PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

SPECTRA, TENSOR PRODUCTS, AND LINEAR OPERATOR EQUATIONS

M. R. EMBRY AND M. ROSENBLUM

Suppose $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are complex Banach spaces with $u_0, \cdots, u_n$ in $\mathscr{L}(\mathfrak{X}_1)$, $v \in \mathscr{L}(\mathfrak{X}_2)$, and suppose $\otimes$ is a uniform crossnorm. The spectra of the operators $\sum_{j=0}^{n} u_j \otimes v^j$ on $\mathfrak{X}_1 \otimes \mathfrak{X}_2$ and $R: x \to \sum_{j=0}^{n} u_j x v^j$, $x \in \mathscr{L}(\mathfrak{X}_2, \mathfrak{X}_1)$, are studied in the context of a general theory. Explicit representations are set down for the resolvents of these and more general operators.

0. Introduction. A classical result of Stephanos [9, p. 83] can be phrased as follows:

Suppose u and v are complex $n \times n$ matrices and $p_0, \cdots, p_m$ are complex polynomials. Let $\otimes$ denote tensor product, and σ spectrum. Then

$$\sigma\Big(\sum_{j=0}^{m} p_j(u) \otimes v^j\Big) = \bigcup \Big\{\sigma\Big(\sum_{j=0}^{m} p_j(u) z^j\Big) : z \in \sigma(v)\Big\} . \tag{1}$$

In 1966 Datuasvili [3] gave the following generalization of Stephanos' result. Let $u_0, \cdots, u_m$ and v be complex $n \times n$ matrices. Then

$$\sigma\Big(\sum_{j=0}^{m} u_j \otimes v^j\Big) = \bigcup \Big\{\sigma\Big(\sum_{j=0}^{m} u_j z^j\Big) : z \in \sigma(v)\Big\} . \tag{2}$$

Stephanos' theorem can be interpreted as a result on linear operator equations. It implies that the operator T on $n \times n$ matrices defined by $Tx = \sum_{j=0}^{m} p_j(u) x v^j$ has

$$\sigma(T) = \bigcup \Big\{\sigma\Big(\sum_{j=0}^{m} p_j(u) z^j\Big) : z \in \sigma(v)\Big\} . \tag{3}$$

Similarly Datuasvili's result yields that the operator R defined by $Rx = \sum_{j=0}^{m} u_j x v^j$ has

$$\sigma(R) = \bigcup \Big\{\sigma\Big(\sum_{j=0}^{m} u_j z^j\Big) : z \in \sigma(v)\Big\} . \tag{4}$$

Lumer and Rosenblum [8] proved that (3) holds if $u, v \in \mathscr{L}(\mathfrak{X})$, where $\mathfrak{X}$ is a complex Banach space and T is considered as an operator on $\mathscr{L}(\mathfrak{X})$ to $\mathscr{L}(\mathfrak{X})$. R. E. Harte [6] has recently shown that (4) holds if $u_0, \cdots, u_m$ and v are in $\mathscr{L}(\mathfrak{H})$, where $\mathfrak{H}$ is a complex Hilbert space.

Brown and Pearcy [1] proved that $\sigma(u \otimes v) = \sigma(u)\sigma(v)$ in case $u, v \in \mathscr{L}(\mathfrak{H})$ and $u \otimes v$ acts on the Hilbert space $\mathfrak{H} \otimes \mathfrak{H}$. This was generalized by Schechter [12] and Dash and Schechter [2]. It was further generalized by Harte in [6].

In this paper we shall set down explicit representations for the resolvent of each of

(i) $\sum_{j=0}^{n} u_j \otimes v^j$,

where $u_0, \cdots, u_n \in \mathscr{L}(\mathfrak{X}_1)$, $v \in \mathscr{L}(\mathfrak{X}_2)$, and $\otimes$ is any uniform crossnorm, and

(ii) $R: x \rightarrow \sum_{j=0}^{n} u_j x v^j$, $x \in \mathscr{L}(\mathfrak{X}_2, \mathfrak{X}_1)$,

where $u_0, \cdots, u_n \in \mathscr{L}(\mathfrak{X}_1)$ and $v \in \mathscr{L}(\mathfrak{X}_2)$. For a survey of explicit solutions of linear matrix equations, see [7].

The theory for the representations is presented in §1. In §2 we prove a spectral mapping theorem that subsumes conclusions such as those of (2) and (4) in one unified theory. In §3 we give some applications.

The notation and terminology used in the paper are as follows. $\mathfrak{A}$ will denote a complex Banach algebra with identity 1 or I. If $a \in \mathfrak{A}$, $\sigma(a \mid \mathfrak{A})$ is the spectrum of a; that is, $\sigma(a \mid \mathfrak{A})$ is the set of complex numbers z for which $z - a$ is singular in $\mathfrak{A}$. In case there is no ambiguity involved we shall use the simpler notation $\sigma(a)$ for the spectrum of a.

If X and Y are Banach spaces, $\mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ is the space of all continuous linear transformations from $\mathfrak{X}$ into $\mathfrak{Y}$, and $\mathscr{L}(\mathfrak{X}) = \mathscr{L}(\mathfrak{X}, \mathfrak{X})$. If Ω is an index set we sometimes use $\bigcup \{a_\lambda : \lambda \in \Omega\}$ to mean $\bigcup_{\lambda \in \Omega} a_\lambda$.

