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STRONGLY SEMISIMPLE ABELIAN GROUPS

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For an abelian group G and a ring R, R is a ring on G if the additive group of R is isomorphic to G. G is nil if the only ring R on G is the zero ring, $R^2 = \{0\}$. G is radical if there is a nonzero ring on G that is radical in the Jacobson sense. Otherwise, G is antiradical. G is semisimple if there is some (Jacobson) semisimple ring on G, and G is strongly semisimple if G is nonnil and every nonzero ring on G is semisimple. It is shown that the only strongly semisimple torsion groups are cyclic of prime order, and that no mixed group is strongly semisimple. The torsion free rank one strongly semisimple groups are characterized in terms of their type, and it is shown that the strongly semisimple and antiradical rank one groups coincide. For torsion free groups it is shown that the property of being strongly semisimple is invariant under quasi-isomorphism and that a strongly semisimple group is strongly indecomposable. Further, for a strongly indecomposable torsion free group G of finite rank, the following are equivalent: (a) G is semisimple, (b) G is strongly semisimple, (c) $G \cong R^+$ where R is a full subring of an algebraic number field K such that $[K,Q] = \text{rank } G$ where Q is the field of rational numbers and $R \doteq J_\pi$, where π is either empty or an infinite set of primes in K, $(d) G$ is nonnil and antiradical.

Introduction. In [4], F. Haimo considered the problem of characterizing those abelian groups G that are the additive groups of nontrivial radical rings, where the radical under consideration is the Jacobson radical. It was observed by the present authors that for several classes of groups, those groups G that did not support nontrivial radical rings (antiradical groups) satisfied a much stronger condition, namely, that every nontrivial ring on G is semisimple (strongly semisimple groups). This suggested the problem of identifying classes of groups for which the antiradical and strongly semisimple groups coincide, and the problem of characterizing strongly semisimple groups.

Section 1 contains the basic definitions. The case of torsion and mixed groups is disposed of in § 2 where it is shown that the only strongly semisimple torsion groups are the cyclic groups of prime order, and that no mixed group is strongly semisimple. In § 3, the torsion free rank one strongly semisimple groups are characterized in terms of their type, and it is shown that the strongly semisimple and antiradical groups coincide. In § 4, it is shown that the property of being strongly semisimple for a torsion free group is invariant under quasi-isomorphism and that a strongly semisimple torsion free group is strongly indecomposable. Applying results on torsion free rings in [1], [2], and [6], we show in §5, that the results for rank one can be recovered for strongly indecomposable torsion free groups of finite rank.

Throughout the paper, group means additive abelian group and ring means associative ring. The notation is standard and generally follows that of [3]. The field of rational numbers as well as its additive group is denoted by Q, Z denotes the integers, and $\mathscr{J}(R)$ is the Jacobson radical of a ring R.

1. **Definitions.** For any ring R, let $\mathscr{N}(R)$ denote the sum of all nilpotent left ideals. $\mathscr{N}(R)$ is a nil ideal and $\mathscr{N}(R)$ contains all nilpotent right ideals. Let $\mathscr{J}(R)$ denote the Jacobson radical of R [5]. Then $\mathscr{J}(R) \supseteq \mathscr{N}(R)$. If G is any group, then R is a *ring on G* if R^+, the additive group of R, is isomorphic to G. The *zero ring on G* is the ring obtained by defining $x \cdot y = 0$ for all $x, y \in G$. If R is the zero ring on G, then $R^2 = \{0\}$, and $R = \mathscr{J}(R) = \mathscr{N}(R)$. A group G is a *nil group* if the only ring on G is the zero ring. Otherwise G is *nonnil.*

DEFINITION 1.1. (Haimo, [4]) A group G is a *radical group* if there is a ring R on G such that $R = \mathscr{J}(R)$ and R is not the zero ring on G. Otherwise, G is an *antiradical group.*

DEFINITION 1.2. A group $G \neq \{0\}$ is a *semisimple group* if there is a ring R on G such that $\mathscr{J}(R) = \{0\}$.

