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INVERSION OF CONDITIONAL EXPECTATIONS

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By its definition a conditional expectation is the Radon-Nikodym derivative of a finite signed measure. In this paper an inversion formula is given for recapturing $E(Y|X)$ as an inverse Fourier transform of the function $E(e^{i(u,X)}Y)$, $u \in R^n$, where X is an random vector and Y is a random variable satisfying some regularity conditions.

1. Introduction. Let $(\Omega, \mathfrak{B}, P)$ be a probability space and let X be a k-dimensional random vector on $(\Omega, \mathfrak{B}, P)$, i.e., a measurable transformation of $(\Omega, \mathfrak{B})$ into $(R^k, \mathfrak{B}^k)$ where $\mathfrak{B}^k$ is the σ-algebra of Borel sets in the k-dimensional Euclidean space R^k. Assume that the probability distribution X is absolutely continuous with respect to the Lebesgue measure m_L on $(R^k, \mathfrak{B}^k)$. For a real valued random variable Y on $(\Omega, \mathfrak{B}, P)$ with $E(|Y|) < \infty$ let $E(Y|X)$ be the conditional expectation of Y given X which is given as a function on the value space R^k of X. For $u \in R^k$ let $(u, X) = \sum_{j=1}^k u_j X_j$. In this paper we show that if $E[e^{i(u,X)}Y]$ is a m_L-integrable function of u on R^k then a version of $E(Y|X)$ is given by

$$(1.1) \qquad E(Y|X)(\xi) = \left(\frac{dP_X}{dm_L}(\xi)\right)^{-1} \frac{1}{(2\pi)^k} \int_{R^k} e^{-i(u,\xi)} E[e^{i(u,X)}Y] m_L(du)$$

for $\xi \in R^k$ assuming that $(dP_X/dm_L)(\xi) > 0$ for a.e. ξ in $(R^k, \mathfrak{B}^k, m_L)$. (Our conditional expectation $E(Y|X)$ given as a function on R^k rather than one on Ω is the "conditional expectation in the wide sense" in the terminology of [2].)

In preparation for (1.1) which is given in Theorem 2 in §3 we show in Theorem 1 in §3 that if the characteristic function (i.e., the Fourier transform) φ of a finite measure Φ on $(R^k, \mathfrak{B}^k)$ is m_L-integrable on R^k then Φ is absolutely continuous with respect to m_L on $(R^k, \mathfrak{B}^k)$ and a version of the Radon-Nikodym derivative of Φ with respect to m_L is given by the inverse Fourier transform of φ. We base this result on the Lévy-Haviland Theorem for the inversion of Fourier transforms of finite measures on $(R^k, \mathfrak{B}^k)$.

The substance of Propositions 1, 2, and 3 in §2 concerning conditional probabilities, conditional expectations and regular conditional distributions given as functions on the value space of X is well-known. We included them here in order to state them in a convenient form.

This research is an attempt at justifying a calculus of Wiener integral originated by M. D. Donsker. Its applications to conditional function space integrals will appear in a subsequent paper.

2. Integration of conditional expectations. Throughout §2 we write $(\Omega, \mathfrak{B}, P)$ for a probability space and X and Y for two measurable transformations of $(\Omega, \mathfrak{B})$ into two arbitrary measurable spaces $(S, \mathfrak{F})$ and $(T, \mathfrak{G})$ respectively unless further specified. We write P_X and P_Y for the probability measures on $(S, \mathfrak{F})$ and $(T, \mathfrak{G})$ determined by X and Y respectively, i.e.,

$$P_X(F) = P(X^{-1}(F)) \qquad \text{for} \quad F \in \mathfrak{F} \tag{2.1}$$

and similarly for P_Y.