1. Integral representation of inverses.

Throughout this section $\{u_j\}_{j=0}^{n}$ and $\{v_j\}_{j=1}^{m}$ are subsets of $\mathfrak{A}$ that satisfy the following commutativity relations: $v_j v_k = v_k v_j$ and $v_j u_i = u_i v_j$ for $j, k = 1, \cdots, m$ and $i = 0, \cdots, n$. It should be noted that we do not require the u_j to pairwise commute. $p_0, \cdots, p_n$ shall be polynomials in m variables.

LEMMA 1.1. *If*

$$\sigma(v_1) \times \cdots \times \sigma(v_m) \subseteq \left\{(z_1, \cdots, z_m) : \sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m) \text{ is invertible}\right\},$$

then $\sum_{j=0}^{n} u_j p_j(v_1, \cdots, v_m)$ *is invertible and its inverse is*

$$(5)\quad \left(\frac{1}{2\pi i}\right)^m \int_{C_1} \cdots \int_{C_m} \left(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m)\right)^{-1} \prod_{k=1}^{m} (z_k - v_k)^{-1} dz_1 \cdots dz_m ,$$

where C_k is the boundary of a Cauchy domain D_k (see Taylor [13]) *that contains $\sigma(v_k)$ for $k = 1, \cdots, m$ and such that $\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m)$ is invertible for $(z_1, \cdots, z_m)$ in $\bar{D}_1 \times \cdots \times \bar{D}_m$.*

Proof. The proof is by induction on m.

Assume $m = 1$. We shall show by direct computation that

$$\left(\sum_{j=0}^{n} u_j p_j(v_1)\right)^{-1} = \frac{1}{2\pi i} \int_{C_1} \left(\sum_{j=0}^{n} u_j p_j(z_1)\right)^{-1} (z_1 - v_1)^{-1} dz_1 .$$

$$\begin{aligned} (6) \qquad & \left(\sum_{k=0}^{n} u_k p_k(v_1)\right) \frac{1}{2\pi i} \int_{C_1} \left(\sum_{j=0}^{n} u_j p_j(z_1)\right)^{-1} (z_1 - v_1)^{-1} dz_1 \\ & = \frac{1}{2\pi i} \int_{C_1} \left(\sum_{k=0}^{n} u_k [p_k(v_1) - p_k(z_1)]\right) \left(\sum_{j=0}^{n} u_j p_j(z_1)\right)^{-1} (z_1 - v_1)^{-1} dz_1 \\ & + \frac{1}{2\pi i} \int_{C_1} \left(\sum_{k=0}^{n} u_k p_k(z_1)\right) \left(\sum_{j=0}^{n} u_j p_j(z_1)\right)^{-1} (z_1 - v_1)^{-1} dz_1 . \end{aligned}$$

Since $v_1 - z_1$ and $\sum_{j=0}^{n} u_j p_j(z_1)$ commute, the penultimate term has an analytic integrand, and thus equals the zero element. The last term reduces to

$$\frac{1}{2\pi i} \int_{C_1} (z_1 - v_1)^{-1} dz_1 = 1 .$$

Thus the right term of (6) is a right inverse of $\sum_{j=0}^{n} u_j p_j(v_1)$. A similar computation shows that it is also a left inverse, which completes the proof for the case $m = 1$.

Assume that the lemma is true when $m = k$, and that $\sigma(v_1) \times \cdots \times \sigma(v_{k+1}) \subseteqq \{(z_1, \cdots, z_{k+1}) \colon \sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_{k+1})$ is invertible$\}$.

Then for each $z_{k+1} \in \bar{D}_{k+1}$ the induction hypothesis yields

$$\begin{aligned} (7) \qquad & \left(\sum_{j=0}^{n} u_j p_j(v_1, \cdots, v_k, z_{k+1})\right)^{-1} \\ & = \left(\frac{1}{2\pi i}\right)^{k} \int_{C_1} \cdots \int_{C_k} \left(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_{k+1})\right)^{-1} \prod_{k=1}^{n} (z_k - v_k)^{-1} dz_1 \cdots dz_k . \end{aligned}$$

However, $(\sum_{j=0}^{n} u_j p_j(v_1, \cdots, v_k, z))^{-1}$ is analytic for z in a neighborhood of $\sigma(v_{k+1})$. Thus, if we multiply (7) by $1/2\pi i(z_{k+1} - v_{k+1})^{-1}$, integrate about C_{k+1}, and apply (6), we deduce that the lemma is true for $m = k + 1$.

We cite one special case of Lemma 1.1.

COROLLARY 1.2. *Suppose* $\{u_j\}_{j=0}^{n}$ *is a subset of the Banach algebra* $\mathfrak{A}$ *and* v *in* $\mathfrak{A}$ *commutes with* $\{u_j\}_{j=0}^{n}$. *If*

$$\sigma(v) \subseteqq \left\{z \colon \sum_{j=0}^{n} u_j z^j \text{ is invertible}\right\} ,$$

then $\sum_{j=0}^{n} u_j v^j$ *is invertible and*

$$\left(\sum_{j=0}^{n} u_j v^j\right)^{-1} = \frac{1}{2\pi i} \int_{C} \left(\sum_{j=0}^{n} u_j z^j\right)^{-1} (z - v)^{-1} dz ,$$

where C *is the boundary of a Cauchy domain* D *that contains* $\sigma(v)$

and such that $\sum_{j=0}^{n} u_j z^j$ *is invertible for each* z *in* $\bar{D}$.

Lemma 1.1 enables us to infer the following general result about spectral inclusion as well as to write an explicit representation for the inverse of $\sum_{j=0}^{n} u_j p_j(v_1, \cdots, v_m) - \lambda$ for certain complex numbers λ.