DEFINITION 1.3. A group G is a *strongly semisimple* group if G is nonnil and $\mathscr{J}(R) = \{0\}$ for every nonzero ring R on G.

We note that a nil group is antiradical, a semisimple group is nonnil and that a strongly semisimple group is semisimple and antiradical. Moreover, if there is a nonzero ring R on G such that $\mathscr{N}(R) \neq \{0\}$, then G is not strongly semisimple.

The cyclic group of order six, $Z(6)$, is semisimple, antiradical, and not strongly semisimple. The direct sum of $2^{\aleph_0}$ copies of the additive group of rational numbers, Q, is radical and semisimple [4].

The following simple observation will be useful.

LEMMA 1.4. *If $G = H \oplus K$, $H \neq \{0\}$, $K \neq \{0\}$, and either H or K is nonnil, then there is a nonzero ring R on G such that $\mathscr{N}(R) \neq \{0\}$.*

Proof. Suppose H is nonnil and let R_H be a nonzero ring on H. Let R_K be the zero ring on K. Then the ring direct sum $R = R_H \dotplus R_K$ is a nonzero ring on $H \oplus K$ such that $\mathscr{N}(R) \supseteqq R_K \neq \{0\}$.

2. Torsion and mixed groups.

As mentioned in the introduction, our purpose is to characterize strongly semisimple groups. This is easily done if G is torsion group or a mixed group.

By Theorem 69.3 [3], there is a ring R on a torsion group G with $\mathscr{N}(R) = \{0\}$ if and only if G is an elementary group. Since an elementary group $G \neq \{0\}$ is the additive group of a direct sum of fields, there is a ring on G with $\mathscr{J}(R) = \{0\}$. Thus, a torsion group G is a semisimple group if and only if G is elementary.

THEOREM 2.1. *The only strongly semisimple torsion groups are the cyclic groups of order p, $Z(p)$, p a prime.*

Proof. If G is strongly semisimple, then G is semisimple, and by the above remarks, G is the direct sum of cyclic groups of order p for various primes p. Since the groups $Z(p)$ are nonnil, it follows from 1.4 that if the direct decomposition of G has more than one component, then G is not strongly semisimple. On the other hand, every nonzero ring on $Z(p)$ is isomorphic to the field with p elements. Thus, $Z(p)$ is strongly semisimple.

THEOREM 2.2. *If G is a mixed group, then there is a nonzero ring R on G such that $\mathscr{N}(R) \neq \{0\}$.*

Proof. Suppose first that G is not reduced. If the maximal divisible subgroup, G_d, of G is not torsion, then $G = Q \oplus G_1$, where $G_1 \neq \{0\}$. By 1.4, there is a nonzero ring R on G with $\mathscr{N}(R) \neq \{0\}$. If G_d is torsion, then since G_d is a nil group and G_d is an ideal in any ring R on G, $\mathscr{N}(R) \supseteqq G_d \neq \{0\}$ for any ring R on G. Since G is mixed there is a nonzero ring R on G. On the other hand, if G is reduced, then $G = \{x\} \oplus G_2$, where $\{x\}$ is a finite cyclic group and $G_2 \neq \{0\}$. Since $\{x\}$ is nonnil, it again follows from 1.4 that there is a ring R on G with $\mathscr{N}(R) \neq \{0\}$.

COROLLARY 2.3. *No mixed group is strongly semisimple.*

3. Rank one torsion free groups.

We first characterize the strongly semisimple torsion free groups of rank 1 in terms of their types. If G is a rank 1 group, we write the type of G as $T(G) = [(k_1, k_2, \cdots, k_n, \cdots)]$, where $(k_1, k_2, \cdots, k_n, \cdots)$ is the height of a nonzero element $g \in G$; that is, for the prime p_n, k_n is the p_n-height of g.

Let π be an arbitrary set of primes, and let m be a fixed positive integer such that $(m, p) = 1$ for every $p \in \pi$. Denote by $S(m, \pi)$ that subring of Q consisting of all rational numbers of the form mr/s, where s is a product of primes in π and r is any integer. Let π^c be the complement of π in the set of all primes, and let π' be the set of all primes $p \in \pi^c$ such that $(m, p) = 1$.