DEFINITION 1. For $G \in \mathfrak{G}$ fixed, the conditional probability of Y being in G given X, written $P(Y \in G \mid X)$, is defined to be any real valued $\mathfrak{F}$-measurable and P_X-integrable function ψ on S such that

$$P(Y^{-1}(G) \cap X^{-1}(F)) = \int_F \psi(\xi) P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{F} .$$

From the Radon-Nikodym Theorem follows that such a function ψ always exists and is determined uniquely up to a null set of $(S, \mathfrak{F}, P_X)$. We shall use $P(Y \in G \mid X)$ to mean either the class of all such functions ψ or a particular member in it depending on the context. Thus

$$P(Y^{-1}(G) \cap X^{-1}(F)) = \int_F P(Y \in G \mid X)(\xi) P_X(d\xi) \ \text{for} \quad F \in \mathfrak{F} . \tag{2.2}$$

DEFINITION 2. Let Z be a real valued random variable on $(\Omega, \mathfrak{B}, P)$ with $E(|Z|) < \infty$. The conditional expectation of Z given X, written $E(Z \mid X)$, is defined to be any real valued $\mathfrak{F}$-measurable and P_X-integrable function ψ on S such that

$$\int_{X^{-1}(F)} Z(\omega) P(d\omega) = \int_F \psi(\xi) P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{F} .$$

The same remark as the one following Definition 1 holds here too and we have

$$\int_{X^{-1}(F)} Z(\omega) P(d\omega) = \int_F E(Z \mid X)(\xi) P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{F} . \tag{2.3}$$

DEFINITION 3. By the regular conditional distribution of Y given X, written $P(Y \mid X)$, we mean a real valued function ψ on $\mathfrak{G} \times S$ such that

1° for every $G \in \mathfrak{G}$, $\psi(G, \cdot)$ is a version of $P(Y \in G \mid X)$

2° for every $\xi \in S$, $\psi(\cdot, \xi)$ is a probability measure on $(T, \mathfrak{G})$.

Thus when we need to indicate the arguments of $P(Y|X)$, we write $P(Y|X)(G, \xi)$ for $(G, \xi) \in \mathfrak{G} \times S$. It is known that a function ψ satisfying the conditions 1° and 2° of Definition 3 always exists whenever the value space $(T, \mathfrak{G})$ of Y is a Borel space and in particular when $(T, \mathfrak{G}) = (R^k, \mathfrak{B}^k)$. For a proof of this statement see [1]. The proof of Proposition 1 below which relates the regular conditional distribution to the conditional expectation is parallel to the proof of the corresponding theorem in which these two are given as functions on Ω rather than as functions on the value space S of X. (See for instance Proposition 4.28 in [1].) We give the proof here for the sake of completeness.

PROPOSITION 1. *Let f be a measurable transformation of $(T, \mathfrak{G})$ into $(R^1, \mathfrak{B}^1)$ and $f \in L_1(T, \mathfrak{G}, P_Y)$. If $P(Y|X)$ exists then*

$$E(f \circ Y|X)(\xi) = \int_T f(\eta)P(Y|X)(d\eta, \xi) \quad \textit{for a.e. } \xi \in (S, \mathfrak{F}, P_X) . \tag{2.4}$$

Proof. Consider the case where $f = \chi_G$ for some $G \in \mathfrak{G}$. Then for every $F \in \mathfrak{F}$ we have by (2.3)

$$\begin{aligned} \int_F E(f \circ Y|X)(\xi)P_X(d\xi) &= \int_{X^{-1}(F)} \chi_G(Y(\omega))P(d\omega) \\ &= P(Y^{-1}(G) \cap X^{-1}(F)) . \end{aligned} \tag{2.5}$$

On the other hand, by 2° and then 1° of Definition 3,

$$\begin{aligned} \int_T f(\eta)P(Y|X)(d\eta, \xi) &= \int_T \chi_G(\eta)P(Y|X)(d\eta, \xi) \\ &= P(Y|X)(G, \xi) = P(Y \in G|X)(\xi) \end{aligned}$$

so that by (2.2) we have

$$\int_F \left\{ \int_T f(\eta)P(Y|X)(d\eta, \xi) \right\} P_X(d\xi) = P(Y^{-1}(G) \cap X^{-1}(F)) . \tag{2.6}$$

Thus the left side of (2.5) is equal to that of (2.6) for every $F \in \mathfrak{F}$ so that (2.4) holds in this case.

