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THEOREMS OF KOROVKIN TYPE FOR L_p-SPACES

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Suppose (X, Σ, μ) is a measure space, $1 < p < \infty$, $p \neq 2$, and that (T_n) is a net of linear contractions on (real or complex) $L_p(X, \Sigma, \mu)$. Let $M = \{x \in L_p: T_n x \to x\}$ (M is the *convergence set* for (T_n)). It is obvious that M is a closed subspace of L_p; indeed this would be true for an arbitrary normed space. In this paper we shall show that M is the range of a contractive projection on L_p and hence is itself isometrically isomorphic to an L_p-space. If $S \subset L_p(X, \Sigma, \mu)$ we can define the *shadow*, $\mathscr{S}(S)$ of S to be the set of all x in L_p such that $T_n x \to x$ for every net of linear contractions (T_n) such that $T_n y \to y$ for all $y \in S$. We shall also give a complete description of $\mathscr{S}(S)$ (for $p \neq 1, 2, \infty$).

Our results are new for finite p not equal to 1 or 2. In the case $p = 2$ the assertions about M are trivial and $\mathscr{S}(S)$ is the closed subspace spanned by S. The case $p = 1$ was first considered by Wulbert [9] for Lebesgue measure on [0, 1]. He showed that if $S = \{1, x, x^2\}$ then $\mathscr{S}(S) = L_1[0, 1]$. (Actually he considered sequences of contractions and required only $T_n 1 \to 1$ and $T_n f$ weakly convergent to f for $f = x$ and $f = x^2$.) Wulbert's results were inspired by and generalized the classical theorem of Korovkin [7] which contains the result that if $S = \{1, x, x^2\}$ then the shadow of S in $C[0, 1]$ is $C[0, 1]$. In [8] Lorentz considered separable L_1 spaces on finite measure spaces. He showed that for sequences of contractions such that $T_n 1 \to 1$ the convergence set is a closed sublattice of L_1. A corollary of this, which he noted, is that for $L_1[0, 1]$, $\mathscr{S}(S) = L_1$ if $S = \{1, x\}$. This last result and some further discussion of $L_1(X, \Sigma, \mu)$, with $\mu(X) = 1$ is also contained in [1].

The methods we use are suggested by the methods used in [3] in discussing contractive projections. I am very grateful to Professors Lorentz and Berens for discussions of this material and for supplying me with preprints of [1], [2], [8]. My first introduction to this circle of ideas was a colloquium lecture by Professor Lorentz in which some of the results from [2] and [8] were presented.

2. **The convergence set.** We shall fix notation as in the first paragraph of the introduction. It does not seem to matter whether our measure space is taken over a σ-ring, σ-algebra or δ-ring. For definiteness we shall assume that Σ is a σ-ring and measurability is as defined by Halmos [5]. We shall let q be the conjugate index to p, defined by $1/p + 1/q = 1$. Since $p \neq 1, 2, \infty$, the same is true

for q and $L_q(X, \Sigma, \mu)$ is the topological dual of $L_p(X, \Sigma, \mu)$ with the usual identifications. We shall consider the complex case; i.e., L_p (and L_q) are (equivalence classes of) complex valued functions. The real case is a little easier, but the methods are the same. If T is a bounded linear operator on L_p, the conjugate operator T^* is defined on L_q by the identity

$$\int x \cdot (T^*y) d\mu = \int (Tx) \cdot y d\mu \, (x \in L_p,\ y \in L_q) .$$

DEFINITION. The *conjugate convergence set* M^* for the net of contractions (T_n) is defined by $M^* = \{y \in L_q\colon T_n^* y \to y\}$.

LEMMA 2.1. (Compare [3, Lemma 2.2].) *Let* $x \in L_p$, *then* $x \in M$ *if and only if* $|x|^{p-1} \operatorname{sgn} \bar{x} \in M^*$.

