PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 2, 1974

FUNDAMENTAL GROUPS OF COMPACT COMPLETE LOCALLY AFFINE COMPLEX SURFACES, II

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The present article is a continuation of a recent paper by J. P. Fillmore and the author on properly-acting groups Γ of complex affine motions of C^2 such that $\Gamma\backslash C^2$ is compact. In that paper, it was proved that such a group has a normal subgroup Γ_0 of finite index which is either free abelian of rank four or has generators A, B, C, D, with relations

$$ABA^{-1}B^{-1} = C^k (k \geqq 1)$$

and C and D central.

Here we build on this description up to finite index to determine the groups Γ themselves.

1. Introduction. A complete locally affine complex surface X has an orbit-space representation $X = \Gamma\backslash C^2$, where the fundamental group Γ of X is a properly-acting group of complex affine transformations of C^2. Two such surfaces $\Gamma\backslash C^2$ and $\Gamma'\backslash C^2$ are isomorphic if and only if Γ and Γ' are conjugate subgroups of the group $A(2, C)$ of all complex affine motions of C^2. Elements of $A(2, C)$ are taken as nonsingular complex matrices $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$ and elements of C^2 are taken as column vectors $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$; then $A(2, C)$ acts on the left on C^2 in the usual way.

Fillmore and Scheuneman [2] have shown the following:

THEOREM 1.1. *Let $\Gamma\backslash C^2$ be a compact complete locally affine complex surface. Then:*

(i) *Γ is conjugate in $A(2, C)$ to a subgroup of the group G of all matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$, and hence may be considered a subgroup of G;*

(ii) *The homomorphism $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix} \mapsto d$, when restricted to Γ, has a kernel Γ_0 which is either free abelian of rank four, or has generators A, B, C, D with relations $ABA^{-1}B^{-1} = C^k (k \geqq 1)$ and C and D central in Γ_0;*

(iii) *The image of Γ under the above homomorphism is a finite cyclic group of order $t = 1, 2, 3, 4, 5, 6, 8, 10,$ or 12.*

In view of this result, we immediately obtain a description as follows: Γ is generated by S, A, B, C, D, where S is a preimage of a generator of Γ/Γ_0, A, B, C, D is a set of generators of Γ_0, $S^t \in \Gamma_0$, and $S\Gamma_0 S^{-1} = \Gamma_0$. This description is too general, for not all groups satisfying these conditions are of the kind we are interested in here. It must be refined by imposing the conditions that Γ be a subgroup of the group G of 1.1 (i), and that it act properly on C^2 and have compact orbit space.

In §2, we normalize Γ_0 up to conjugation. This normalization leads to the elimination of the cases $t = 5, 8, 10, 12$ in 1.1 (iii). In succeeding sections we describe the generators and relations for Γ using the normalization. Our study of the problem at hand uses a blend of abstract group theory, which is more efficient than matrix calculations, together with calculations involving embedded groups (groups of matrices), this being an essential ingredient in the problem.

2. Normalization of Γ_0.

LEMMA 2.1. *Let* $T = \begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ *in* $A(2, C)$ *be different from the identity. Then* T *has a fixed point in* C^2 *if and only if*

(1) $b \neq 0$ *and* $s = 0$, *in case* $d = 1$;

(2) $bs - (d - 1)r = 0$, *in case* $d \neq 1$.

Proof. A point (x, y) of C^2 is fixed under T if $x + by + r = x$ and $dy + s = y$. It is easy to check that if such a point exists, the conditions hold. Conversely, if the conditions hold, the points $(0, -r/b)$ and $(0, -s/(d - 1))$ are fixed in the respective cases.

Let Γ_0 denote a subgroup of

$$G_0 = \left\{ \begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \middle| b, r, s \text{ complex} \right\} \subset A(2, C)$$

which acts properly on C^2 and such that $\Gamma_0 \backslash C^2$ is compact.

PROPOSITION 2.2.

(i) *Suppose* Γ_0 *is generated by four element* A, B, C, *and* D *with relations* $ABA^{-1}B^{-1} = C^k$ *and* C *and* D *central for some fixed* $k \geqq 1$. *Then* Γ_0 *is conjugate in* $A(2, C)$ *to a group where*

$$A = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & \lambda b - k & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \textit{ and } D = \begin{pmatrix} 1 & 0 & d' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) *Suppose Γ_0 is free abelian of rank four with generators A, B, C, and D. Then Γ_0 is conjugate in $A(2, C)$ to a group where*