Now that (2.4) holds when f is the characteristic function of a member of $\mathfrak{G}$ we can follow the usual procedure in integration theory to show that (2.4) holds for nonnegative simple functions on T, nonnegative $\mathfrak{G}$-measurable function on T and finally real valued $\mathfrak{G}$-measurable functions on T. Since $f \in L_1(T, \mathfrak{G}, P_Y)$, both sides of (2.4) always exist and are finite. In passing from nonnegative simple functions on T to nonnegative $\mathfrak{G}$-measurable function on T we use the Monotone Convergence Theorem for the conditional expectation which states that if $\{Z_n, n = 1, 2, \cdots\} \subset L_1(\Omega, \mathfrak{B}, P)$ and $Z_n(\omega) \uparrow Z_0(\omega)$ for a.e.

$\omega \in (\Omega, \mathfrak{B}, P)$ then $E(Z_n|X)(\xi) \uparrow E(Z_0|X)(\xi)$ for a.e. $\xi \in (S, \mathfrak{F}, P_X)$ and which can be proved readily.

PROPOSITION 2. *Let* $\sigma(\mathfrak{F} \times \mathfrak{G})$ *be the* σ*-algebra of subsets of* $S \times T$ *generated by the semialgebra* $\mathfrak{F} \times \mathfrak{G}$ *and let* $P_{[X,Y]}$ *be the probability measure on* $(S \times T, \sigma(\mathfrak{F} \times \mathfrak{G}))$ *determined by the measurable transformation* $[X, Y]$ *of* $(\Omega, \mathfrak{B})$ *into* $(S \times T, \sigma(\mathfrak{F} \times \mathfrak{G}))$. *Let* f *be a measurable transformation of* $(S \times T, \sigma(\mathfrak{F} \times \mathfrak{G}))$ *into* $(R^1, \mathfrak{B}^1)$. *If* $P(Y|X)$ *exists then*

$$\begin{aligned} E(f \circ [X, Y]) &= \int_{S \times T} f(\xi, \eta) P_{[X,Y]}(d(\xi, \eta)) \\ &= \int_S \left\{ \int_T f(\xi, \eta) P(Y|X)(d\eta, \xi) \right\} P_X(d\xi) \end{aligned} \tag{2.7}$$

in the sense that the existence of any member in (2.7) *implies that of the other and the equality of all.*

Proof. The first equality in (2.7) is standard. Let us prove the second. Consider the case where

$$f(\xi, \eta) = \chi_{F \times G}(\xi, \eta) = \chi_F(\xi)\chi_G(\eta) \quad \text{for} \quad (\xi, \eta) \in S \times T$$

where $F \in \mathfrak{F}$ and $G \in \mathfrak{G}$. Then by 2° and 1° of Definition 3 and by (2.2)

$$\begin{aligned} &\int_S \left\{ \int_T f(\xi, \eta) P(Y|X)(d\eta, \xi) \right\} P_X(d\xi) \\ &\quad = \int_S \chi_F(\xi) \left\{ \int_T \chi_G(\eta) P(Y|X)(d\eta, \xi) \right\} P_X(d\xi) \\ &\quad = \int_S \chi_F(\xi) P(Y|X)(G, \xi) P_X(d\xi) \\ &\quad = \int_F P(Y \in G|X)(\xi) P_X(d\xi) \\ &\quad = P(Y^{-1}(G) \cap X^{-1}(F)) \end{aligned}$$

while

$$\begin{aligned} &\int_{S \times T} f(\xi, \eta) P_{[X,Y]}(d(\xi, \eta)) \\ &\quad = \int_{S \times T} \chi_{F \times G}(\xi, \eta) P_{[X,Y]}(d(\xi, \eta)) \\ &\quad = P_{[X,Y]}(F \times G) \\ &\quad = P\{\omega \in \Omega; X(\omega) \in F \text{ and } Y(\omega) \in G\} \\ &\quad = P(Y^{-1}(G) \cap X^{-1}(F)) \end{aligned}$$

so that the second equality in (2.7) holds for this particular case.

We then proceed as in the proof of the Fubini Theorem to an arbitrary real valued $\sigma(\mathfrak{F} \times \mathfrak{S})$-measurable function f on $S \times T$ to complete the proof.