THEOREM 1.3.

$$\sigma\Big(\sum_{j=0}^{n} u_j p_j(v_1, \cdots, v_m)\Big) \subseteqq \Delta , \tag{8}$$

where

$$\Delta \stackrel{\text{def}}{=} \bigcup \Big\{\sigma\Big(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m)\Big): z_k \in \sigma(v_k),\ k = 1, \cdots, m\Big\} .$$

If $\lambda \notin \Delta$, *then*

$$\begin{aligned} &\Big(\sum_{j=0}^{n} u_j p_j(v_1, \cdots, v_m) - \lambda\Big)^{-1} \\ &\quad = \Big(\frac{1}{2\pi i}\Big)^m \int_{C_1} \cdots \int_{C_m} \Big(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m) - \lambda\Big)^{-1} \\ &\quad\quad \times \prod_{k=1}^{m} (z_k - v_k)^{-1} dz_1 \cdots dz_m , \end{aligned} \tag{9}$$

where C_k *is the boundary of a Cauchy domain* D_k *that contains* $\sigma(v_k)$ *for* $k = 1, \cdots, m$ *and such that* $\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m) - \lambda$ *is invertible for* $(z_1, \cdots, z_m) \in \bar{D}_1 \times \cdots \times \bar{D}_m$.

Proof. If $\lambda \notin \Delta$, then it is immediate that $\sigma(v_1) \times \cdots \times \sigma(v_m) \subseteqq \{(z_1, \cdots, z_m): \sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m) - \lambda$ is invertible$\}$. Define $u_{n+1} = -\lambda$ and $p_{n+1} = 1$. Lemma 1.1 is now applicable to $\sum_{j=0}^{n+1} u_j p_j(v_1, \cdots, v_m)$. Thus the theorem follows from that lemma.

Simple finite dimensional examples show that the spectral containment conclusion of (8) need not hold, if the v_j do not commute with $\{u_k\}$. Consider

$$u_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ u_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \text{and}\ v_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

In this case $\sigma(u_0 + u_1 v_1) = \{0, 2\}$, but $\bigcup \{\sigma(u_0 + u_1 z): z \in \sigma(v_1)\} = \{0, 1\}$.

Even when the required commutativity relations hold one cannot in general hope for equality in (8). For example, consider commuting elements u and v of a Banach algebra, set $v_1 = v_2 = v$. Then $\sigma(uv_1 - uv_2) = \{0\}$, but $\bigcup \{\sigma(uz_1 - uz_2): z_1, z_2 \in \sigma(v)\}$ is in general not $\{0\}$.

2. A spectral mapping theorem. In §1 we showed that under certain commutativity conditions (8) holds, but that in general equality does not hold. In this section we find conditions sufficient to imply equality in (8). For a different attack, see Harte [5], [6].

DEFINITION 2.1. Let $\mathfrak{A}$ be a complex Banach algebra with closed subalgebras $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_m (m \geqq 1)$ such that the identity 1 in $\mathfrak{A}$ is also in $\mathfrak{A}_j$, $j = 0, \cdots, m$. Then $\mathfrak{A}_0, \cdots, \mathfrak{A}_m$ are *independent* algebras in $\mathfrak{A}$ if the following conditions hold for $j, k = 0, \cdots, m$:

(i) If $a \in \mathfrak{A}_j$, $b \in \mathfrak{A}_k$, $j \neq k$, then $ab = ba$;

(ii) There exists a real number M such that whenever $a_j \in \mathfrak{A}_j$, $j = 0, \cdots, m$, then

$$\prod_{j=0}^{m} ||a_j|| \leqq M||a_0 a_1 \cdots a_m||;$$

(iii) If $a_j \in \mathfrak{A}_j$, then $\sigma(a_j | \mathfrak{A}_j) = \sigma(a_j | \mathfrak{A})$.

LEMMA 2.2. *Let $\mathfrak{A}_0, \cdots, \mathfrak{A}_m$ be independent algebras in $\mathfrak{A}$ with $\{u_j\}_{j=0}^{n} \subseteqq \mathfrak{A}_0$ and $v_k \in \mathfrak{A}_k$ for $k = 1, \cdots, m$. Let each of p_j, $j = 0, \cdots, n$ be a polynomial in m variables. If*

$$0 \in \bigcup \left\{\sigma\Bigl(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m)\Bigr) : z_k \in \sigma(v_k),\ k = 1, \cdots, m\right\},$$

then there exist $\lambda_k \in \sigma(v_k)$, $k = 1, \cdots, m$ such that $\sum_{j=0}^{n} u_j p_j(\lambda_1, \cdots, \lambda_m)$ is singular in $\mathfrak{A}$ and either

(i) *$v_k - \lambda_k$ is the limit of invertible elements of $\mathfrak{A}_k$ for $k = 1, \cdots, m$, or*

(ii) *$\sum_{j=0}^{n} u_j p_j(\lambda_1, \cdots, \lambda_m)$ is the limit of invertible elements of $\mathfrak{A}_0$.*

Proof. Select a point $(\zeta_1, \cdots, \zeta_m)$ in $\sigma(v_1) \times \cdots \times \sigma(v_m)$ for which $\sum_{j=0}^{n} u_j p_j(\zeta_1, \cdots, \zeta_m)$ is singular. The select components W_k of $\sigma(v_k)$ containing ζ_k for $k = 1, \cdots, m$, and set $W = W_1 \times \cdots \times W_m$. Clearly W is a connected set in complex m-space. Let K be the set of all points $(z_1, \cdots, z_m)$ in W for which $\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m)$ is invertible in $\mathfrak{A}_0$. Note that K is open in W and $K \neq W$. Thus since W is connected, either K is empty or there exists a point $(\lambda_1, \cdots, \lambda_m)$ in $\bar{K} - K$.