$$A = \begin{pmatrix} 1 & \lambda & a' \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & \lambda b & b' \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & \lambda c & c' \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},\ \text{and}\ D = \begin{pmatrix} 1 & \lambda d & d' \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. (i) The commutator subgroup of G consists of matrices of the form $\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, thus C^k is a matrix of this form. The 12- and 23- entries of matrices in G add when the matrices are multiplied, so we have that C itself is of the form $C = \begin{pmatrix} 1 & 0 & c' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $c' \neq 0$. Conjugate C by $P = \begin{pmatrix} 1/c' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in $A(2, C)$; this gives $PCP^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and preserves G, $PGP^{-1} = G$. Hence we may assume that C has $c' = 1$. Note that C is central. Set $A = \begin{pmatrix} 1 & \lambda & a' \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & \mu & b' \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & \delta & d' \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}$. From $ABA^1B^{-1} = C^k$ we obtain $\lambda b - \mu a = k \neq 0$. Since D commutes with A and B, we have, respectively, $\lambda d = \delta a$ and $\mu d = \delta b$. Where $d \neq 0$, we would have $k = \lambda b - \mu a = (\delta ab - \delta ba)/d = 0$, a contradiction. Hence, $d = 0$. By 2.1 we must have $\delta = 0$ also. Now C and D have the desired form. From Lemma 3.1 of [2] we have $a \neq 0$. Conjugate by $P = \begin{pmatrix} 1 & -a'/a & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{pmatrix}$. P commutes with C and D, so these are unchanged; $PAP^{-1} = \begin{pmatrix} 1 & \lambda a & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and PBP^{-1} has the same form as B. Hence we may assume A has $a = 1$ and $a' = 0$. Set $P = \begin{pmatrix} 1 & \lambda s & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$. P commutes with A, C, and D, while $PBP^{-1} = \begin{pmatrix} 1 & \mu & b' + (\lambda b - \mu)s \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$. Now $ABA^{-1}B^{-1} = C^k$ has its 13-entry equal to $\lambda b - \mu = k \neq 0$. Set $s = -b'/(\lambda b - \mu)$ and $r = 0$. Conjugation by P leaves A, C, and D unchanged and replaces B by a matrix of the same form with $b' = 0$. We still have $\lambda b - \mu = k$, so $\mu = \lambda b - k$.

(ii) Select an arbitrary set of generators of Γ_0. By Lemma 3.1 of [2], one of these generators must have its 23-entry different from zero. Call this one $A = \begin{pmatrix} 1 & \lambda & a' \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$. For $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we have

$PAP^{-1} = \begin{pmatrix} 1 & \lambda a & a' \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Hence we may take $a = 1$, so $A = \begin{pmatrix} 1 & \lambda & a' \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. A matrix of G which commutes with A necessarily has the form $\begin{pmatrix} 1 & \lambda x & x' \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$, and all such matrices commute. Hence B, C, and D have the form indicated.

It is easily verified that conversely, if A, B, C, D are as in 2.2, then the group they generate acts properly on $\boldsymbol{C}^2$ provided b and d' are not real in case (i) and $\left\{\begin{pmatrix} a' \\ 1 \end{pmatrix}, \begin{pmatrix} b' \\ b \end{pmatrix}, \begin{pmatrix} c' \\ c \end{pmatrix}, \begin{pmatrix} d' \\ d \end{pmatrix}\right\}$ is linearly independent over the real numbers in case (ii).

Proposition 2.2 leads to an improvement of 1.1 and the result in [2] as follows:

THEOREM 2.3. *Let $\Gamma\backslash\boldsymbol{C}^2$ be a compact complete locally affine complex surface. Then* (i) *and* (ii) *of Theorem* 1.1 *hold; and* (iii) *t, which is the index of Γ_0 in Γ, can take only the values* 1, 2, 3, 4, *and* 6.

Proof. We refer to the proof of Theorem 4.1 in [2] where it is shown that $t = 1, 2, 3, 4, 5, 6, 8, 10$, or 12. We must eliminate the cases $t = 5, 8, 10, 12$. Now, the restriction on t comes from the inequality $\varphi(t) \leqq r$, where is Euler's totient and r is the rank of the additive subgroup of $\boldsymbol{C}$ generated by the 12-entries of elements of Γ_0. The cases $t = 5, 8, 10, 12$ can occur only when $r = 4$. By Proposition 2.2, this can occur only when Γ_0 is abelian and, in the notation there, $\lambda \neq 0$ and $1, b, c, d$ are linearly independent over the integers. The only element of such a Γ_0 having its 12- and 23-entries zero is the identity. Suppose now that such a Γ_0 occurs as a subgroup of Γ of index $t > 1$. Let $S = \begin{pmatrix} 1 & f & u \\ 0 & w & v \\ 0 & 0 & 1 \end{pmatrix}$ be the element of Γ described after Theorem 1.1; then S^t is in Γ_0. We have $S^t = \begin{pmatrix} 1 & 0 & u' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $u' = tu + tfu/(1 - w)$, since $w^t = 1$. Hence $u' = 0$ and $fv - (w - 1)u = 0$. This contradicts Lemma 2.1. Thus, such a Γ_0 has no proper extensions to a group Γ.

3. Generators and relations. In this section, we shall use the matrix description of the generators S, A, B, C, D of Γ that we have obtained in order to delimit the relations that can occur for our fundamental groups Γ, as opposed to the general relations implied by the observation following Theorem 1.1.

THEOREM 3.1. *Let Γ be a subgroup of* $A(2, \boldsymbol{C})$ *which acts properly on* $\boldsymbol{C}^2$ *and has compact orbit space. Then Γ has generators S, A, B, C, D with relations among the following*:

(1) *If the group Γ_0 generated by A, B, C, D is free abelian, we have three cases as follows*:

(a) *If the parameter λ of Proposition* 2.2 (ii) *is zero, then the groups are given in* [1].