PROPOSITION 3. *Let Z be a real valued random variable on $(\Omega, \mathfrak{B}, P)$ with $E(|Z|) < \infty$ and let g be a measurable transformation of $(S, \mathfrak{F})$ into $(R^1, \mathfrak{B}^1)$. Then*

$$E[(g \circ X)Z] = \int_S g(\xi)E(Z\,|\,X)(\xi)P_X(d\xi) \tag{2.8}$$

in the sense that the existence of one side implies that of the other and the equality of the two.

Proof. Let us define a set function Φ on $\mathfrak{B}$ by

$$\Phi(B) = \int_B Z(\omega)P(d\omega) \qquad \text{for} \quad B \in \mathfrak{B}\,.$$

Since $E(|Z|) < \infty$, Φ is a finite signed measure on $(\Omega, \mathfrak{B})$ which is absolutely continuous with respect to P and has Z as its Radon-Nikodym derivative with respect to P. Thus for the real valued random variables $g \circ X$ and Z on $(\Omega, \mathfrak{B}, P)$ we have

$$E[(g \circ X)Z] = \int_\Omega g(X(\omega))Z(\omega)P(d\omega) = \int_\Omega g(X(\omega))\Phi(d\omega)$$

in the sense that the existence of one member implies that of the others and the equality of all. Then, to prove (2.8) it suffices to show that

$$\int_\Omega g(X(\omega))\Phi(d)\omega = \int_S g(\xi)E(Z\,|\,X)(\xi)P_X(d\xi) \tag{2.9}$$

in the sense that the existence of one side implies that of the other and the equality of the two.

Let us consider the case where $g = \chi_F$ for some $F \in \mathfrak{F}$. Then

$$\begin{aligned}\int_\Omega g(X(\omega))\Phi(d\omega) &= \int_\Omega \chi_F(X(\omega))\Phi(d\omega) = \int_{X^{-1}(F)} \Phi(d\omega)\\ &= \int_{X^{-1}(F)} Z(\omega)P(d\omega) = \int_F E(Z\,|\,X)(\xi)P_X(d\xi)\\ &= \int_S g(\xi)E(Z\,|\,X)(\xi)P_X(d\xi)\end{aligned}$$

by (2.3) so that (2.9) holds. Following the standard procedure in integration theory we proceed from this particular case to nonnegative simple functions on S, nonnegative $\mathfrak{F}$-measurable functions on S and finally real valued $\mathfrak{F}$-measurable functions on S to complete the proof.

3. Inversion of conditional expectations. It is well-known that if the characteristic function φ of a distribution function F on R^1 is m_L-integrable on R^1, then F is absolutely continuous and

$$F'(\xi) = \frac{1}{2\pi}\int_{R^1} e^{-i(\xi,\eta)}\varphi(\eta)m_L(d\eta) \quad \text{for a.e. } \xi \in (R^1, \mathfrak{B}^1, m_L)\,.$$

Let Φ be a finite measure on $(R^k, \mathfrak{B}^k)$ and let φ be its characteristic function, i.e.,

$$\varphi(\eta) = \int_{R^k} e^{i(\xi,\eta)}\Phi(d\xi) \qquad \text{for} \quad \eta \in R^k \tag{3.1}$$

where $(\xi, \eta) = \sum_{j=1}^k \xi_j\eta_j$. According to the Lévy-Haviland Inversion Theorem (see [3] and [4])

$$\begin{aligned}&\int_{R^k}\prod_{j=1}^k \tilde{\chi}_{a_j,b_j}(\xi_j)\Phi(d\xi)\\ &\quad = \lim_{h\to\infty}\frac{1}{(2\pi)^k}\int_{C_h}\prod_{j=1}^k \frac{e^{-ib_j\eta_j} - e^{-ia_j\eta_j}}{-i\eta_j}\varphi(\eta)m_L(d\eta)\end{aligned} \tag{3.2}$$

for any $a_j, b_j \in R^1$, $a_j < b_j$, $j = 1, 2, \cdots, k$, where

$$C_h = (-h, h) \times \cdots \times (-h, h) \subset R^k \qquad \text{with} \quad h > 0\,, \tag{3.3}$$

and the modified characteristic function $\tilde{\chi}_{a_j,b_j}$ is defined by

$$\tilde{\chi}_{a_j,b_j}(\eta_j) = \begin{cases} 1 & \text{for} \quad \eta_j \in (a_j, b_j) \\ 0 & \text{for} \quad \eta_j \in [a_j, b_j]^c \\ \dfrac{1}{2} & \text{for} \quad \eta_j = a_j \text{ and for } \eta_j = b_j\,. \end{cases} \tag{3.4}$$