Case (a). If K is empty, then $\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m)$ is singular for all $(z_1, \cdots, z_m)$ in W. In particular it is singular for $(z_1, \cdots, z_m)$ chosen so that z_k is in the boundary of $\sigma(v_k) = \sigma(v_k | \mathfrak{A}_k)$, $k = 1, \cdots, m$. Thus, (i) follows. See Rickart ([10], p. 22).

Case (b). Assume $(\lambda_1, \cdots, \lambda_m) \in \bar{K} - K$. This means that

$$\sum_{j=0}^{n} u_j p_j(\lambda_1, \cdots, \lambda_m)$$

is singular, but is the limit of invertible elements of $\mathfrak{A}_0$. Thus (ii) holds.

We shall use the following terminology in the remainder of this section. An element u of $\mathfrak{A}_j$ is an $\mathfrak{A}_j$ *generalized divisor of zero* if there exists a sequence $\{x_j\}$ of unit vectors in $\mathfrak{A}_j$ such that $\lim_{j\to\infty} ux_j = 0$ or $\lim_{j\to\infty} x_j u = 0$. In the first case we say that $\{x_j\}$ *right zero divides* u and in the second $\{x_j\}$ *left zero divides* u.

THEOREM 2.3. *Assume that* $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_m$ *are independent algebras in* $\mathfrak{A}$ *and that each singular element of* $\mathfrak{A}_j$ *is an* $\mathfrak{A}_j$ *generalized divisor of zero. If* $\{u_j\}_{j=0}^{n} \subseteqq \mathfrak{A}_0$, $v_k \in \mathfrak{A}_k$, $k = 1, \cdots, m$, *and each of* $p_1, \cdots, p_m$ *is a polynomial, then*

$$\begin{aligned}(10)\qquad &\sigma\Big(\sum_{j=0}^{n} u_j p_j(v_1, \cdots, v_m)\Big)\\ &= \bigcup\left\{\sigma\Big(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m)\Big): z_k \in \sigma(v_k),\ k = 1, \cdots, m\right\}.\end{aligned}$$

Proof. Theorem 1.3 gives the containment $\subseteqq$ in (10). To prove the reverse containment it is sufficient to assume that $0 \in \sigma(v_k)$, $k = 1, \cdots, m$, $p_j(0, \cdots, 0) = 0$ if $j \geqq 1$, and $0 \in \sigma(u_0)$, and deduce that $R = \sum_{j=0}^{n} u_j p_j(v_1, \cdots, v_m)$ is not invertible. By hypothesis we know that there exist left or right zero-dividing sequences $\{y_j^{(0)}\} \subseteqq \mathfrak{A}_0$ of u_0 and $\{y_j^{(k)}\} \subseteqq \mathfrak{A}_k$ of v_k, $k = 1, \cdots, m$. By Lemma 2.2 and the nature of limits of invertible operators (Rickart [10], p. 22) we may assume that either $y_j^{(0)}$ left divides u_0 or $\{y_j^{(k)}\}$ right divides v_k for $k = 1, \cdots, m$. Thus the following two cases exhaust all the possibilities.

Case (a). $\{y_j^{(0)}\}$ left divides u_0, $\{y_j^{(k)}\}$ left divides v_k for $k = 1, \cdots, r$, and $y_j^{(k)}$ right divides v_k for $k = r + 1, \cdots, m$.

Assume that R is invertible and set $g_j = R^{-1} y_j^{(r+1)} \cdots y_j^{(m)}$. Then if $k = r + 1, \cdots, m$, $R(v_k g_j) = v_k R g_j = v_k y_j^{(r+1)} \cdots y_j^{(m)} \to 0$ as $j \to \infty$. Thus since R is invertible $\lim_{j\to\infty} v_k g_j = 0$ for $k = r + 1, \cdots, m$. Then

$$\begin{aligned} y_j^{(0)} \cdots y_j^{(m)} &= y_j^{(0)} \cdots y_j^{(r)} R g_j \\ &= y_j^{(0)} \cdots y_j^{(r)} \Big[u_0 + \sum_{j=1}^{n} u_j p_j(v_1, \cdots, v_m)\Big] g_j \longrightarrow 0 \text{ as } j \longrightarrow \infty . \end{aligned}$$

However, by condition (ii) of Definition 2.1,

$$1 = \prod_{k=0}^{m} \| y_j^{(k)} \| \leqq M \| y_j^{(0)} \cdots y_j^{(m)} \| ,$$

which is a contradiction, so R cannot be invertible.

For the remaining case u_0 does not have a left zero dividing sequence.

Case (b). $\{y_j^{(k)}\}$ right divides v_k for $k = 1, \cdots, m$ and $\{y_j^{(0)}\}$ right divides u_0.

In this case

$$R(y_j^{(0)} \cdots y_j^{(m)}) = u_0 y_j^{(0)} \cdots y_j^{(m)} + \sum_{j=1}^{n} u_j y_j^{(0)} p_j(v_1, \cdots, v_m) y_j^{(1)} \cdots y_j^{(m)} \longrightarrow 0 \text{ as } j \longrightarrow \infty .$$

This shows that R is not invertible since, as shown in the proof of case (a), $y_j^{(0)} \cdots y_j^{(m)}$ is bounded away from 0.