(b) *If $\lambda \neq 0$, we must have $S^2 = D$, $SD = DS$, and* (i) $SAS^{-1} = A^{-1}D^{\alpha}$, $SBS^{-1} = B^{-1}D^{\beta}$, $SCS^{-1} = C^{-1}D^{\gamma}$, *where α, β, γ are* 0 *or* 1, *or* (ii) $SC = CS$, $SAS^{-1} = A^{-1}C^{\alpha}D^{\beta}$, $SBS^{-1} = B^{-1}C^{\gamma}D^{\delta}$, *where $\alpha, \beta, \gamma, \delta$ are all* 0 *or* 1.

(2) *If the group Γ_0 generated by A, B, C, D has relations $ABA^{-1}B^{-1} = C^k$, C and D central in Γ_0, then the other relations may be taken as follows*: $SC = CS$, $SD = DS$, $S^t = C^rD^s$ (where, as before, $t = [\Gamma : \Gamma_0]$), *and*

(a) *if $t = 2$ then $SAS^{-1} = A^{-1}C^eD^f$ and $SBS^{-1} = B^{-1}C^gD^h$*,

(b) *if $t = 3$ then $SAS^{-1} = BC^eD^f$ and $SBS^{-1} = A^{-1}B^{-1}C^gD^h$*,

(c) *if $t = 4$ then $SAS^{-1} = BC^eD^f$ and $SBS^{-1} = A^{-1}C^gD^h$, and*

(d) *if $t = 6$ then $SAS^{-1} = BC^eD^f$ and $SBS^{-1} = A^{-1}BC^gD^h$.*

Here r may be chosen from among $0, 1, 2, \cdots, t-1$ and s may be chosen from among $0, 1 \cdots, r-1$, and in addition there are only finitely many choices for e, f, g, h.

REMARK. Not all the abstract groups above are of interest here; they must be subjected to the further condition that they act properly on $\boldsymbol{C}^2$. This is discussed in the next section.

Proof. We know from 1.1 that we may assume Γ is contained in the group G, and that Γ_0 is as in 2.2. Hence we take $S = \begin{pmatrix} 1 & x & z \\ 0 & \alpha & y \\ 0 & 0 & 1 \end{pmatrix}$, where α is a primitive tth root of unity, and study the implications of the conditions that $S^t \in \Gamma_0$ and $S\Gamma_0S^{-1} = \Gamma_0$.

(1) If Γ_0 is abelian, and (a) the parameter λ is zero, then the elements of Γ are hermitian, and this case is fully discussed in [1].

If Γ_0 is abelian and $\lambda \neq 0$, notice that since $S = \begin{pmatrix} 1 & x & z \\ 0 & \alpha & y \\ 0 & 0 & 1 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} 1 & -\alpha^{-1}x & \alpha^{-1}xy - z \\ 0 & \alpha^{-1} & -\alpha^{-1}y \\ 0 & 0 & 1 \end{pmatrix}$ and $S\begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}S^{-1} = \begin{pmatrix} 1 & \alpha^{-1}u & -\alpha^{-1}uy + w + vx \\ 0 & 1 & \alpha v \\ 0 & 0 & 1 \end{pmatrix}$, we must have $\alpha = \alpha^{-1} = -1$ if $S\Gamma_0S^{-1} = \Gamma_0$ (because elements of Γ_0 have the property that their 12-entries equal their 23-entries multiplied by λ).

Next notice that 2.2 says that when the rank of the additive subgroup of C generated by $1, b, c, d$ is 4, the group Γ_0 admits no proper extensions. Also, the rank of this group cannot be 1, for then $\Gamma_0\backslash C^2$ would not be compact (see [2], §3). We are therefore reduced to the following cases for an abelian Γ_0:

$$\text{(i)}\quad A = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & \lambda b & b' \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & \lambda c & b' \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{(ii)}\quad A = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & \lambda b & b' \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\lambda \neq 0$ and $1, b, c$ linearly independent over Z in case (a) and $1, b$ linearly independent over Z in case (b). Also, it has been determined that $S = \begin{pmatrix} 1 & x & z \\ 0 & -1 & y \\ 0 & 0 & 1 \end{pmatrix}$.

In case (b), we automatically have $SD = DS$; furthermore $SAS^{-1} = \begin{pmatrix} 1 & -\lambda & \lambda y + x \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $SBS^{-1} = \begin{pmatrix} 1 & -\lambda b & \lambda by + b' + bx \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$, and

$$SCS^{-1} = \begin{pmatrix} 1 & -\lambda c & \lambda cy + c' + cx \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

Knowing that $S\Gamma_0S^{-1} = \Gamma_0$, we must therefore have $SAS^{-1} = A^{-1}D^{\alpha}$, $SBS^{-1} = B^{-1}D^{\beta}$, $SCS^{-1} = C^{-1}D^{\gamma}$, as we see from a glance at the 23-entries of the above matrix. Now, the group Γ is generated by S, AD^m, BD^p, CD^q, D as well as by S, A, B, C, D; seeing that $SAD^mS^{-1} = A^{-1}D^{\alpha}D^m = A^{-1}D^{-m}D^{\alpha+2m}$, etc., we may assume that α, β, γ above are reduced independently modulo 2.

Now impose the condition $S^2 \in \Gamma_0$. Since $S^2 = \begin{pmatrix} 1 & 0 & 2z + xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we must have $S^2 = D^k$ for some k. Γ is generated by SD^m, A, B, C, D as well as by S, A, B, C, D, and $(SD^m)^2 = D^{k+2m}$, so k can be reduced modulo 2 also. If $k = 0$ then $S^2 = \mathrm{id}$, so $2z + xy = 0$, in which case $S \in \Gamma$ has a fixed point (see 2.1). Hence we may take $k = 1$ for purposes of this paper. This takes care of 3.1.1.b. (i).