LEMMA 3.1. *$\mathscr{J}(S(m, \pi)) = \bigcap_{p \in \pi'} pS(m, \pi)$ if $\pi^c \neq \phi$. Otherwise $\mathscr{J}(S(m, \pi)) = \{0\}$.*

Proof. If $\pi^c = \phi$, then $m = 1$ and $S(m, \pi) = Q$. Thus, $\mathscr{J}(S(m, \pi)) = \{0\}$. We show that if $\pi^c \neq \phi$, then $\{pS(m, \pi) \mid p \in \pi'\}$ is the collection of maximal modular ideals in $S(m, \pi)$. Since $p \in \pi' \subseteqq \pi^c$, the ideal $pS(m, \pi) \neq S(m, \pi)$, e.g., $m \notin pS(m, \pi)$. If $mr/s \in S(m, \pi)$ and $mr/s \notin pS(m, \pi)$, we have $(r, p) = 1$. Then $xr + yp = 1$ for $x, y \in Z$, and $xmr + ymp = m$. Thus, m is in the ideal generated by $pS(m, \pi)$ and mr/s. That is, this ideal is $S(m, \pi)$. Hence $pS(m, \pi)$ is maximal. Since $p \in \pi'$, $(m, p) = 1$. Thus, $xm + yp = 1$ for $x, y \in Z$. If $mr/s \in S(m, \pi)$, then $(xm)(mr/s) + (yp)(mr/s) = mr/s$, or $mr/s - (xm)(mr/s) = pymr/s \in pS(m, \pi)$. Hence xm is an identity modulo $pS(m, \pi)$. Therefore, $pS(m, \pi)$ is modular.

Suppose that $\mathscr{I} \neq \{0\}$ is an ideal in $S(m, \pi)$. If mr is the least positive integer in $\mathscr{I}$, then every element of $\mathscr{I}$ is a multiple of mr. Note that $(mr, p) = 1$ for $p \in \pi$. If $r = 1$, $\mathscr{I} = S(m, \pi)$. If $r \neq 1$, $\mathscr{I} \subseteqq rS(m, \pi) \subseteqq pS(m, \pi)$ for some $p \in \pi^c$. Thus, the maximal ideals in $S(m, \pi)$ are the ideals $pS(m, \pi)$ for $p \in \pi^c$. If $pS(m, \pi)$ is modular, then in particular, there is an element $mr/s \in S(m, \pi)$ such that $m - (mr/s)m = pmr'/s' \in pS(m, \pi)$. This equation yields $ss' - mrs' = pr's$. Since $(ss', p) = 1$ for $p \in \pi^c$, it follows that $(m, p) = 1$. That is, $p \in \pi'$.

Note that if $\pi' = \phi$, then the collection of maximal modular ideals is vacuous and $\mathscr{J}(S(m, \pi)) = S(m, \pi) = \bigcap_{p \in \pi'} pS(m, \pi)$ [5, p. 9].

THEOREM 3.2. *Let G be a torsion free group of rank 1. Then the following statements are equivalent:*

(a) *G is semisimple.*

(b) *G is strongly semisimple.*

(c) *$T(G) = [(k_1, k_2, \cdots, k_n, \cdots)]$, where $k_n = 0$ or ∞ for all n, and either $k_n = \infty$ for all n or $k_n = 0$ for infinitely many n.*

(d) *G is nonnil and antiradical.*

Proof. It follows from Definitions 1.2 and 1.3 that (b) implies (a).

To prove that (c) implies (b), we note that since $k_n = 0$ or ∞ for all n, it follows [3, p. 269] that G is nonnil and any nonzero ring R on G is isomorphic to a subring $S(m, \pi)$ of Q, where π is the set of all primes for which $k_n = \infty$. If $k_n = \infty$ for all n, then $R \cong S(m, \pi) = Q$, so that G is strongly semisimple. If $k_n = 0$ for infinitely many n, then π^c is infinite, so that π', the set of all primes p in π^c such that $(m, p) = 1$, is also infinite. If $mr/s \in \bigcap_{p \in \pi'} pS(m, \pi)$, then $p \mid r$ for all $p \in \pi'$. Hence, $mr/s = 0$. By Lemma 3.1, $\mathscr{J}(S(m, \pi)) = \{0\}$. Therefore, G is strongly semisimple.