We note that the "uniform crossnorm" condition (ii) of Definition 2.1 cannot be omitted in the hypotheses of Theorem 2.3. For, consider a Hilbert space $\mathfrak{H}_1$ and let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_1$. Let $\mathfrak{A}_0$ and $\mathfrak{H}_1$ be defined by

$$\mathfrak{A}_0 = \left\{\begin{pmatrix} A & 0 \\ 0 & \alpha I \end{pmatrix} : A \in \mathscr{L}(\mathfrak{H}_1),\ \alpha \text{ complex}\right\} ,$$
$$\mathfrak{A}_1 = \left\{\begin{pmatrix} \beta I & 0 \\ 0 & B \end{pmatrix} : B \in \mathscr{L}(\mathfrak{H}_1),\ \beta \text{ complex}\right\} ,$$

and let $\mathfrak{A} = \mathscr{L}(\mathfrak{H})$. Clearly $\mathfrak{H}$, $\mathfrak{H}_0$, and $\mathfrak{A}_1$ satisfy all of the conditions of Definition 2.1 except possibly (ii). If we let $u_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$, A, $B \in \mathscr{L}(\mathfrak{H}_1)$, then $\sigma(u_0 v_1) = \{0\}$, but $\bigcup \{\sigma(u_0 z) : z \in \sigma(v_1)\} = \sigma(A)\cdot\sigma(B) \cup \{0\}$. Thus in general $\sigma(u_0 v_1) \neq \sigma(u_0)\sigma(v_1)$, and for this simple example the conclusion of Theorem 2.3 does not hold.

3. Applications.

Our first two applications of Theorems 1.3 and 2.3 generalize results of Rosenblum [11, Theorem 3.1] and Lumer and Rosenblum [8, Theorem 10].

THEOREM 3.1. *Suppose that* $\{u_j\}_{j=0}^n \subseteqq \mathfrak{A}$ *and suppose* $\{v_k\}_{k=1}^m$ *is a commutative subset of* $\mathfrak{A}$. *Let each of* p_j, $j = 0, \cdots, n$, *be a polynomial in* m *variables. Define* $R: \mathfrak{A} \to \mathfrak{A}$ *by* $Rx = \sum_{j=0}^n u_j x p_j(v_1, \cdots, v_m)$. *Then* $\sigma(R) \subseteqq \Delta$, *where*

$$\Delta \overset{\text{def}}{=} \bigcup \left\{\sigma\left(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m)\right) : z_k \in \sigma(v_k),\ k = 1, \cdots, m\right\} ,$$

and if $\lambda \notin \Delta$, $x \in A$,

$$(R - \lambda)^{-1} x = \left(\frac{1}{2\pi i}\right)^m \int_{C_1} \cdots \int_{C_m} \left(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m) - \lambda\right)^{-1} \times x \prod_{k=1}^{m} (z_k - v_k)^{-1} dz_1 \cdots dz_m ,$$

where the C_k *are chosen as in Theorem* 1.3.

Proof. Let $\mathfrak{B} = \mathscr{L}(\mathfrak{A})$,

$$\mathfrak{B}_0 = \{u^L : u \in \mathfrak{A} \text{ and } u^L : x \longrightarrow ux,\ x \in \mathfrak{A}\}$$
$$\mathfrak{B}_1 = \{v^R : v \in \mathfrak{A} \text{ and } v^R : x \longrightarrow xv,\ x \in \mathfrak{A}\} .$$

By hypothesis $\mathfrak{B}_1$ is commutative, and clearly each element of $\mathfrak{B}_0$ commutes with each element of $\mathfrak{B}_1$. Thus we may apply Theorem 1.3 to $\sum_{j=0}^n u_j^L p_j(v_1^R, \cdots, v_m^R)$. Since $\sigma(R) = \sigma(\sum_{j=0}^n u_j^L p_j(v_1^R, \cdots, v_m^R))$ and $\sigma(v_k) = \sigma(v_k^R)$, we have the desired conclusions.

A result analogous to that in Theorem 3.1 can be obtained if one fixes complex Banach spaces $\mathfrak{X}_0$ and $\mathfrak{X}_1$ and defines R on $\mathscr{L}(\mathfrak{X}_1, \mathfrak{X}_0)$ by $Rx = \sum_{j=0}^n u_j x p_j(v_1, \cdots, v_m)$, where $\{u_j\}_{j=0}^n$ is a subset of $\mathscr{L}(\mathfrak{X}_0)$ and $\{v_k\}_{k=1}^m$ is a commutative subset of $\mathscr{L}(\mathfrak{X}_1)$. Indeed, if we consider the case where $m = 1$ we get a stronger result.

THEOREM 3.2. *Let* $\mathfrak{X}_0$ *and* $\mathfrak{X}_1$ *be complex Banach spaces,* $\{u_j\}_{j=0}^n \subseteqq \mathscr{L}(\mathfrak{X}_0)$ *and* $v \in \mathscr{L}(\mathfrak{X}_1)$. *Define* $R: \mathscr{L}(\mathfrak{X}_1, \mathfrak{X}_0) \rightarrow \mathscr{L}(\mathfrak{X}_1, \mathfrak{X}_0)$ *by*

$$Rx = \sum_{j=0}^n u_j x v^j .$$

Then

$$\sigma(R) = \bigcup \left\{\sigma\Big(\sum_{j=0}^n u_j z^j\Big) : z \in \sigma(v)\right\} , \tag{11}$$

and if $\lambda \notin \sigma(R)$,

$$(R - \lambda)^{-1}x = \frac{1}{2\pi i}\int_C \Big(\sum_{j=0}^n u_j z^j - \lambda\Big)^{-1} x(z - v)^{-1}dz , \tag{12}$$

where C *is the boundary of a Cauchy domain* D *that contains* $\sigma(v)$ *and such that* $\sum_{j=0}^n u_j z^j$ *is invertible for* z *in* $\bar{D}$.