Turning now to case (ii) where Γ_0 is abelian and $\lambda \neq 0$, we first impose the condition that $S^2 \in \Gamma_0$. This being the case, we must have $S^2 = C^aD^b$. Replacing S by $S_1 = SC^mD^n$, we have $S_1^2 = S^2C^{2m}D^{2n} = C^{a+2m}D^{b+2n}$, since $SC = CS$ and $SD = DS$. Hence a and b may be chosen to be 0 or 1. If $a = b = 0$ then S again has a fixed point. The other three cases give the same group, so we may take $S^2 = D$.

The other relations to contend with are those giving the form of SAS^{-1} and SBS^{-1}. A glance at 23-entries shows that we must have $SAS^{-1} = A^{-1}C^{a_1}D^{a_2}$ and $SBS^{-1} = B^{-1}C^{b_1}D^{b_2}$, and replacement of the generators S, A, B, C, D by S, AC^mD^n, BC^pD^q, C, D shows that a_1, a_2, b_1, b_2 may be independently taken to be 0 or 1. The case where Γ_0 is abelian is now taken care of.

When Γ_0 is nonabelian, we have $A = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & ab-k & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so that $AB = BAC^k$ and C and D are central in Γ_0. Then Γ_0 consists of all words $A^aB^bC^cD^d$. Two such words are equal only when the corresponding exponents are equal, and

$$(A^aB^bC^cD^d)(A^rB^sC^tD^u) = A^{a+r}B^{b+s}C^{c+t-rbk}D^{t+u} .$$

The elements $A_1 = A^mB^nC^eD^f$, $B_1 = A^pB^qC^gD^h$, $C_1 = C^{mq-np}$, and $D_1 = C^eD^{\pm 1}$ of Γ_0 may be seen to generate Γ_0 provided $mq - np = \pm 1$, and then they satisfy the same relations as A, B, C, D. If we were to choose these different generators for Γ_0, we could, according to 2.2, still conjugate them back to the canonical form above, but with possibly different values of the parameters a, b, and d.

Now look for $S = \begin{pmatrix} 1 & x & z \\ 0 & \alpha & y \\ 0 & 0 & 1 \end{pmatrix}$ such that $S^t \in \Gamma_0$(i.e., $\alpha^t = 1$) for $t = 2, 3, 4, 6$, and $S\Gamma_0S^{-1} = \Gamma_0$.

We must have $S^t = C^rD^s$. Replacing S by $S_1 = SC^mD^n$, we get $S_1^t = S^tC^{mt}D^{nt} = C^{r+mt}D^{s+nt}$, so that we may take r and s to be reduced modulo t. Then replacing C by $CD^p = C_1$, we have $S^t = C^rD^s = (CD^p)^rD^{s-pr} = C_1^rD^{s-pr}$, so that we may further take s to be reduced modulo r.

Next impose the condition $S\Gamma_0S^{-1} = \Gamma_0$. We automatically have $SC = CS$ and $SD = DS$, so this condition is the same as saying that $SAS^{-1} = A^mB^nC^eD^f$, and $SBS^{-1} = A^pB^qC^gD^h$, with $mq - np = 1$. (Since $SCS^{-1} = C$ and conjugation by S is an automorphism of Γ_0, $mq - np \neq -1$.) Then $\begin{pmatrix} m & n \\ p & q \end{pmatrix}$ is an integer matrix whose tth power is the identity ($t = 2, 3, 4$, or 6). There are lots of these; however it is well known that there is an integer matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $xw - yz = \pm 1$ such that $M\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} m & n \\ p & q \end{pmatrix}$, where $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ if $t = 2$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ if $t = 4$, $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ if $t = 3$, and $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ if $t = 6$. Let $S_1 = S$, $A_1 = A^xB^y$, $B_1 = A^zB^w$, $C_1 = C^{xw-yz}$, $D_1 = D$. Then, as noted above, these generate Γ, and A_1, B_1, C_1, D, satisfy the same relation as A, B, C, D. Now, however, we have

$$\begin{aligned} SA_1S^{-1} &= SA^xB^yS^{-1} \\ &= (SAS^{-1})^x(SBS^{-1})^y \\ &= (A^mB^n)^x(A^pB^q)^yP , \end{aligned}$$

where P involves powers of C and D. Modulo powers of C and D, we have $SA_1S^{-1} \equiv A^{mx+py}B^{nx+qy}$ and similarly, $SB_1S^{-1} \equiv A^{mz+pw}B^{nz+qw}$.

The following table then describes SA_1S^{-1} and SB_1S^{-1} according to $]\Gamma : \Gamma_0] = t$:

	$t=2$	$t=3$	$t=4$	$t=6$
SA_1S^{-1}	$A_1^{-1}C_1^{e_1}D_1^{f_1}$	$B_1C_1^{e_1}D_1^{f_1}$	$B_1C_1^{e_1}D_1^{f_1}$	$B_1C_1^{e_1}D_1^{f_1}$
SB_1S^{-1}	$B_1^{-1}C_1^{g_1}D_1^{h_1}$	$A_1^{-1}B_1^{-1}C_1^{g_1}D_1^{h_1}$	$A_1^{-1}C_1^{g_1}D_1^{h_1}$	$A_1^{-1}B_1C_1^{g_1}D_1^{h_1}$

We may thus normalize conjugation of A and B by S as above; the other relations among S, A, B, C, D remain as before.