We next show that (a) implies (c). Assume that (c) is not satisfied. Then either $0 < k_n < \infty$ for infinitely many n, or $k_n = 0$ or ∞ for all n and $k_n = \infty$ for almost all n, but not all n. In the first case G is a nil group, and therefore not semisimple. In the second case, any ring R on G is isomorphic to an $S(m, \pi)$, where π^c is finite and not empty. Therefore, π' is finite. By Lemma 3.1, $\mathscr{J}(S(m, \pi)) = \bigcap_{p \in \pi'} pS(m, \pi) = S(m, \pi)$ if $\pi' = \phi$, and $\mathscr{J}(S(m, \pi)) = p_1 p_2 \cdots p_k S(m, \pi) \neq \{0\}$ if $\pi' = \{p_1, p_2, \cdots, p_k\} \neq \phi$. Therefore, G is not semisimple.

Since (b) $\Rightarrow$ (d) by Definitions 1.1 and 1.3, we complete the chain of implications, by showing that (d) implies (c). Here we observe from the above argument that if (c) is not satisfied, then either G is a nil group, or any ring R on G is either radical or $p_1 p_2 \cdots p_k R$ is radical. But since $G \cong R^+ \cong (p_1 p_2 \cdots p_k R)^+$, G is isomorphic to a radical group, and hence is radical. Haimo [4, Theorem 4] proves that (c) and (d) are equivalent in a somewhat different manner.

COROLLARY 3.3. *Z and Q are strongly semisimple.*

4. Quasi-isomorphism.

We show that for torsion free groups the property of being strongly semisimple is invariant under quasi-isomorphism. This follows from Thorem 2.6 and Corollary 2.7 in [2]. Theorem 2.6 in [2] states that if G and H are quasi-isomorphic, and if R is a ring on G, then there is a ring S on H and a positive integer n such that S is isomorphic to a subring T of R and $nR \subseteq T$. Corollary 2.7 in [2] states that if G and H are quasi-isomorphic and R is a ring on G, then there is a ring S on H such that the rational algebras $Q \otimes_Z R$ and $Q \otimes_Z S$ are isomorphic. It follows at once from this result that if G and H are quasi-isomorphic and G is a nil group, then H is a nil group.

THEOREM 4.1. *Let G and H be quasi-isomorphic torsion free groups. If G is strongly semisimple, then so is H.*

Proof. Assume that G is strongly semisimple and H is not strongly semisimple. Then either H is nil or there is a nonzero ring R on H such that $\mathscr{J}(R) \neq \{0\}$. If H is nil, then G is nil and hence not strongly semisimple. In the second case, it follows from the above remarks, that there is a nonzero ring S on G such that S is isomorphic to a subring T of R and $nR \subseteqq T$ for some positive integer n. Now $n\mathscr{J}(R)$ is an ideal in R and $n\mathscr{J}(R) \subseteqq nR \subseteqq T$. Thus, $n\mathscr{J}(R)$ is an ideal in T. Therefore, $\mathscr{J}(T) \supseteqq \mathscr{J}(T) \cap n\mathscr{J}(R) = \mathscr{J}(n\mathscr{J}(R))$. Moreover, $\mathscr{J}(n\mathscr{J}(R)) = \mathscr{J}(R) \cap n\mathscr{J}(R) = n\mathscr{J}(R)$. Hence $\mathscr{J}(T) \supseteqq n\mathscr{J}(R) \neq \{0\}$. Therefore, T is not semisimple, and consequently S is not semisimple. This contradicts the hypothesis that G is strongly semisimple.