Proof. Suppose $x \in M$ and write $u = |x|^{p-1} \operatorname{sgn} \bar{x}$. Then $\|u\|_q = \|x\|_p^{p/q}$ and $(T_n^* u)$ is a bounded net in L_q. Let w be a weak-* limit point of this net. We have

$$\begin{aligned} \int x \cdot (w - T_n^* u) d\mu &= \int x \cdot w d\mu - \int (T_n x) \cdot u d\mu \longrightarrow \int x \cdot w d\mu - \int x \cdot u d\mu \\ &= \int x \cdot (w - u) d\mu . \end{aligned}$$

Taking a subnet such that $T_n^* u \to w$ (weak $- *$), we conclude that

$$\int x \cdot w d\mu = \int x \cdot u d\mu = \int |x|^p = \|x\|_p^p .$$

Since the T_n^* are contractions, $\|T_n^* u\|_q \leqq \|u\|_q = \|x\|_p^{p/q}$ and hence $\|w\|_q \leqq \|x\|_p^{p/q}$. Hölders' inequality now gives

$$\|x\|_p^p = \int x \cdot w d\mu \leqq \|x\|_p \|w\|_q \leqq \|x\|_p \|x\|_p^{p/q} = \|x\|_p^p .$$

This gives equality throughout so [6, § 13.5] we have

$$w = |x|^{p-1} \operatorname{sgn} \bar{x} = u .$$

Thus u is the unique weak-* limit point of the net $(T_n^* u)$.

Since every subnet of $(T_n^* u)$ has a convergent subnet (by weak-* compactness of the unit ball in L_q), we see that $T_n^* u$ is weak-* convergent to u. Hence

$$\|u\|_q \leqq \liminf \|T_n^* u\|_q \leqq \limsup \|T_n^* u\|_q \leqq \|u\|_q ,$$

because the T_n^* are contractions; and we also have $\|u\| = \lim \|T_n^* u\|$. Because L_q is uniformly convex [4; 6, § 15.17] it follows that $T_n^* u \to u$

in the norm of L_q, which gives $|x|^{p-1}\operatorname{sgn}\bar{x} \in M^*$ as required.

The same argument applied to L_q shows that if

$$u = |x|^{p-1}\operatorname{sgn}\bar{x} \in M^*\ , \quad x = |u|^{q-1}\operatorname{sgn}\bar{u} \in M^{**} = M\ ,$$

so we are done.

We now apply differentiation arguments like those in [3, Lemma 2.3]. Recall that if z, w are complex, λ is real and $h(\lambda) = |z + \lambda w|$ then, if $z + \lambda w \neq 0$, h is differentiable at λ with

$$h'(\lambda) = \operatorname{Re}[w \operatorname{sgn}(\overline{z + \lambda w})]\ .$$

LEMMA 2.2. (Compare [3, Lemma 2.3(i)].) *If* $x, y \in M$, *then* $|x|\operatorname{sgn} y \in M$.

Proof. Assume first that $p > 2$ and define, for $\lambda \in R$, and $0 < \lambda < 1$,

$$\begin{aligned} z_\lambda &= \lambda^{-1}[|x + \lambda y|^{p-1}\operatorname{sgn}(\overline{x + \lambda y}) - |x|^{p-1}\operatorname{sgn}\bar{x}] \\ &= \lambda^{-1}[(|x + \lambda y|^{p-2} - |x|^{p-2})(\overline{x + \lambda y})] + |x|^{p-2}\bar{y}\ . \end{aligned}$$

Now, make a fixed choice of functions from the equivalence classes determined by x, y and observe that, except for the null-set where x or y is infinite, our differentiation result quoted above shows that as $\lambda \to 0$,

$$z_\lambda \to (p-2)|x|^{p-3}\cdot\operatorname{Re}[y \operatorname{sgn}\bar{x}]\cdot\bar{x} + |x|^{p-2}\bar{y}$$

at all points where $x \neq 0$. Also, since $p > 2$, $z_\lambda \to 0 (\lambda \to 0)$ at points where $x = 0$. Let z_1 denote this (almost everywhere) pointwise limit of z_λ as $\lambda \to 0$.

At points where $2|\lambda y| < |x|$, the mean value theorem gives a θ, $0 < \theta < 1$ such that

$$z_\lambda = (p-2)|x + \theta\lambda y|^{p-3}\operatorname{Re}[y \operatorname{sgn}(\overline{x + \theta\lambda y})](\overline{x + \lambda y}) + |x|^{p-2}\bar{y}\ .$$

Since $|x|/2 < |x + \theta\lambda y| < 2|x|$, we have

$$\begin{aligned} |z_\lambda| &\leqq (p-2)2^{|p-3|}|x|^{p-3}|y|\cdot 2|x| + |x|^{p-2}|y| \\ &\leqq (2^p(p-2) + 1)|x|^{p-2}|y| \in L_q\ . \end{aligned}$$

At points where $|x| \leqq 2|\lambda y|$, we have

$$\begin{aligned} |z_\lambda| &\leqq \lambda^{-1}(|3\lambda y|^{p-1} + |2\lambda y|^{p-1}) \\ &= (3^{p-1} + 2^{p-1})|y|^{p-1}|\lambda|^{p-2} \\ &\leqq (3^{p-1} + 2^{p-1})|y|^{p-1} \in L_q\ . \end{aligned}$$