We next wish to normalize powers of C and D that occur when A and B are conjugated by S. To this end, assume that $SAS^{-1} = A^mB^nC^eD^f$ and $SBS^{-1} = A^pB^qC^gD^h$. Replace the generators S, A, B, C, D by S, $A_1 = AC^xD^y$, $B_1 = BC^zD^w$, C, and D. All relations remain the same except conjugation of A_1 and B_1 by S where we get $SA_1S^{-1} = A^mB^nC^{e+x}D^{f+y}$ and $SB_1S^{-1} = A^pB^qC^{g+z}D^{h+w}$, or,

$$SA_1S^{-1} = (AC^xD^y)^m(BC^zD^w)^nC^{e+x-mx-nz}D^{f+y-my-nw}$$

and

$$SB_1S^{-1} = (AC^xD^y)^p(BC^zD^w)^qC^{g+z-px-qz}D^{h+w-py-qw} .$$

We may rewrite this as $SA_1S^{-1} = A_1^mB_1^nC^{e_1}D^{f_1}$ and $SB_1S^{-1} = A_1^pB_1^qC^{g_1}D^{h_1}$, where

$$\begin{aligned}\begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix} &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} - \begin{pmatrix} m & n \\ p & q \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} \\ &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} - \begin{pmatrix} m-1 & n \\ p & q-1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} .\end{aligned}$$

We are thus led to the matrix congruence relation $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \equiv \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}$ provided their difference is a multiple of $\begin{pmatrix} m-1 & n \\ p & q-1 \end{pmatrix}$. Viewing the set M of all 2×2 integer matrices as a $\boldsymbol{Z}$-module (under matrix addition, of course), we see that the set of all multiples of

$$\begin{pmatrix} m-1 & n \\ p & q-1 \end{pmatrix}$$

is a submodule M_1, and the equivalence classes above are the same as the elements of M/M_1. One may easily verify that the matrices $\begin{pmatrix} m-1 & n \\ p & q-1 \end{pmatrix}$ that arise when $t = 2, 3, 4$, and 6 are nonsingular. Hence M and M_1 have the same rank, which means that M/M_1 is finite. In other words, there are only finitely many choices of e, f, g, and h that give nonisomorphic groups.

This concludes the proof of Theorem 3.1.

4. Topological results. In this section, we discuss certain results having to do with transformation groups which are of use in the further study of the groups of 3.1.

Recall that a group Γ of homeomorphisms of a space X is said to act properly on X if each $x \in X$ has a neighborhood U such that $\gamma U \cap U = \emptyset$ for each $\gamma \neq \text{id.}$ in Γ. Notice that if Γ acts properly then Γ acts freely (i.e., no element of Γ except the identity has any fixed point). A subset F of the space X will be called a fundamental domain for Γ if F contains exactly one point of each Γ-orbit.

PROPOSITION 4.1. *Suppose that Γ is a freely-acting group of homeomorphisms of a space X, and that Γ has a subgroup Γ_0 of finite index that acts properly. Let F be a fundamental domain for Γ_0 and assume that there is an open set $G \supset F$ such that the cardinality of $\{\beta \in \Gamma_0 \mid \beta G \cap G \neq \emptyset\}$ is finite. Then Γ acts properly.*

The proof of 4.1 will be omitted, as it is not part of the main point of the present article.

It is not difficult to see that the groups Γ_0 of 2.3 have as fundamental domain a product of two half-open parallelograms spanned by the translation parts of the generators A, B, C, D. An open set G of the kind mentioned above in 4.1 is then readily obtained. As we have already remarked, the groups Γ_0 act properly on C^2. Hence, to prove that the groups of 3.1 act properly, it suffices to prove that they act freely. It is easily verified that the groups Γ_0 of 2.3 act freely; this being the case, there is a convenient abstract condition that insures that Γ acts freely, as follows.

PROPOSITION 4.2. *Let Γ be group of affine motions having a subgroup Γ_0 of finite index which acts freely. Then Γ acts freely if and only if Γ has no elements of finite order.*

The proof of 4.2 will also be omitted.

5. Conclusion. In light of §4, we may conclude our classification by seeing which of the groups of Theorem 3.1 are torsionfree and then checking to see that these can indeed be embedded in the group G of Theorem 1.1. It will be seen that the task of carrying out these two steps is quite lengthy (though elementary), so that many results will be given without proof. However, a method of arriving at them will be described completely and application in some representative cases will be given.

First we take care of the case where Γ_0 is abelian.