A torsion free group G is *strongly indecomposable* if whenever G is quasi-isomorphic to a direct sum $G_1 \oplus G_2$, G_1 and G_2 torsion free, then either $G_1 = \{0\}$ or $G_2 = \{0\}$. Otherwise G is *quasi-decomposable.*

THEOREM 4.2. *A strongly semisimple torsion free group G is strongly indecomposable.*

Proof. Assume that G is quasi-decomposable. Then G is quasi-isomorphic to $G_1 \oplus G_2$, where G_1 and G_2 are nonzero torsion free groups. By Theorem 4.1, $G_1 \oplus G_2$ is strongly semisimple. If either G_1 or G_2 is nonnil, then by Lemma 1.4, $G_1 \oplus G_2$ is not strongly semisimple. Hence we may assume that both G_1 and G_2 are nil groups. Moreover, $G_1 \oplus G_2$, being strongly semisimple, is nonnil.

Let $*$ be a nontrivial associative multiplication on $G_1 \oplus G_2$. Let π_{G_1} and π_{G_2} be the projections of $G_1 \oplus G_2$ onto G_1 and G_2, respectively. For (x_1, y_1), (x_2, y_2) in $G_1 \oplus G_2$ define a multiplication $\circ$ on $G_1 \oplus G_2$ by $(x_1, y_1) \circ (x_2, y_2) = (0, \pi_{G_2}[(x_1, 0)*(x_2, 0)])$. If $\pi_{G_2}[(x_1, 0)*(x_2, 0)] \neq 0$ for some $x_1, x_2 \in G_1$, then $\circ$ is an associative multiplication on $G_1 \oplus G_2$ such that $(G_1 \oplus G_2)^2 \neq \{0\}$ and $(G_1 \oplus G_2)^3 = \{0\}$. Therefore, $G_1 \oplus G_2$ is a radical group, contradicting the fact that $G_1 \oplus G_2$ is strongly semisimple. If $\pi_{G_2}[(x_1, 0)*(x_2, 0)] = 0$ for all $x_1, x_2 \in G_1$, define $(x_1, y_1) \times (x_2, y_2) = (\pi_{G_1}[(0, y_1)*(0, y_2)], 0)$. As above, if $\pi_{G_1}[(0, y_1)*(0, y_2)] \neq 0$ for some $y_1, y_2 \in G_2$, $\times$ is an associative multiplication on $G_1 \oplus G_2$ such that $(G_1 \oplus G_2)^2 \neq \{0\}$ and $(G_1 \oplus G_2)^3 = \{0\}$, again contradicting the fact that $G_1 \oplus G_2$ is strongly semisimple.

We may now assume that $\pi_{G_2}[(x_1, 0)*(x_2, 0)] = 0$ for all $x_1, x_2 \in G_1$ and that $\pi_{G_1}[(0, y_1)*(0, y_2)] = 0$ for all $y_1, y_2 \in G_2$. It follows that $(x_1, 0)*(x_2, 0) = (x(x_1, x_2), x_1), 0)$ for all $x_1, x_2 \in G_1$ and that $(0, y_1)*(0, y_2) = (0, y(y_1, y_2))$ for all $y_1, y_2 \in G_2$. Then $x_1 * x_2 = x(x_1, x_2)$ and $y_1 * y_2 = y(y_1, y_2)$ are associative multiplications on G_1 and G_2, respectively. Since G_1 and G_2 are nil groups, $x(x_1, x_2) = y(y_1, y_2) = 0$ for all $x_1, x_2 \in G_1$ and all $y_1, y_2 \in G_2$. That is, $(x_1, 0)*(x_2, 0) = (0, 0)$ and $(0, y_1)*(0, y_2) = (0, 0)$ for all $x_1, x_2 \in G_1$ and all $y_1, y_2 \in G_2$, so that under the multiplication $*$, G_1 and G_2 are subrings of $G_1 \oplus G_2$ such that $G_1^2 = \{0\}$ and $G_2^2 = \{0\}$.