Proof. Let $\mathfrak{A} = \mathscr{L}(\mathscr{L}(\mathfrak{X}_1, \mathfrak{X}_0))$,

$$\mathfrak{A}_0 = \{u^L : u \in \mathscr{L}(\mathfrak{X}_0) \text{ and } u^L : x \longrightarrow ux,\ x \in \mathscr{L}(\mathfrak{X}_1, \mathfrak{X}_0)\} ,$$

and

$$\mathfrak{A}_0 = \{v^R : v \in \mathscr{L}(\mathfrak{X}_1) \text{ and } v^R : x \longrightarrow xv,\ x \in \mathscr{L}(\mathfrak{X}_1, \mathfrak{X}_0)\} .$$

It is easily checked that conditions (i) and (iii) of Definition 2.1 are satisfied by $\mathfrak{A}$, $\mathfrak{A}_0$, and $\mathfrak{A}_1$. The following argument will show that the "uniform crossnorm" condition (ii) is also satisfied and thus by

Theorem 2.3

$$\sigma(R) = \sigma\Big(\sum_{j=0}^{n} u_j^L (v^R)^j\Big) = \bigcup \Big\{\sigma\Big(\sum_{j=0}^{n} u_j^L z^j\Big): z \in \sigma(v^R)\Big\} .$$

This is the desired conclusion since $\sigma(v^R) = \sigma(v)$ and $\sigma(\sum_{j=0}^{n} u_j^L z^j) = \sigma(\sum_{j=0}^{n} u_j z^j)$.

Choose unit vectors $\{\alpha_n\}$ in X_0 and $\{\beta_n\}$ in $\mathfrak{X}_1^*$ so that $\|u\alpha_n\| \to \|u\|$ and $\|v^*\beta_n\| \to \|v\|$. Then, upon setting $x_n = \langle \cdot, \beta_n \rangle \alpha_n$, we have

$$\begin{aligned} \|u^L v^R\| &= \sup\{\|uxv\|: x \in \mathscr{L}(\mathfrak{X}_1, \mathfrak{X}_0), \|x\| = 1\} \\ &\geqq \limsup_n \|u x_n v\| \\ &= \limsup_n (\|v^*\beta_n\| \, \|u\alpha_n\|) = \|u\| \, \|v\| . \end{aligned}$$

Consequently we have $\|u^L v^R\| \geqq \|u^L\| \, \|v^R\|$, which proves that Theorem 2.3 is applicable.

(11) was proved by Harte ([6], Theorem 3.5) under the assumption that $\mathfrak{X}_0 = \mathfrak{X}_1$ is a Hilbert space.

Next we give an application of Theorem 2.3 similar to the one above to obtain a generalization of a result of Brown and Pearcy [1].

THEOREM 3.3. *Let $\mathfrak{H}$ be a complex Hilbert space and let c_p be the class of compact operators u in $\mathscr{L}(\mathfrak{H})$ for which*

$$\|u\|_p = [tr(u^*u)^{p/2}]^{1/p} < \infty \quad if \quad 1 \leqq p < \infty$$

and $\|u\|_\infty = \|u\|$. Fix $u_0, \cdots, u_n, v$ in $\mathscr{L}(\mathfrak{H})$ and define $R: \mathscr{L}(c_p) \to \mathscr{L}(c_p)$, $1 \leqq p \leqq \infty$ by $Rx = \sum_{j=0}^{n} u_j x v^j$. Then (11) *holds and if $\lambda \notin \sigma(R)$, so does* (12).

Indication of proof. Let $\mathfrak{A} = \mathscr{L}(c_p)$ and proceed as in the proof of Theorem 3.2.

The remaining applications deal with tensor products. The authors were led to the formulation of Definition 2.1 and Theorem 2.3 through efforts to unify these results and the preceding applications. In the next theorem (13) can be deduced from Harte ([6], Theorem 2.3), and (14) is new.

THEOREM 3.4. *Let $\mathfrak{X}_0, \mathfrak{X}_1, \cdots, \mathfrak{X}_m$ be complex Banach spaces and let $\mathfrak{X}$ be the completion of $\mathfrak{X}_0 \otimes \mathfrak{X}_1 \otimes \cdots \otimes \mathfrak{X}_m$ with respect to some uniform crossnorm. Let $\{u_j\}_{j=0}^{n} \subseteqq \mathscr{L}(\mathfrak{X}_0)$, $v_k \in \mathscr{L}(\mathfrak{X}_k)$, $k = 1, \cdots, m$ and let each of $p_0, \cdots, p_n$ be a polynomial in m variables.*
Define

$$\begin{aligned} u_j^{(0)} &= u_j \otimes \overbrace{I \otimes \cdots \otimes I}^{m \text{ terms}}, \ j = 0, \cdots, n \\ v^{(1)} &= I \otimes v_1 \otimes \cdots \otimes I \\ &\vdots \\ v^{(m)} &= I \otimes \cdots \otimes I \otimes v_m . \end{aligned}$$

Then

$$\begin{aligned} &\sigma\Big(\sum_{j=0}^{n} u_j^{(0)} p_j(v^{(1)}, \cdots, v^{(m)}) \Big| \mathscr{L}(\mathfrak{X})\Big) \\ &= \bigcup \Big\{\sigma\Big(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m) \Big| \mathscr{L}(\mathfrak{X}_0)\Big) \colon z_k \in \sigma(v_k \,|\, \mathscr{L}(\mathfrak{X}_k)) , \\ &\quad k = 1, \cdots, m\Big\} . \end{aligned} \tag{13}$$