THEOREM 5.1. *The following groups* (and only those) *from* (3.1.1b) *are torsion-free*:

(i) $S^2 = D$, $SD = DS$, $SAS^{-1} = A^{-1}$, $SBS^{-1} = B^{-1}$, $SCS^{-1} = C^{-1}$;

(ii) $S^2 = D$, $SD = DS$, $SC = CS$, SAS^{-1}
$= A^{-1}C^{\alpha}D^{\beta}$, $SBS^{-1} = B^{-1}C^{\gamma}D^{\delta}$,

where (1) $\alpha = \beta = \gamma = \delta = 0$, (2) $\alpha = 1$, $\beta = \gamma = \delta = 0$, (3) $\gamma = 1$, $\alpha = \beta = \delta = 0$, (4) $\alpha = \beta = 1$, $\gamma = \delta = 1$, (5) $\alpha = \gamma = 1$, $\beta = \delta = \delta = 0$, (6) $\alpha = \beta = 0$, $\gamma = \delta = 1$ *and* (7) $\alpha = \beta = \gamma = \delta = 1$. *Furthermore, each of these can be embedded in the group* G *of* 1.1.

Proof. We restrict attention to case (i). If $X \in \Gamma$ has finite order, then $X \in S\Gamma_0$ and $X^2 = 1$. Writing $X = SA^mB^nC^pD^q$, we have $X^2 = D^{m\alpha+n\beta+p\gamma+2q+1}$. Clearly the exponent is always nonzero if and only if $\alpha = \beta = \gamma = 0$ (recall that α, β, γ are 0 or 1). Hence only one case gives a torsion-free group.

As for the embedding problem, recall that S, A, B, C, D have the form $\begin{pmatrix}1 & x & z\\ 0 & -1 & y\\ 0 & 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & a & a'\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & ab & b'\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & ac & c'\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & ad & d'\\ 0 & 1 & d\\ 1 & 0 & 1\end{pmatrix}$ respectively. Then $S^2 = \begin{pmatrix}1 & 0 & xy+2z\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = D$ is easily satisfied and $SD = DS$ is automatic. The other three relations lead to four equations involving the ten parameters $x, y, z, a, a', b, b', c, c, d$ and d' and there are many solutions.

From here on, we deal only with the cases where Γ_0 is nonabelian. Therefore, Γ has generators S, A, B, C, D and relations as follows: $AB = BAC^k$ with $k \geqq 1$; C and D central in Γ; S has order $t = 2, 3, 4$ or 6 modulo Γ_0 and $S^t = C^rD^s$, where r and s are reduced modulo t, and s is further reduced modulo r; finally, we have

(a) if $t = 2$, $SAS^{-1} = A^{-1}C^eD^f$ and $SBS^{-1} = B^{-1}C^gD^h$,

(b) if $t = 3$, $SAS^{-1} = BC^eD^f$ and $SBS^{-1} = A^{-1}B^{-1}C^gD^h$,

(c) if $t = 4$, $SAS^{-1} = BC^eD^f$ and $SBS^{-1} = A^{-1}C^gD^h$, and

(d) if $t = 6$, $SAS^{-1} = BC^eD^f$ and $SBS^{-1} = A^{-1}BC^gD^h$.

Recall from the proof of 3.1 that for each t, only finitely many choices of e, f, g, h give distinct groups. More precisely, $\begin{pmatrix}e & f\\ g & h\end{pmatrix} \sim \begin{pmatrix}e' & f'\\ g' & h'\end{pmatrix}$ provided $\begin{pmatrix}e & f\\ g & h\end{pmatrix} - \begin{pmatrix}e' & f'\\ g' & f'\end{pmatrix} = T\begin{pmatrix}x & y\\ z & w\end{pmatrix}$ for some $x, y, z, w \in \boldsymbol{Z}$, where $T = \begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}$ if $t = 2$, $\begin{pmatrix}-1 & 1\\ -1 & -2\end{pmatrix}$ if $t = 3$, $\begin{pmatrix}-1 & 1\\ -1 & -1\end{pmatrix}$ if $t = 4$, and $\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}$ if $t = 6$. Accordingly, we have the following possibilities for e, f, g, h:

(a) if $t = 2$, e, f, g, h independently chosen to be 0 or 1;

(b) if $t = 3$, $g = h = 0$ and e, f chosen to be 0, 1 or 2;

(c) if $t = 4$, $g = h = 0$ and e, f chosen to be 0 or 1;

(d) if $t = 6$, $e = f = g = h = 0$. Therefore, when Γ_0 is nonabelian, there are 133 cases to study (ignoring the cases where $r = s = 0$ and the infinite dependence on k).

To determine which of these groups are torsion-free, we must check each coset $S^i\Gamma_0$, $i = 1, \cdots, t-1$, in each group for elements of finite order. All together, there are 330 cosets to be studied. In each coset we take a general element $X = S^iA^mB^nC^pD^q$ whose order u, if finite, is easily determined. Then X^u is always of the form C^vD^w, where v and w depend on k, m, n, p and q (and i and Γ, also, of course), so $X^u = 1$ if and only if v and w can be made zero by some choice of integers k, m, n, p and q.