Let $r(G_1)$ and $r(G_2)$ be the right annililators of G_1 and G_2, respectively in the ring $G_1 \oplus G_2$ with multiplication $*$. Then $r(G_1) \supseteqq G_1$ and

$r(G_2) \supseteqq G_2$. If $r(G_1) = G_1$, then $r(G_1)$ is a nonzero nilpotent right ideal in $G_1 \oplus G_2$. Hence $\mathscr{J}(G_1 \oplus G_2) \supseteqq r(G_1) = G_1 \neq \{0\}$. Therefore, the group $G_1 \oplus G_2$ is not strongly semisimple, contrary to hypothesis. If $r(G_1) \supset G_1$, let $g \in r(G_1)$, $g \notin G_1$. Since $r(G_1) \supseteqq G_1$ and $r(G_2) \supseteqq G_2$, $r(G_1) + r(G_2) = G_1 \oplus G_2$. Therefore, $g = g_1 + g_2$, where $g_1 \in G_1$, $g_2 \in G_2$, and $g_2 \neq 0$. Thus, $g_2 = g - g_1 \in r(G_1) \cap G_2$. It follows that the infinite cyclic group (g_2) is a nonzero nilpotent left ideal in $G_1 \oplus G_2$. Indeed, if $x + y \in G_1 \oplus G_2$, $x \in G_1$, $y \in G_2$, then $(x + y)^* g_2 = x^* g_2 + y^* g_2 = 0 + 0 = 0$, since $g_2 \in r(G_1) \cap G_2$. Therefore, $\mathscr{J}(G_1 \oplus G_2) \supseteqq (g_2) \neq \{0\}$, and again we have contradicted the fact that $G_1 \oplus G_2$ is strongly semisimple, completing the proof.

5. **Strongly indecomposable torsion free groups.** Theorem 4.2 allows us to restrict our attention to strongly indecomposable torsion free groups in our investigation of strongly semisimple groups. It is possible to generalize Theorem 3.2 for rank one groups to strongly indecomposable groups of finite rank. To do this, we rely heavily on results in [1] and [2].

If H is a subgroup of the torsion free group G such that G/H is a torsion group, then H is a *full subgroup* of G. A subring S of a torsion free ring R is a *full subring* of R if S^+ is a full subgroup of R^+. We recall that each torsion free ring R is naturally embedded as a full subring of the rational algebra $Q \otimes_Z R$ [2].

In the following lemmas, G is a strongly indecomposable torsion free group of finite rank n.

LEMMA 5.1. *If R is any ring on G, then either $\mathscr{J}(R) = \{0\}$ or $R/\mathscr{J}(R)$ is finite.*

Proof. It follows frow Theorem 1.4, Corollary 3.6, and Theorem 1.13 in [2], that if R is any ring on G, then the rational algebra $Q \otimes_Z R$ is either nilpotent or is an algebraic number field of dimension n over Q. In the first case, R is a nilpotent ring, so that $\mathscr{J}(R) = R$. In the second case, if I is a nonzero ideal in R, then R/I is finite [1, p. 206]. Thus, if $Q \otimes_Z R$ is an algebraic number field, then either $\mathscr{J}(R) = \{0\}$ or $R/\mathscr{J}(R)$ is finite.

A torsion free group G is *quotient divisible* (or a q.d. group) if G contains a full, free subgroup F such that G/F is divisible. Each torsion free group G of rank n is embedded in a rational vector space V of dimension n. Let $\mathscr{L}(V)$ be the ring of all linear transformations of V. Then

$$\mathscr{E}(G) = \{\phi \in \mathscr{L}(V) \mid n\phi(G) \subseteqq G \text{ for some } n \neq 0 \text{ in } Z\}$$

is a subring of $\mathscr{L}(V)$ called the *ring of quasi-endomorphisms* of G.

LEMMA 5.2. *If G is semisimple, then G is a q.d. group, $\mathscr{E}(G)$ is an algebraic number field K such that $[K: Q] = n$, and there is a ring R on G such that $Q \otimes_Z R \cong K$.*

Proof. Let S be a (Jacobson) semisimple ring on G. If N is the radical of $Q \otimes_Z S$, then $N \cap S$ is the maximum nilpotent ideal in S, and the rank of $N \cap S$ is equal to the dimension of N [2, p. 71]. Since $N \cap S \subseteqq \mathscr{J}(S) = \{0\}$, it follows that $N = \{0\}$. Thus, $Q \otimes_Z S$ is a semisimple algebra. By [2, Corollary 4.9] G is a q.d. group. By [1, Corollary 4.6], $\mathscr{E}(G)$ is an algebraic number field K such that $[K: Q] = n$. Finally, by [1, Theorem 4.1], there is a ring R on G such that $Q \otimes_Z R \cong K$.