Moreover, if $\lambda \notin \sigma(\sum_{j=0}^{n} u_j^{(0)} p_j(v^{(1)}, \cdots, v^{(m)}) \,|\, \mathscr{L}(\mathfrak{X}_0))$, *then*

$$\begin{aligned} &\Big(\sum_{j=0}^{n} u_j^{(0)} p_j(v^{(1)}, \cdots, v^{(m)}) - \lambda\Big)^{-1} \\ &= \Big(\frac{1}{2\pi i}\Big)^m \int_{C_1} \cdots \int_{C_m} \Big(\sum_{j=0}^{n} u_j p_j(z_1, \cdots, z_m) - \lambda\Big)^{-1} \\ &\quad \otimes (z_1 - v_1)^{-1} \otimes \cdots \otimes (z_m - v_m)^{-1} dz_1 \cdots dz_m \end{aligned} \tag{14}$$

where $C_1, \cdots, C_m$ *is in Theorem* 1.3.

Proof. Let $\mathfrak{A} = \mathscr{L}(\mathfrak{X})$ and

$$\begin{aligned} \mathfrak{A}_0 &= \{u \otimes I \otimes \cdots \otimes I \colon u \in \mathscr{L}(\mathfrak{X}_0)\} \\ \mathfrak{A}_1 &= \{I \otimes v_1 \otimes \cdots \otimes I \colon v_1 \in \mathscr{L}(\mathfrak{X}_1)\} \\ &\vdots \\ \mathfrak{A}_m &= \{I \otimes \cdots \otimes I \otimes v_m \colon v_m \in \mathscr{L}(\mathfrak{X}_m)\} . \end{aligned}$$

Each of $\mathfrak{A}_0, \cdots, \mathfrak{A}_m$ is a closed subalgebra of $\mathfrak{A}$ containing the identity $I \otimes \cdots \otimes I$. Since the crossnorm is uniform,

$$\|a_0 \cdots a_m\| = \|a_0\| \cdots \|a_m\| \text{ for } a_j \in \mathfrak{A}_j, \ j = 0, \cdots, m$$

and thus it is easily seen that $\mathfrak{A}_0, \cdots, \mathfrak{A}_m$ are independent algebras in $\mathfrak{A}$. Each singular element of $\mathfrak{A}_j$ is an $\mathfrak{A}_j$ generalized zero divisor (Rickart [10], p. 279). Then by Theorem 2.3

$$\begin{aligned} &\sigma\Big(\sum_{j=0}^{n} u_j^{(0)} p_j(v^{(1)}, \cdots, v^{(m)}) \Big| \mathscr{L}(\mathfrak{X})\Big) \\ &= \bigcup \Big\{\sigma\Big(\sum_{j=0}^{n} u_j^{(0)} p_j(z_1, \cdots, z_m) \Big| \mathscr{L}(\mathfrak{X})\Big) \colon z_k \in \sigma(v^{(k)} \,|\, \mathscr{L}(\mathfrak{X})) , \\ &\quad k = 1, \cdots, m\Big\} . \end{aligned}$$

The result now follows since $\sigma(v^{(k)} | \mathscr{L}(\mathfrak{X})) = \sigma(v_k | \mathscr{L}(\mathfrak{X}_k))$, $k = 1, \cdots, m$ and $\sigma(u^{(0)} | \mathfrak{A}) = \sigma(u | \mathfrak{A}_0)$ for any $u^{(0)}$ in $\mathfrak{A}$ of the form

$$u^{(0)} = u \otimes I \otimes \cdots \otimes I, \; u \in \mathscr{L}(\mathfrak{X}_0) .$$

As in the proofs of the preceding applications the representation formula (14) is a consequence of Theorem 1.3.

If in Theorem 3.4 we choose $u_0 = v_0 \in \mathscr{L}(\mathfrak{X}_0)$ and $u_j = 0$ for $j = 1, \cdots, n$, we obtain Schechter's result [12, Theorem 2.1]:

$$\begin{aligned}\sigma(p(v^{(0)}, v^{(1)}, \cdots, v^{(m)})) = \bigcup \{\sigma(p(v_0, z_1, \cdots, z_m) | \mathscr{L}(\mathfrak{X}_0)) &: \\ z_j \in \sigma(v_j | \mathscr{L}(\mathfrak{X}_j)), j = 1, \cdots, m\} &= p(\sigma(v_0), \sigma(v_1), \cdots, \sigma(v_m))\end{aligned}$$

for any polynomial p of $m + 1$ variables. More specifically we have the following result.

COROLLARY 3.5. *Let* $\mathfrak{X}_1, \cdots, \mathfrak{X}_m$ *satisfy the hypotheses of Theorem* 3.4 *and let* $v_j \in \mathscr{L}(\mathfrak{X}_j)$, $j = 1, \cdots, m$. *Then*

$$\sigma(v_1 \otimes \cdots \otimes v_m) = \prod_{k=1}^{m} \sigma(v_k | \mathscr{L}(\mathfrak{X}_k)) ,$$

and if $\lambda \notin \sigma(v_1 \otimes \cdots \otimes v_m)$, *then*

$$\begin{aligned}(v_1 \otimes \cdots \otimes v_m - \lambda)^{-1} &= \left(\frac{1}{2\pi i}\right)^m \int_{C_1} \cdots \int_{C_m} (z_1 \cdots z_m - \lambda)^{-1}) \\ &\otimes (z_1 - v_0)^{-1} \otimes \cdots \otimes (z_m - v_m)^{-1} dz_1 \cdots dz_m\end{aligned}$$

where $C_1, \cdots, C_m$ *is as in Theorem* 1.3.