Computation gives the following answers for v and w:

(a) if $t = 2$ and $X \in S\Gamma_0$, then $v = mnk + me + ng + 2p + r$ and $w = mf + nh + 2q + s$;

(b) if $t = 3$ and $X \in S\Gamma_0$, then $v = (1/2)k(m + n)(m + n + 1) + 2me - ne + 3p + r$ and $w = 2mf - nf + 3q + s$;

if $X \in S^2\Gamma_0$, then $v = (1/2)km(m + 1) + (1/2)kn(n + 1) + 2me - ne + 3p + 2r$ and $w = 2mf - nf + 3q + 2s$;

(c) if $\mathrm{t} = 4$ and $X \in S\Gamma_0$, then $v = k(m + n)^2 + 2me - 2ne + 4p + r$ and $w = 2mf - 2nf + 4q + s$;

if $X \in S^2\Gamma_0$, then $v = kmn + me - ne + 2p + r$ and $w = mf - nf + 2q + s$;

if $X \in S^3\Gamma_0$, then $v = -k(m + n)^2 + 2me - 2ne + 4p + 3r$ and $w = 2mf - 2nf + 4q + s$;

(d) if $t = 6$ and $X \in S\Gamma_0$, then $v = 3k(m + n)^2 - 3nk - 3mk + 6p + r$ and $w = 6q + s$;

if $X \in S^2\Gamma_0$, then $v = 2kmn + (1/2)km(m - 1) + (1/2)kn(n + 1) - 2nk - mk + 3p + r$ and $w = 3q + s$;

if $X \in S^3\Gamma_0$, then $v = mnk - mk - nk + 2p + r$ and $w = 2q + s$;

if $X \in S^4\Gamma_0$, then $v = -(1/2)k(m + n)(m + n + 3) + 3p + 2r$ and $w = 3q + 2s$;

if $X \in S^5\Gamma_0$, then $v = -3k(m^2 + m + n^2 + n) + 6p + 5r$ and $w = 6q + 5s$.

From this, some cases are easily seen to be torsionfree. For example, when $t = 3$, $f = 0$, $s = 1$, or when $t = 6$ and $s = 1$, no $X \in \Gamma$ has finite order since $3q + 1$, $3q + 2$, $6q + 1$, etc. are never zero. Other cases require a bit more; for example, consider the case $t = 3$, $e = f = 1$, $r = 0$, $s = 1$. Then $S\Gamma_0$ has an element of finite order if and only if $(1/2)k(m+n)(m+n+1)+2m-n+3p=0$ and $2m-n+3q+1=0$, and $S^2\Gamma_0$ has an element of finite order if and only if $(1/2)km(m+1)+(1/2)kn(n+1)+2m-n+3p=0$ and $2m-n+3q+2=0$, for some k, m, n, p, q. The first system is equivalent to finding $(1/2)k(m+n)(m+n+1)-3q+3p-1=0$, which can happen if and only if $k \equiv 1 \pmod 3$, and the second system is equivalent to $(1/2)km(m+1)+(1/2)kn(n+1)+3p-3q-2=0$, which can happen if and only if $k \equiv 1$ or $2 \pmod 3$. Therefore the group in this case is torsion-free if and only if $k \equiv 0 \pmod 3$.

The complete search for elements of finite order gives the following result.

THEOREM 5.2. *The following groups* (and only those) *from* 3.1.2 *are torsion-free.*

(a) (when $t=2$) *The following cases are torsion-free without restriction on* k: $r=0, s=1$ *and*: $e=f=g=h=0$, *or* $e=1$ *and* $f=g=h=0$, *or* $g=1$ *and* $e=f=h=0$, *or* $e=g=1$ *and* $f=h=0$, *or* $e=f=g=h=1$; $r=1, s=0$ *and*: $f=1$ *and* $e=g=h=0$, *or* $h=1$ *and* $e=f=g=0$, *or* $e=f=1$ *and* $g=h=0$, *or* $e=f=0$ *and* $g=h=1$, *or* $e=0$ *and* $f=g=h=1$, *or* $e=f=h=1$ *and* $g=0$. *When k is even, the following are torsion-free*: $r=1, s=0$ *and*: $e=f=g=h=0$, *or* $e=g=0$ *and* $f=h=1$, *or* $e=f=g=h=1$. *When k is odd, the following are torsion-free*: $r=0, s=1$ *and*: $e=g=h=1$ *and* $f=0$, *or* $e=f=g=1$ *and* $h=0$.

(b) (when $t=3$) *The following cases are torsion-free without restriction on* k: $r=0, s=1$ *and* $e=f=0$, *or* $e=1$ *and* $f=0$, *or* $e=2$ *and* $f=0$; $r=0, s=2$ *and* $e=f=0$, *or* $e=1$ *and* $f=0$, *or* $e=2$ *and* $f=0$; $r=2, s=1$ *and* $e=f=0$, *or* $e=1$ *and* $f=0$, *or* $e=2$ *and* $f=0$. *When* $k \equiv 0 \pmod 3$, *the following cases are torsionfree*: $r=0, s=1$ *and* $e=f=1$, *or* $e=2$ *and* $f=1$, *or* $e=1$ *and* $f=2$; $r=0, s=2$ *and* $e=f=1$, *or* $e=2$ *and* $f=1$, *or* $e=1$ *and* $f=2$; $r=1, s=0$ *and* $e=f=0$, *or* $e=0$ *and* $f=1$, *or* $e=f=1$, *or* $e=2$ *and* $f=1$, *or* $e=1$ *and* $f=2$, *or* $e=f=2$; $r=2, s=0$ *and* $e=f=1$, *or* $e=0$ *and* $f=1$, *or* $e=f=1$, *or* $e=2$ *and* $f=1$, *or* $e=1$ *and* $f=2$; *or* $e=f=2$; $r=2, s=1$ *and* $e=f=1$. *When* $k \equiv 0$ *or* $1 \pmod 3$, *the following cases are torsion-free*: $r=0, s=2$ *and* $e=f=2$; $r=1, s=0$ *and* $e=0, f=2$. *When* $k \equiv 0$ *or* $2 \pmod 3$, *the following are torsion-free*: $r=0, s=1$ *and* $e=f=2$; $r=2, s=0$ *and* $e=0, f=2$; $r=2, s=1$ *and* $e=f=0$. *When* $k \equiv 1$ or $2 \pmod 3$, *the following is torsion-free*: $r=0, s=2$, $e=0, f=2$.