LEMMA 5.3. *If G is semisimple, then every ring R on G is isomorphic to a full subring of a single algebraic number field K such that $[K: Q] = n$.*

Proof. As in the proof of Lemma 5.1, if R is any ring on G, then $Q \otimes_Z R$ is either nilpotent or is an algebraic number field K such that $[K: Q] = n$. Wickless [6, Theorem 2.3] shows that it is impossible for both alternatives to hold. By Lemma 5.2, there is a ring R on G such that $Q \otimes_Z R \cong K = \mathscr{E}(G)$. Therefore, by Wickless' result, if S is any ring on G, then $Q \otimes_Z S$ is an algebraic number field L such that $[L, Q] = n$. But by [1, Theorem 4.1] $L = \mathscr{E}(G) = K$. That is, every ring R on G is isomorphic to a full subring of $\mathscr{E}(G)$.

Two subrings R and S of an algebraic number field K are *quasi-equal* ($R \doteq S$) if there is a positive integer n such that $nR \subseteqq S$ and $nS \subseteqq R$.

LEMMA 5.4. *Let R and S be subrings of an algebraic number field K such that $R \doteq S$. Suppose further that R^+ is strongly indecomposable. If R is semisimple, then so is S.*

Proof. Assume that S is not semisimple. Note that since $R \doteq S$, then R^+ and S^+ are quasi-isomorphic, so that S^+ is strongly indecomposable. Since $\mathscr{J}(S) \neq \{0\}$, it follows from Lemma 5.1, that there is a positive integer n such that $nS \subseteqq \mathscr{J}(S)$. Since $R \doteq S$, there is a positive integer m such that $mR \subseteqq S$ and $mS \subset R$. Hence $nmR \subseteqq nS \subseteqq \mathscr{J}(S)$. Since $mS \subseteqq R$, $nm^2R \subseteqq nmS \subseteqq m\mathscr{J}(S)$. Each element of nm^2R has a quasi-inverse in the quasi-regular ideal $m\mathscr{J}(S)$ of S. Moreover, $m\mathscr{J}(S) \subseteqq mS \subseteqq R$. Thus, each element of nm^2R has a quasi-inverse in R. But nm^2R is an ideal in R, hence a quasi-regular ideal. Therefore, $\mathscr{J}(R) \supseteqq nm^2R \neq \{0\}$, contradicting the hypothesis that R is semisimple.

Let J be the ring of integers in an algebraic number field K. In [1] it is shown that the quasi-equality classes of full subrings of K are in one-to-one correspondence with the sets of prime ideals in J. If P is any prime ideal in J, let $J_P = \{x/y \mid x, y \in J, y \notin P\}$. Also, if π is any set of prime ideals in J, define $J_\pi = \bigcap_{P \in \pi} J_P$. Then every quasi-equality class of full subrings of K contains one of rings J_π, J_π is integrally closed and is the integral closure of every ring in its class. It should be noted that the prime ideals of J_π are precisely the ideals PJ_π and that nonzero prime ideals in J_π are maximal. It follows that $\mathscr{J}(J_\pi) = \bigcap_{P \in \pi} PJ_\pi$.

THEOREM 5.5. *Let G be a strongly indecomposable torsion free group of finite rank n. Then the following statements are equivalent.*

(a) *G is semisimple.*

(b) *G is strongly semisimple.*

(c) *$G \cong R^+$, where R is a full subring of an algebraic number field K such that $[K, Q] = n$, and $R \doteq J_\pi$, where π is either empty or infinite.*

(d) *G is nonnil and antiradical.*

Proof. By definition, (b) implies (a) and (b) implies (d). We show that (d) implies (b), (a) implies (c), and (c) implies (b).