Thus the pointwise convergence of z_λ to

$$z_1 = (p-2)\,|x|^{p-3}\bar{x}\,\mathrm{Re}\,(y\,\mathrm{sgn}\,\bar{x}) + |x|^{p-2}\bar{y}$$

is dominated by an element of L_q. The dominated convergence theorem then shows that $\|z_\lambda - z_1\| \to 0$. Now, by Lemma 2.1, $z_\lambda \in M^*$ and M^* is closed. Hence $z_1 \in M^*$.

Apply this result to x and $-iy$ to see that

$$z_2 = (p-2)\,|x|^{p-3}\bar{x}\,\mathrm{Re}\,(-iy\,\mathrm{sgn}\,\bar{x}) + i\,|x|^{p-2}\bar{y} \in M^* .$$

Thus $z_1 - iz_2 \in M^*$ and

$$\begin{aligned} z_1 - iz_2 = (p-2)\,|x|^{p-3}\bar{x}\,&[\mathrm{Re}\,y\,\mathrm{sgn}\,\bar{x} - i\,\mathrm{Im}\,y\,\mathrm{sgn}\,\bar{x}] \\ &+ 2\,|x|^{p-2}\bar{y} = p\,|x|^{p-2}\bar{y} . \end{aligned}$$

Use Lemma 2.1 again to conclude that

$$|\,|x|^{p-2}\bar{y}\,|^{q-1}\,\mathrm{sgn}\,(\overline{|x|^{p-2}\bar{y}}) = |x|^{1-(q-1)}\,|y|^{q-1}\,\mathrm{sgn}\,y \in M .$$

Let $k_n = |x|^{1-(q-1)^n}\,|y|^{(q-1)^n}\,\mathrm{sgn}\,y$. Observe that for each n, $k_{n+1} = |x|^{1-(q-1)}\,|k_n|^{q-1}\,\mathrm{sgn}\,k_n$. The argument we have just provided shows inductively that $k_n \in M$ for all n. Since $0 < q-1 < 1$, $k_n \to |x|\,\mathrm{sgn}\,y$, μ-almost everywhere, and clearly

$$|k_n| \leqq \max\,(|x|, |y|) \in L_p .$$

By dominated convergence again, $\|k_n - |x|\,\mathrm{sgn}\,y\|_p \to 0$ and hence, $|x|\,\mathrm{sgn}\,y \in M$ as required.

If $1 < p < 2$, then by Lemma 2.1, $x_1 = |x|^{p-1}\,\mathrm{sgn}\,\bar{x}$ and $y_1 = |y|^{p-1}\,\mathrm{sgn}\,\bar{y}$ are in M^*. Since $q > 2$, our proof above shows that $|x_1|\,\mathrm{sgn}\,y_1 = |x|^{p-1}\,\mathrm{sgn}\,\bar{y} \in M^*$. Apply Lemma 2.1 again to get $|x|\,\mathrm{sgn}\,y = |\,|x|^{p-1}\,\mathrm{sgn}\,\bar{y}\,|^{q-1}\,\mathrm{sgn}\,(\overline{|x|^{p-1}\,\mathrm{sgn}\,\bar{y}}) \in M$.

For our next result we need some terminology from [3]. A subspace N of $L_p(X, \Sigma, \mu)$ is a *vector sublattice* if for each $x \in N$, $(\mathrm{Re}\,x)^+ \in N$; this means that N is closed under taking real (or imaginary) parts and that the set of real functions in N is a real subspace and a sublattice of L_p. For a nonempty subset K of L_p, the *polar* $K^\perp = \{x \in L_p : |x| \wedge |y| = 0 (y \in K)\}$. A *band* in L_p is a subset K such that $K = K^{\perp\perp}$ (a band is necessarily a solid vector sublattice). If K is a band in L_p there is a natural direct sum decomposition $L_p = K \oplus K^\perp$ and the associated projections are positive and contractive. We write J_K for the band projection on K. If $K = y^{\perp\perp}$ (the only case we need) we write J_y for the band projection and note that J_y is multiplication by the characteristic function of the set on which y is nonzero.

LEMMA 2.3. *If $y \in M$ and J_y is the associated band projection, then $J_y M \subset M$.*

Proof. Let $x \in M$, then $J_y x = ||x| \operatorname{sgn} y| \operatorname{sgn} x$ and this element is in M by two applications of Lemma 2.2.