(c) (when $t=4$) *The following are torsion-free without restriction on* k: $r=0, s=1$ *and* $e=f=0$, *or* $e=1$ *and* $f=0$; $r=0, s=$

2 and $e = f = 0$, *or* $e = 1$ *and* $f = 0$; $r = 0$, $s = 3$ *and* $e = f = 0$, *or* $e = 1$ *and* $f = 0$; $r = 2$, $s = 1$ *and* $e = f = 0$, *or* $e = 1$ *and* $f = 0$; $r = 3$, $s = 1$ *and* $e = f = 0$, *or* $e = 0$ *and* $f = 1$, *or* $e = 1$ *and* $f = 0$; $r = 3$, $s = 2$ *and* $e = f = 0$ *or* $e = 1$ *and* $f = 0$. *When* k *is even, the following are torsion-free*: $r = 0$, $s = 1$, $e = f = 1$; $r = 0$, $s = 3$, $e = f = 1$; $r = 1$, $s = 0$, $e = f = 0$ *or* $e = 0$ *and* $f = 1$, *or* $e = f = 1$; $r = 2$, $s = 1$, $e = f = 1$; $r = 3$, $s = 0$, $e = f = 0$, or $e = 0$ *and* $f = 1$, *or* $e = f = 1$; $r = 3$, $s = 2$, $e = 0$ *and* $f = 1$, *or* $e = f = 1$.

(d) (when $t = 6$) *The following are torsion-free regardless of the value of* k: $r = 0$, $s = 1$; $r = 0$, $s = 5$; $r = 2$, $s = 1$; $r = 3$, $s = 1$; $r = 4$, $s = 1$; $r = 5$, $s = 1$. *When* k *is even, the following are torsion-free*: $r = 3$, $s = 2$; $r = 5$, $s = 2$; $r = 5$, $s = 4$. *When* $k \equiv 0 (\mathrm{mod}\, 3)$, *the following are torsion-free*: $r = 4$, $s = 3$; $r = 5$, $s = 3$. *When* $k \equiv 0 (\mathrm{mod}\, 6)$, *the following are torsion-free*: $r = 1$, $s = 0$; $r = 5$, $s = 0$.

We shall now see that all the groups above can be embedded in the group G of 1.1. In fact, we have the following result.

THEOREM 5.3. *All the groups of* 3.1.2 *can be embedded in the group* G *of Theorem* 1.1.

Proof. We take $S = \begin{pmatrix} 1 & x & z \\ 0 & w & y \\ 0 & 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & ab - k & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = B$, $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where w is a primitive tth root of unity, and try to choose the parameters x, y, z, a, b, d so that the relations of 3.1.2 hold: $AB = BAC^k$, C and D central in Γ, $S^t = C^r D^s$, and SAS^{-1} and SBS^{-1} as given in 3.1.2 as a function of t. The first two of these are consequences of 2.2 and the shape of S. The remaining relations are treated separately for $t = 2, 3, 4, 6$. For purposes of calculation, note that

$$S^t = \begin{pmatrix} 1 & 0 & t\left(z - \dfrac{xy}{w-1}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & x & -z - xy \\ 0 & -1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & -a & a \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

and $B^{-1} = \begin{pmatrix} 1 & -ab\,k & b(ab - k) \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$.

In case $t = 2$, the relations lead to the following system of equations:

$$2\left(z - \frac{xy}{-2}\right) = r + ds$$

$$ay + x = a + e + fd$$

$$y(ab - k) + bx = b(ab - k) + g + hd\ .$$

These can be solved uniquely for x, y and z, so that when $t = 2$, there is a unique S for each value of a, b, d so that the required relations hold.

The other cases $t = 3, 4, 6$ are slightly different in that the parameters a, b are uniquely determined and then for each d, S is uniquely determined. To illustrate, in case $t = 3$, the relations in question lead to the following system:

$$3\left(z - \frac{xy}{w - 1}\right) = r + ds \tag{1}$$

$$w^{-1}a = ab - k \tag{2}$$

$$w = b \tag{3}$$

$$-w^{-1}ay + x = e + fd \tag{4}$$

$$w^{-1}(ab - k) = -ab + k - a \tag{5}$$

$$bw = -b - 1 \tag{6}$$

$$-w^{-1}y(ab - k) + bx = b(ab - k) + g + hd + ab + a\ . \tag{7}$$

In view of (3), (6) is automatic. Solve (2) for a and then (5) is automatic. Then solve (4) and (7) for y and x, and after that, solve (1) for z.

We now have in Theorems 5.1 and 5.2, together with the groups in [1], a complete list of the fundamental groups of compact complete locally affine complex surfaces. It should be noted in closing that some of the groups in this list may be isomorphic; however, it does not seem that the list of isomorphism classes is markedly shorter than the list above.

References

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Received August 12, 1973.

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