(d) *implies* (b). Assume that G is not strongly semisimple. Then G is either a nil group or there is a nonzero ring R on G such that $\mathscr{J}(R) \neq 0$. In the latter case, it follows from Lemma 5.1 that there is a positive integer m such that $mR \subseteq \mathscr{J}(R)$. Since mR is an ideal in R, we have $\mathscr{J}(mR) = \mathscr{J}(R) \cap mR = mR$. Hence mR is a nonzero radical ring, so that mG is a radical group. Since $G \cong mG$, it follows that G is a radical group. Thus, if G is not strongly semisimple, then G is either nil or radical.

(a) *implies* (c). By Lemma 5.3, every ring R for which $G \cong R^+$ is a full subring of an algebraic number field K such that $[K, Q] = n$. By the remarks preceding the theorem $R \doteq J_\pi$ for some set π of prime ideals in J. Suppose that π is nonempty and finite, and let $\pi = \{P_1, P_2, \cdots, P_k\}$. Then $\mathscr{J}(J_\pi) = \bigcap_{P \in \pi} PJ_\pi \supseteqq P_1P_2 \cdots P_kJ_\pi \neq \{0\}$. Hence J_π is not semisimple. By Lemma 5.4, R is not semisimple. But at least one ring R such that $G \cong R^+$ is semisimple. For that ring R, $R \doteq J_\pi$, where π is either empty or infinite.

(c) *implies* (b). If π is the empty set of prime ideals in J, then $J_\pi = \bigcap_{P \in \pi} J_P = K$. Hence J_π is semisimple. If π is an infinite set of prime ideals in J, then $\mathscr{J}(J_\pi) = \bigcap_{P \in \pi} PJ_\pi = \{0\}$ since J_π is a Dedekind ring. Again J_π is semisimple. Thus, if (c) is satisfied, it follows from Lemma 5.4, that there is a semisimple ring R on G. That is, G is semisimple. By Lemma 5.3, every ring R on G is isomorphic to a full subring of K, and hence is quasi-equal to a ring J_π for some set of prime ideals π in J. Suppose R_1 and R_2 are rings on G, $R_1 \doteq J_{\pi_1}$, $R_2 \doteq J_{\pi_1}$, $R_2 \doteq J_{\pi_2}$. Then

$$J_{\pi_1}^+ \doteq R_1^+ \cong R_2^+ \doteq J_{\pi_2}^+ ,$$

so that $J_{\pi_1}^+$ is quasi-isomorphic to $J_{\pi_2}^+$.

Let π' be the set of rational primes p such that $p \notin P$ for all $P \in \pi$. Then J_π^+ is p-divisible if and only if $p \in \pi'$. If $p \in \pi'$, then $1/p \in J_\pi$. Hence, if $x/y \in J_\pi$, $x/y = (px/y)(1/p) = p(x/yp)$. Therefore, J_π^+ is p-divisible. On the other hand, if $p \notin \pi'$, $p \in P$ for some $P \in \pi$. If J_π were p-divisible, $1 = p(x/y)$ for some $x/y \in J$. But then $y = px \in P$. This is a contradiction, since if $x/y \in J_\pi$, $y \notin P \in \pi$.

If G and H are quasi-isomorphic torsion free groups, G is p-divisible if and only if H is p-divisible. Thus, it follows from the result of the preceding paragraph that if $J_{\pi_1}^+$ and $J_{\pi_2}^+$ are quasi-isomorphic, $\pi_1' = \pi_2'$. If π_1 is empty, then π_1', and consequently π_2', is the set of all primes. Hence π_2 is empty. If π_1 is infinite, then since each rational prime has only a finite number of prime ideal divisors in J, it follows that the complement of π_1' in the set of all primes is infinite. Since $\pi_1' = \pi_2'$, the complement of π_2' is infinite. If π_2 were finite, then since each prime ideal P in J contains exactly one rational prime, it follows that π_2' contains almost all primes. But then the complement of π_2' would be finite, a contradiction.

We have shown that if R is any nonzero ring on G, then $R \doteq J_\pi$, where π is either empty or infinite. We have seen that every such J_π is semisimple. By Lemma 5.4, every nonzero ring R on G is semisimple. Hence G is strongly semisimple.

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