LEMMA 2.4. *If $x, y \in M$, then* $(\operatorname{Re}(x \operatorname{sgn} \bar{y}))^+ \operatorname{sgn} y \in M$.

Proof. Suppose $\lambda \in R, \lambda \neq 0$, then by Lemma 2.2, $v_\lambda = \lambda^{-1}(|y + \lambda x| - |y|) \operatorname{sgn} y \in M$. Since $v_\lambda = 0$ at points where $y = 0$ we see that as $\lambda \to 0$, v_λ converges pointwise (μ-almost everywhere) to $(\operatorname{Re}(x \operatorname{sgn} \bar{y})) \operatorname{sgn} y$. Since

$$|v_\lambda| \leqq |\lambda|^{-1} ||y + \lambda x| - |y|| \leqq |\lambda|^{-1} |y + \lambda x - y| = |x| ,$$

dominated convergence shows that $\| v_\lambda - (\operatorname{Re}(x \operatorname{sgn} \bar{y})) \operatorname{sgn} y \|_p \to 0$; so that $(\operatorname{Re}(x \operatorname{sgn} \bar{y})) \operatorname{sgn} y \in M$. Another application of Lemma 2.2 gives $|\operatorname{Re}(x \operatorname{sgn} \bar{y})| \operatorname{sgn} y \in M$ and our results follows.

THEOREM 2.5. *The convergence set M is the range of a contractive projection on L_p.*

Proof. For $y \in M$, Lemma 2.4 shows that the map $U_y: L_p \to L_p$, defined by $U_y x = x \operatorname{sgn} \bar{y}$, is norm decreasing, linear, and maps M onto a closed vector sublattice of L_p. Such a map was called a *unitary multiplication operator* in [3]. Choose, by Zorn's lemma, a maximal subset Y of M such that $|y_1| \wedge |y_2| = 0$ if $y_1, y_2 \in Y$ and $y_1 \neq y_2$ (a maximal pairwise disjoint subset of M). If $f \in L_p$ the set $\{t \in X : f(t) \neq 0\}$ is σ-finite so the set $\{y \in Y: U_y f \neq 0\}$ is countable. Thus we can define the direct sum U of the unitary multiplications $U_y (y \in Y)$ by $Uf = \sum_{y \in Y} U_y f$, and the defining sum has at most countably many nonzero terms and is convergent in L_p norm. Clearly UM is a closed vector sublattice of L_p. We show that U is isometric on M.

Suppose $x \in M, x \neq 0$. Let $y_1, \cdots, y_n \cdots$ be an enumeration of the countable set of $y \in Y$ such that $|y| \wedge |x| \neq 0$. Let $y_0 = \sum 2^{-n} \| y_n \|^{-1} y_n$. Then $y_0 \in M$ and, by Lemma 2.3, $J_{y_0} x \in M$. Hence $x - J_{y_0} x \in M$ and $|x - J_{y_0} x| \wedge |y| = 0 (y \in Y)$. By maximality of Y $x = J_{y_0} x$. Hence

$$x = \sum_{y \in Y} J_y x \text{ and } \| x \|^p = \sum_{y \in Y} \| J_y x \|^p = \sum_{y \in Y} \| U_y x \|^p = \| Ux \|^p .$$

It now follows by Theorem 4.1 of [3] that M is the range of a contractive projection on L_p and, which is an equivalent condition, that M is isometrically isomorphic to some $L_p(X_0, \Sigma_0, \mu_0)$.

3. The shadow of subset S. In a certain sense characterization of shadows is trivial. Call a subspace M of L_p an *exchange subspace* if $|x| \operatorname{sgn} y \in M$ for any $x, y \in M$. Clearly an intersection of (closed) exchange subspaces is again a (closed) exchange subspace. Hence for a subset S of L_p we can determine the closed exchange subspace of L_p generated by S as the intersection of all closed exchange subspaces of L_p which contains S.

THEOREM 3.1. *If $S \subset L_p$ then the shadow, $\mathscr{S}(S)$, of S is the closed exchange subspace of L_p generated by S.*

Proof. Lemma 2.2 shows that $\mathscr{S}(S)$ is a closed exchange subspace of L_p which contains S. If M denotes the closed exchange subspace of L_p generated by S a careful check of the proofs of Lemmas 2.3 and 2.4 and Theorem 2.5 show that these are valid for any closed exchange subspace of L_p. Hence M is the range of a contractive projection, say P, on L_p. Define a sequence (T_n) of linear contractions of L_p by $T_n = P (n = 1, 2 \cdots)$. Then M is the convergence set for (T_n) and hence, $\mathscr{S}(S) \subset M$. This proves our theorem.