From (3.2) we derive the following:

THEOREM 1. *If the characteristic function φ of a finite measure Φ on $(R^k, \mathfrak{B}^k)$ is m_L-integrable on R^k, then Φ is absolutely continuous with respect to m_L on $(R^k, \mathfrak{B}^k)$ and a version of the Radon-Nikodym derivative of Φ with respect to m_L is given by*

$$\frac{d\Phi}{dm_L}(\xi) = \frac{1}{(2\pi)^k}\int_{R^k} e^{-i(\xi,\eta)}\varphi(\eta)m_L(d\eta) \qquad \textit{for} \quad \xi \in R^k\,. \tag{3.5}$$

Proof. Since the jth factor of the product in the integrand on the right side of (3.2) is a bounded continuous function of $\eta_j \in R^1$, if we assume the m_L-integrability of φ on R^k then the integrand on the right side of (3.2) is m_L-integrable on R^k so that (3.2) reduces to

$$
\begin{aligned}
(3.6)\qquad & \int_{R^k} \prod_{j=1}^{k} \tilde{X}_{a_j,b_j}(\xi_j)\Phi(d\xi) \\
& = \frac{1}{(2\pi)^k} \int_{R^k} \prod_{j=1}^{k} \frac{e^{-ib_j\eta_j} - e^{-ia_j\eta_j}}{-i\eta_j} \varphi(\eta) m_L(d\eta) .
\end{aligned}
$$

To show that Φ is absolutely continuous with respect to m_L on $(R^k, \mathfrak{B}^k)$ let $A \in \mathfrak{B}^k$ and $m_L(A) = 0$. We proceed to show that $\Phi(A) = 0$. Let $\mathfrak{A}$ be the algebra of subsets of R^k which are unions of finitely many disjoint half open and half closed intervals $(a_1, b_1] \times \cdots \times (a_k, b_k]$ in R^k. Then the σ-algebra of subsets of R^k generated by $\mathfrak{A}$ is precisely our $\mathfrak{B}^k$. Let $\varepsilon > 0$ be arbitrarily given. Since Φ is a finite measure on $\mathfrak{B}^k$ and since $m_L(A)$ is finite (in fact equal to zero), $(\Phi + m_L)(A)$ is finite so that there exists some $B \in \mathfrak{A}$ such that

$$
(3.7)\qquad (\Phi + m_L)(A \Delta B) < \varepsilon
$$

where $A \Delta B$ is the symmetric difference between A and B. Now (3.7) implies that $\Phi(A \Delta B) < \varepsilon$ so that

$$
(3.8)\qquad \Phi(A) \leqq \Phi(B) + \Phi(A \Delta B) < \Phi(B) + \varepsilon .
$$

It also implies that $m_L(A \Delta B) < \varepsilon$ so that in view of $m_L(A) = 0$ we have

$$
m_L(B) < \varepsilon .
$$

Since B is the union of finitely many, say m, disjoint half open half closed intervals, there exist m open intervals $B^{(n)}$, $n = 1, 2, \cdots, m$, such that

$$
(3.9)\qquad B \subset \bigcup_{n=1}^{m} B^{(n)} \quad \text{and} \quad m_L(B) < \sum_{n=1}^{m} m_L(B^{(n)}) < \varepsilon .
$$

Let each $B^{(n)}$ be given as

$$
(3.10)\qquad B^{(n)} = \zeta^{(n)} + C^{(n)}
$$

where

$$
(3.11)\qquad \zeta^{(n)} \in R^k \text{ and } C^{(n)} = (-h_1^{(n)}, h_1^{(n)}) \times \cdots \times (-h_k^{(n)}, h_k^{(n)}) \subset R^k .
$$