Proof. Let $\mathfrak{X}_0$ be a one dimensional Hilbert space and set $u_j = 0$, $j \geqq 1$, $u_0 = 1$, $p_0(z_1, \cdots, z_m) = z_1 \cdots z_m$ in Theorem 3.4.

Next we consider a complex Hilbert space $\mathfrak{C}$ and let $H^2_{\mathfrak{C}}(U^m)$ be the Hardy space of $\mathscr{L}(\mathfrak{C})$-valued functions holomorphic in $U^m = U \times \cdots \times U$ (m factors), where U is the unit disk in the complex plane.

COROLLARY 3.6. *Let n be a nonnegative integer, and assume* $\{c_{j_1,\cdots,j_m} : 0 \leqq j_1, \cdots, j_m < \infty\} \subseteqq \mathscr{L}(\mathfrak{C})$, *where all but a finite numbers of the* $c_{j_1,\cdots,j_m}$ *are equal to* 0. *Define* $T: H^2_{\mathfrak{C}}(U^m) \rightarrow H^2_{\mathfrak{C}}(U^m)$ *by*

$$T: f(z_1, \cdots, z_m) \longrightarrow \sum_{j_1,\cdots,j_m} c_{j_1,\cdots,j_m} z_1^{j_1} \cdots z_m^{j_m} f(z_1, \cdots, z_m) .$$

Then

$$\sigma(T) = \left\{\sigma(\sum_{j_1,\cdots,j_m} c_{j_1,\cdots,j_m} z_1^{j_1} \cdots z_m^{j_m} | \mathscr{L}(\mathfrak{C})) : |z_k| \leqq 1, k = 1, \cdots, m\right\} .$$

Proof. $H^2_{\mathfrak{C}}(U^m)$ is the completion of $\mathfrak{C} \otimes H^2(U) \otimes \cdots \otimes H^2(U)$ under the Hilbert tensor product norm. If S is the unilateral shift on $H^2(U)$ defined by $(Sf)(z) = zf(z)$, then one can view T as

$$T = \sum c_{j_1,\cdots,j_m} S^{j_1} \otimes \cdots \otimes S^{j_m} .$$

The corollary now follows by applying (13) of Theorem 3.4 and noting that $\sigma(S) = \{z: |z| \leqq 1\}$.

Theorem 3.4 also leads to the following result:

COROLLARY 3.7. *Let $\mathfrak{C}$ be a complex Hilbert space, $\{u_j\}_{j=0}^n \subseteqq \mathscr{L}(\mathfrak{C})$ and define V on $\sum_{j=0}^\infty {}^\oplus \mathfrak{C}$ by*

$$V\colon \{c_j\}_{j=0}^\infty \longrightarrow \left\{\sum_{k=0}^n u_k c_{k+j}\right\}_{j=0}^\infty ,$$

so

$$V = \begin{pmatrix} u_0 & u_1 & \cdots & & u_n & 0 & 0 & \cdots \\ 0 & u_0 & u_1 & \cdots & & u_n & 0 & \cdots \\ 0 & 0 & u_0 & u_1 & \cdots & & u_n & \cdots \\ \vdots & \vdots & \ddots & & & & \ddots & \end{pmatrix} .$$

Then $\sigma(v) = \bigcup \{\sigma(\sum_{j=0}^n u_j z^j): |z| \leqq 1\}$.

Proof. $\sum_{j=0}^\infty {}^\oplus \mathfrak{C}$ is isomorphic to the Hilbert space $\mathfrak{C} \otimes H^2(U)$ under the isomorphism that sends $\{c_j\}_{j=0}^\infty$ into $\sum_{j=0}^\infty c_j \otimes z^j$. If S is the unilateral shift on $H^2(U)$, $Sf(z) = zf(z)$ then V is mapped into

$$\sum_{j=0}^n u_j \otimes S^{*j} .$$

Thus the corollary follows from Theorem 3.4. (14) can be used to set down a formula of $(v - \lambda)^{-1}$ if $\lambda \notin \sigma(v)$.

REFERENCES

1. A. Brown and C. Pearcy, *Spectra of tensor products of operators*, Proc. Amer. Math. Soc., **17** (1966), 162-169.
2. A. T. Dash and M. Schechter, *Tensor products and joint spectra*, Israel J. Math., **8** (1970), 191-193.
3. G. S. Datuasvili, *The spectrum of a generalized matrix polynomial* (in Russian, with Georgian summary), Sakharth SSR Mecn. Akad. Moambi, **44** (1966), 7-9.
4. J. Dixmier, *Les algèbres d'opèrateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1957.
5. R. E. Harte, *Spectral mapping theorems*, Proc. Royal Irish Acad., **72**, Section A, number 7 (1972), 89-107.

6. R. E. Harte, *Tensor products, multiplication operators and the spectral mapping theorem*, (preprint).
7. Peter Lancaster, *Explicit solutions of linear matrix equations*, SIAM Review, **12** (1970), 544-566.
8. G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc., **10** (1959), 32-41.
9. C. C. MacDuffee, *The theory of matrices*, Ergebnisse der Math., 1933.
10. C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, Princeton, 1960.
11. M. Rosenblum, *On the operator equation* $BX - XA = Q$, Duke Math. J., **23** (1956), 263-269.
12. M. Schechter, *On the spectra of operators on tensor products*, J. Functional Analysis, **4** (1969), 95-99.
13. A. Taylor, *Introduction to Functional Analysis*, Wiley, New York, 1958.
14. F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.

Received July 24, 1973. M. Rosenblum was supported in part by NSF Grant GP-31483X.

UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE
AND
UNIVERSITY OF VIRGINIA