As a corollary of the proof of Theorem 3.1 we digress to state the following result.

THEOREM 3.2. *Suppose $1 \leqq p < \infty$, $p \neq 2$, a subspace M of $L_p(X, \Sigma, \mu)$ is the range of a contractive projection if and only if M is a closed exchange subspace of $L_p(X, \Sigma, \mu)$.*

Proof. If M is a closed exchange subspace of L_p then just as in Theorem 3.1, Theorem 2.5 is valid for M. (This is equally true for $p = 1$ and $p = 2$ as can easily be checked.) By [3, Theorem 4.1] it follows that M is the range of a contractive projection on L_p. The converse result is the statement of [3, Lemma 2.3(i)] if $p \neq 1$ and an easy consequence of [3, Lemma 3.3] if $p = 1$.

Returning to shadows we note that Theorem 3.1 is difficult to apply in practice. The following alternative seems a little more useful.

Let $\mathscr{B}$ be the smallest sub σ-ring of Σ such that the functions $J_y x/y$ are $\mathscr{B}$-measurable for all $x, y \in S$. (To be precise here, we consider all choices of functions x, y in equivalence classes in S. The ratios are zero, by definition, wherever the denominators are zero.) Choose, by Zorn's lemma, a subset $\mathscr{K}$ of $\mathscr{B} \times S$ which is maximal with respect to the properties: (i) if $(A, y) \in \mathscr{K}$, then $A \subset \{t \in X : y(t) \neq 0\}$; (ii) if $(A, y) \in \mathscr{K}$, $\mu A > 0$; (iii) if (A_1, y_1), (A_2, y_2) are distinct elements of $\mathscr{K}$, then $\mu(A_1 \cap A_2) = 0$. Define a measure λ on $\mathscr{B}$ by

$$\lambda(B) = \sum_{(A,y)\in\mathscr{K}} \int_{A\cap B} |y|^p d\mu . \quad (B \in \mathscr{B}) .$$

Also define a map $V: L_p(X, \mathscr{B}, \lambda) \to L_p(X, \Sigma, \mu)$ by $Vf = \sum_{(A,y)\in\mathscr{K}} f\cdot y\chi_A$ $(f \in L_p(X, \mathscr{B}, \lambda))$. We note that the sets in $\mathscr{B}$ all have σ-finite μ-measure; that the sum defining Vf has, therefore, only countably many nonzero terms and is convergent in the norm of $L_p(X, \Sigma, \mu)$. Furthermore, V is an isometry of $L_p(X, \mathscr{B}, \lambda)$.

THEOREM 3.3. *The shadow* $\mathscr{S}(S) = VL_p(X, \mathscr{B}, \lambda)$.

Proof. Let U be a direct sum of unitary multiplications such that U is an isometry of $\mathscr{S}(S)$ and $U\mathscr{S}(S)$ is a closed vector sublattice of $L_p(X, \Sigma, \mu)$. For $f \in L_p$ write $T(f) = \{t \in X: f(t) \neq 0\}$, (we allow the ambiguity of sets of measure zero here); and let $\mathscr{B}_S = \{T(f): f \in \mathscr{S}(S)\}$. I claim that $\mathscr{B}_S$ is a sub σ-ring of Σ.

For this, observe that $T(f) = T(Uf)$ so we may assume that $\mathscr{S}(S)$ is a closed vector sublattice of L_p. Now $T(f) = T(|f|) = T(|\mathrm{Re}\, f|) \cup T(|\mathrm{Im}\, f|)$ and $|f| \in \mathscr{S}(S)$. Hence

$$\bigcup_{n=1}^{\infty} T(f_n) = T\Big(\sum_{n=1}^{\infty} 2^{-n} \|f_n\|^{-1} |f_n|\Big) \in \mathscr{B}_S$$

so $\mathscr{B}_S$ is closed under countable union. If $f, g \in \mathscr{S}(S)$, $J_f|g| = \lim_{n\to\infty} |g| \wedge n|f| \in \mathscr{S}(S)$ so $T(g) \sim T(f) = T(|g| - J_f|g|) \in \mathscr{B}_S$. This proves our claim about $\mathscr{B}_S$.