In view of the openness of $C^{(n)}$ and the definition of $\tilde{\chi}_{a_j,b_j}$ by (3.4) we have from (3.6)

$$
\begin{aligned}
& \Phi(\zeta^{(n)} + C^{(n)}) \\
(3.12)\qquad & \leqq \frac{1}{(2\pi)^k} \int_{R^k} \prod_{j=1}^{k} \frac{e^{-i(\zeta_j^{(n)}+h_j^{(n)}\eta_j} - e^{-i(\zeta_j^{(n)}-h_j^{(n)}\eta_j)}}{-i\eta_j} \\
& = \frac{m_L(C^{(n)})}{(2\pi)^k} \int_{R^k} \prod_{j=1}^{k} \frac{\sin \eta_j h_j^{(n)}}{\eta_j h_j^{(n)}} e^{-i\zeta_j^{(n)}\eta_j} \varphi(\eta) m_L(d\eta) .
\end{aligned}
$$

Since $|(\eta_j h_j^{(n)})^{-1} \sin \eta_j h_j^{(n)}| \leqq 1$ for $\eta_j \in R^1$ and since $m_L(C^{(n)}) = m_L(B^{(n)})$ we have

$$(3.13) \qquad \Phi(B^{(n)}) = \Phi(\zeta^{(n)} + C^{(n)}) \leqq \frac{m_L(B^{(n)})}{(2\pi)^k} \int_{R^k} |\varphi(\eta)| \, m_L(d\eta) \, .$$

From (3.9) and (3.13) we obtain

$$(3.14) \qquad \Phi(B) \leqq \sum_{n=1}^{m} \Phi(B^{(n)}) \leqq \frac{\varepsilon}{(2\pi)^k} \int_{R^k} |\varphi(\eta)| \, m_L(d\eta) \, .$$

Using (3.14) in (3.8) we have

$$\Phi(A) < \varepsilon \left\{ \frac{1}{(2\pi)^k} \int_{R^k} |\varphi(\eta)| \, m_L(d\eta) + 1 \right\} .$$

From the arbitrariness of $\varepsilon > 0$ we have $\Phi(A) = 0$. This proves the absolute continuity of Φ with respect to m_L on $(R^k, \mathfrak{B}^k)$.

To obtain the Radon-Nikodym derivative of Φ with respect to m_L on $(R^k, \mathfrak{B}^k)$, let us observe first that the absolute continuity of Φ with respect to m_L implies that the Φ measure of the boundary of the open interval $C^{(n)}$ in (3.11) is equal to zero. Thus in (3.12) the strict equality actually holds. If we apply this improved (3.12) to $\zeta + C_h$ where ζ is an arbitrary point in R^k and C_h is an open interval in R^k as given by (3.3) then we have

$$(3.15) \qquad \begin{aligned} \int_{\zeta + C_h} \frac{d\Phi}{dm_L}(\xi) m_L(d\xi) &= \Phi(\zeta + C_h) \\ &= \frac{m_L(C_h)}{(2\pi)^k} \int_{R^k} \prod_{j=1}^{k} \frac{\sin \eta_j h}{\eta_j h} e^{-i\zeta_j \eta_j} \varphi(\eta) m_L(d\eta) \, . \end{aligned}$$

Let $h \to 0$ on both sides of (3.15). On the one hand we have

$$(3.16) \qquad \begin{aligned} \lim_{h \to 0} \frac{1}{m_L(C_h)} \int_{\zeta + C_h} \frac{d\Phi}{dm_L}(\xi) m_L(d\xi) \\ = \frac{d\Phi}{dm_L}(\zeta), \text{ for a.e. } \zeta \in (R^k, \mathfrak{B}^k, m_L) \end{aligned}$$

and on the other hand by the Dominated Convergence Theorem

$$(3.17) \qquad \begin{aligned} \lim_{h \to 0} \frac{1}{(2\pi)^k} \int_{R^k} \prod_{j=1}^{k} \frac{\sin \eta_j h}{\eta_j h} e^{-i\zeta_j \eta_j} \varphi(\eta) m_L(d\eta) \\ = \frac{1}{(2\pi)^k} \int_{R^k} e^{-(\zeta, \eta)} \varphi(\eta) m_L(d\eta) \, . \end{aligned}$$

