P(D)]^r V_1 = g , \quad \text{on} \quad R^n , \tag{3.9}$$

where $g = -\Sigma \partial^h f(\cdot, 0)/h! \in C_0^\infty(R^n)$, $h = 1, \cdots, r$. The solution of system (3.9) with boundary condition (3.8) is provided by Corollary 2.2. Hence, V_1 is a smooth function with compact support and is the unique solution of (3.9). By system (3.6),

$$V_{h+1} = \{[P(D)]^h V_1 + \Sigma \partial^j f(\cdot, 0)/j!\}/(h + 1) , \qquad j = 1, \cdots, r ,$$

and these functions are smooth and compactly supported. This yields the last assertion and the proof of the theorem is complete.

The rest of the section will show that conditions (3.4) and (3.7) are crucial to assertions about the homogeneous and nonhomogeneous systems, respectively, in Theorem 3.1. The following theorem is an application of Theorem 2.3 and Corollary 2.4 to hyperbolic systems.

Let S be an $N \times N$ matrix with finite distributions as entries, and satisfy conditions (I), (II), and (III) in the last section with $k > 0$. Then, det $\tilde{S}$ can be written in the form

$$\det \tilde{S} = \tilde{T}_0 \Pi [\tilde{T}_j]^{r_j} , \quad r_j \geqq 1 , \quad j = 1, \cdots, s ,$$

with $\mathcal{N}(\tilde{T}_0) = \phi$ and $T_j \in C_{k_j}$, but $T_j \notin C_{k_j+1}$ for some $k_j > 0$, $j = 1, \cdots, s$. Hence, k is the minimum of k_j, $j = 1, \cdots, s$.

THEOREM 3.2. *Let $f(x, t)$ be a solution of system* (1.2) *and preserve properties* (3.2) *and* (3.3). *For each integer μ_j, $0 \leqq \mu_j < r_h r_j$, $j = 1, \cdots, s$, the system of the N convolution equations*

$$\delta'(t) * u = S * u + f(x, t) \quad on \quad R^n \times R^1 \tag{3.10}$$

has a nontrivial solution $u(x, t)$ with the properties (2), (3), *and* (1.5) *such that u_p and u_h are solutions of* (3.10) *and its corresponding homogeneous system, respectively. The function u_h can be represented in the form* (1.6) *with vecter-valued functions U_j satisfying the system*

$$S * U_{j-1} = j U_j, \ j = 1, \cdots, r_h; \ S * U_{r_h} = 0 \quad on \quad R^n . \tag{3.11}$$

Moreover, u_p *can be represented in the form* (1.8) *with* $V_j \in C_0^\infty(R^n)$ *as solutions of the system*

$$\text{(3.12)}\qquad \begin{cases} S * V_j = (j+1)V_{j+1} - \partial^j f(\cdot, 0)/j!\,, & j = 1, \cdots, r_p; \\ S * V_j = -\partial^j f(\cdot, 0)/j\,, \quad j = r_p + 1\,, & \text{on} \quad R^n\,. \end{cases}$$

Furthermore, the initial state behaves at infinity as

$$\text{(3.13)}\quad \partial^j u(x, 0) = o(|x|^{\mu - k/2})\,, \quad \text{but} \quad \partial^j u(x, 0) \neq o(|x|^{\mu - k/2})\,, \quad j = 0, 1\,,$$

with μ *as the maximum of* μ_j, $j = 1, \cdots, s$; *in particular, there is a solution* $u(x, t)$ *of the homogeneous system of* (3.10) *which preserves property* (3.13) *with* $j = 0$.

Proof. For each integer μ_j, $0 \leqq \mu_j < r_h r_j$, $j = 1, \cdots, s$, the assertion of Theorem 2.3 with $N = 1$ implies that the convolution equation

$$\text{(3.14)}\qquad [T_j]_*^\alpha * u_\beta = 0\,, \quad (\alpha = r_j r_h,\ \beta = \mu_j) \quad \text{on} \quad R^n\,,$$

has an analytic solution u_β which behaves at infinity like

$$\text{(3.15)}\qquad u_\beta(x) = o(|x|^{\beta - kj/2})\,, \quad \text{but} \quad u_\beta(x) \neq o(|x|^{\beta - kj/2})\,.$$

Let $\omega = (1, \cdots, 1) \in R^n$ and let U_0 be the inverse Fourier transform of $[({}^{co}S)^\sim]^{r_h}[(\Sigma u_\beta)\omega]$ with $\beta = \mu_1, \cdots, \mu_s$. Since $\tilde{U}_0(\xi)$ has compact support, $U_0(x)$ can be extended to an entire function of finite exponential type and, from the proof of Theorem 2.3, satisfies property (2.7). Set $U_j = [S]_*^j * U_0/j!$, $j = 1, \cdots, r_h$. By simple computation and by (3.14), $S * U_j = 0$ with $j = r_h$ and $S * U_{j-1} = jU_j$, $j = 1, \cdots, r_h$; i.e., U_j, $j = 1, \cdots, r_h$, satisfies system (3.1). Let $u_h(x, t) = \Sigma U_j(x)t^j$, $j = 0, \cdots, r_h$, which completes the construction of (1.6).

Next, the assertion of Corollary 2.4 implies the existence of $V_j \in C_0^\infty(R^n)$, $j = 1, \cdots, r_p + 1$, which are solutions of system (3.12). Let

$$\text{(3.16)}\qquad u_p(x, t) = \Sigma V_j(x)t^j\,, \qquad j = 1, \cdots, r_p + 1\,.$$

Hence, $u(x, t) = u_h(x, t) + u_p(x, t)$ satisfies the assertions. This completes the proof of the theorem.

REMARK 3.3. Set $S = P(D)\Delta$ with $P(\xi) \in K(k, p_0, N)$. Then Theorem 3.2 gives some counterexamples to Theorem 3.1 if either condition (3.4) or (3.7) is relaxed with $d < k/2$. For the case $k/2 \leqq d < n - 1 - k/2$, the problem is still open except $k = n - 1$.

In conclusion, a final comment is in order: Even though the conditions of Theorem 3.1 are naturally imposed as an extension of the results for the Cauchy-Riemann equation, it seems that they still are too restrictive, particularly condition IV. It requires that the eigenvalues of $P(\xi)$ are real. But the wave equation $\partial^2 v/\partial t^2 = \Delta v + g(x, t)$ can be represented as a system $\partial u/\partial t = P(D)u + f(x, t)$, where $u = (v, \partial v/\partial t)$, $f(x, t) = (0, g(x, t))$, and the matrix $P(\xi)$ consists of $(0, 1)$ and $(-|\xi|^2, 0)$ as first and second rows, respectively. Here $P(\xi)$ has only nonreal eigenvalues $\pm i|\xi|$. Yet the system satisfies Theorem 3.2 in that $u_h = 0$ and u_p can be constructed as in (3.16).

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Received September 28, 1973.

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