If $x, y \in S$, $\{t \in X: \mathrm{Re}\, J_y x/y > \alpha > 0\} = T((\mathrm{Re}\,(J_y x - \alpha y))^+) \in \mathscr{B}_S$. Hence, every $J_y x/y$ $(x, y \in S)$ is $\mathscr{B}_S$-measurable and $\mathscr{B}_S \supset \mathscr{B}$.

Suppose $B \in \mathscr{B}$ then $B = T(f_B)$ for some $f_B \in \mathscr{S}(S)$. If $\lambda(B) < \infty$, then $V\chi_B = \sum_{(A,y)\in\mathscr{K}} y \cdot \chi_{B\cap A}$. Since B is σ-finite we can enumerate the countable set of pairs (A, y) in $\mathscr{K}$ such that $\mu(B \cap A) \neq 0$ as (A_n, y_n) and choose $f_n \in \mathscr{S}(S)$ such that $T(f_n) = A_n \cap B$. Then $V\chi_B = \sum_{n=1}^{\infty} J_{f_n} y_n$. Since the A_n are disjoint and

$$\begin{aligned}\sum_{n=1}^{\infty} \|J_{f_n} y_n\|_{L_p(\mu)}^p &= \sum \int_{A_n\cap B} |y_n|^p d\mu \\ &= \sum \lambda(A_n \cap B) \\ &= \lambda B \\ &< \infty ,\end{aligned}$$

the series for $V\chi_B$ converges in $L_p(X, \Sigma, \mu)$. Since each $y_n \in S$ and each $f_n \in \mathscr{S}(S)$, each $J_{f_n} y_n \in \mathscr{S}(S)$ and $V\chi_B \in \mathscr{S}(S)$. Extending linearly to simple functions in $L_p(X, \mathscr{B}, \mu)$ and then taking closure we conclude that $VL_p(X, \mathscr{B}, \mu) \subset \mathscr{S}(S)$.

If $x \in S$ and $(A, y) \in \mathscr{K}$ then $\chi_A x/y$ is $\mathscr{B}$-measurable and

$$\int |\chi_A x/y|^p d\lambda = \int_A |x|^p d\mu < \infty .$$

Hence $z = \sum_{(A,y)\in\mathscr{K}} \chi_A x/y \in L_p(X, \mathscr{B}, \lambda)$ and $x = Vz \in VL_p(X, \mathscr{B}, \lambda)$. This shows that $VL_p(X, \mathscr{B}, \lambda) \supset S$.

If $f, g \in L_p(X, \mathscr{B}, \lambda)$, then $|f| \operatorname{sgn} g \in L_p(X, \mathscr{B}, \lambda)$ and $|Vf| \operatorname{sgn} Vg = V(|f| \operatorname{sgn} g)$. Thus $VL_p(X, \mathscr{B}, \lambda)$ is a closed exchange subspace of $L_p(X, \Sigma, \mu)$ which contains S. By Theorem 3.1 it follows that $\mathscr{S}(S) = VL_p(X, \mathscr{B}, \lambda)$ as required.

For applications of this result we recall that a subset S of a Banach space E is a *Korovkin set* for contractions if $\mathscr{S}(S) = E$.

In our closing results, μ is Lebesgue measure and Σ the σ-ring of Lebesgue measurable subsets of R, or R^n as appropriate. Also, remember that $1 < p < \infty$, $p \neq 2$.

THEOREM 3.4. *If* $-1/p < \alpha < \beta$ *then* $\{t^\alpha, t^\beta\}$ *is a Korovkin set for* $L_p([0, 1], \Sigma, \mu)$.

Proof. In the construction preceding Theorem 3.3, we see that $\mathscr{B}$ is the σ-ring of Borel subsets of $[0, 1]$. Take $\mathscr{K} = \{([0, 1], t^\beta)\}$ and observe that V is an isometry of $L_p([0, 1], \Sigma, \mu)$ and $L_p([0, 1], \mathscr{B}, \lambda)$.

Observe that we could also use $\{\cos t, \sin t\}$ on $[0, \pi/2]$ or on $[0, \pi]$ or, of course, many other doubleton sets on many other finite intervals.

THEOREM 3.5. *Let* $X = [0, 1]^n \subset R^n$ *and let* S *contain the constant function* 1 *and the* n *coordinate projections, then* S *is a Korovkin set for* $L_p(X, \Sigma, \mu)$.

One last result for a case when μ is not finite will now suffice.

THEOREM 3.6. *Let* $S = \{e^{-t}, e^{-t^2}\}$, *then* S *is a Korovkin set for* $L_p([0, \infty], \Sigma, \mu)$.

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