Using (3.16) and (3.17) in (3.15) we have

$$\frac{d\Phi}{dm_L}(\zeta) = \frac{1}{(2\pi)^k} \int_{R^k} e^{-(\zeta, \eta)} \varphi(\eta) m_L(d\eta) \text{ for a.e. } \zeta \in (R^k, \mathfrak{B}^k, m_L) \, .$$

This completes the proof of the theorem.

By means of Proposition 2 and Theorem 1 our inversion theorem for conditional expectation can be derived now.

THEOREM 2. *Let Y be a real valued random variable on a probability space $(\Omega, \mathfrak{B}, P)$ with $E(|Y|) < \infty$ and let X be a k-dimensional random vector i.e., a measurable transformation of $(\Omega, \mathfrak{B})$ into $(R^k, \mathfrak{B}^k)$. Assume that the probability distribution P_X of X is absolutely continuous with respect to m_L on $(R^k, \mathfrak{B}^k)$. If $E[e^{i(u,X)}Y]$ is a m_L-integrable function of u on R^k then a version of the conditional expectation of Y given X, $E(Y|X)$, is given by*

$$E(Y|X)(\xi)\frac{dP_X}{dm_L}(\xi) = \frac{1}{(2\pi)^k}\int_{R^k} e^{-i(u,\xi)}E[e^{i(u,X)}Y]m_L(du) \tag{3.18}$$

for $\xi \in R^k$.

Proof. Since Y is a measurable transformation of $(\Omega, \mathfrak{B})$ into $(R^1, \mathfrak{B}^1)$ which is a Borel space, the regular conditional distribution of Y given X, $P(Y|X)$, exists. With fixed $u \in R^k$ consider a complex valued function f on $R^k \times R^1$ defined by

$$f(\xi, \eta) = e^{i(u,\xi)}\eta \qquad \text{for } \xi \in R^k \text{ and } \eta \in R^1 .$$

Applying (2.7) of Proposition 2 and (2.4) of Proposition 1 to the real and the imaginary parts of f we obtain

$$\begin{aligned} E[e^{i(u,X)}Y] &= \int_{R^k} e^{i(u,\xi)}\left\{\int_{R^1}\eta P(Y|X)(d\eta, \xi)\right\}P_X(d\xi) \\ &\int_{R^k} e^{i(u,\xi)}E(Y|X)(\xi)P_X(d\xi) . \end{aligned} \tag{3.19}$$

Consider a set function Φ defined on $\mathfrak{B}^k$ by

$$\Phi(F) = \int_F E(Y|X)(\xi)P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{B}^k . \tag{3.20}$$

Since $E(Y|X)$ is P_X integrable on R^k, Φ is a finite signed measure on $(R^k, \mathfrak{B}^k)$ which is absolutely continuous with respect to P_X on $(R^k, \mathfrak{B}^k)$ and has $E(Y|X)$ as its Radon-Nikodym derivative with respect to P_X. According to (3.19), $E[e^{i(u,X)}Y]$, $u \in R^k$, is the characteristic function of Φ. Under the hypothesis of the theorem, this characteristic function is m_L-integrable over R^k. Applying Theorem 1 to the positive and the negative part of Φ we obtain

$$\begin{aligned}(3.21)\quad &\frac{d\Phi}{dm_L}(\xi) \\ &= \frac{1}{(2\pi)^k}\int_{R^k} e^{-i(u,\xi)} E[e^{i(u,X)}Y] m_L(du) \text{ for a.e. } \xi \in (R^k, \mathfrak{B}^k, m_L) \,.\end{aligned}$$

Since by (3.20)

$$\frac{d\Phi}{dm_L}(\xi) = E(Y|X)(\xi)\frac{dP_X}{dm_L}(\xi) \quad \text{for a.e. } \xi \in (R^k, \mathfrak{B}^k, m_L)$$

we have (3.18).

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