PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 1, 1974

CHAIN BASED LATTICES

G. EPSTEIN AND A. HORN

In recent years several weakenings of Post algebras have been studied. Among these have been P_0-lattices by T. Traczyk, Stone lattice of order n by T. Katrinak and A. Mitschke, and P-algebras by the present authors. Each of these system is an abstraction from certain aspects of Post algebras, and no two of them are comparable. In the present paper, the theory of P_0-lattices will be developed further and two new systems, called P_1-lattices and P_2-lattices are introduced. These systems are referred to as chain based lattices. P_2-lattices form the intersection of all three weakenings mentioned above. While P-algebras and weaker systems such as L-algebras, Heyting algebras, and B-algebras, do not require any distinguished chain of elements other than 0, 1, chain based lattices require such a chain.

Definitions are given in § 1. A P_0-lattice is a bounded distributive lattice A which is generated by its center and a finite subchain containing 0 and 1. Such a subchain is called a chain base for A. The order of a P_0-lattice A is the smallest number of elements in a chain base of A. In § 2, properties of P_0-lattices are given which are used in later sections. If a P_0-lattice A is a Heyting algebra, then it is shown in § 3, that there exists a unique chain base $0 = e_0 < e_1 < \cdots < e_{n-1} = 1$ such that $e_{i+1} \rightarrow e_i = e_i$ for all $i > 0$. A P_0-lattice with such a chain base is called a P_1-lattice. Every P_1-lattice of order n is a Stone lattice of order n. If a P_1-lattice is pseudo-supplemented then it is called a P_2-lattice. It turns out that P_2-lattices of order n are direct products of finitely many Post algebras whose maximum order is n. In § 4, properties of P_2-lattices are studied. In § 5, equational axioms are given for P_2-lattices. P_2-lattices share many of the properties of Post algebras and have application to computer science. Among examples of P_2-lattices are direct products of finitely many p-rings. These further remarks on P_2-lattices are in § 6. In § 7, prime ideals in P_0-lattices are studied. It is shown that the order of a P_0-lattice is one more than the number of elements in a chain of prime ideals of maximum length. A characterization of P_1-lattices by properties of their prime ideals is given. Such a characterization of P_2-lattices is also indicated.

1. DEFINITIONS. We use ϕ for the empty set. Let A be a distributive lattice which is *bounded*, that is, has a largest element 1 and a smallest element 0. The dual of A is denoted by A^d. The

complement of x is denoted by $\bar{x}$ or $-x$. The *center* of A is the set B of all complemented elements of A. We use $x \vee y$ for the join, and $x \wedge y$ or xy for the meet of two elements x, y in A. $x \rightarrow y$ denotes the largest $z \in A$ (if it exists) such that $xz \leqq y$. A is called a *Heyting algebra* if $x \rightarrow y$ exists for all $x, y \in A$. $\neg x = x \rightarrow 0$ is called the *pseudo-complement* of x (when it exists). An element x is called *dense* if $\neg x = 0$. A is called *pseudo-complemented* if $\neg x$ exists for all $x \in A$. $x \Rightarrow y$ denotes the largest $z \in B$ such that $xz \leqq y$. A is called a *B-algebra* if $x \Rightarrow y$ exists for all $x, y \in A$. $!x = 1 \Rightarrow x$ is called the *pseudo-supplement* of x. A is called pseudo-supplemented if $!x$ exists for all $x \in A$.

A *Stone lattice* is a pseudo-complemented lattice satisfying the identity $\neg x \vee \neg\neg x = 1$. An *L-algebra* is a Heyting algebra satisfying $(x \rightarrow y) \vee (y \rightarrow x) = 1$. A *P-algebra* is a B-algebra satisfying $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$. We denote the interval $\{z : x \leqq z \leqq y\}$ by $[x, y]$. A is an L-algebra if and only if every interval in A is a Stone lattice [1, 3.11]. The identity $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$ is satisfied in an L-algebra. The identity $x \Rightarrow (y \vee z) = (x \Rightarrow y) \vee (x \Rightarrow z)$ is satisfied in a P-algebra.

2. *P_0-lattices.* Let A be a bounded distributive lattice and let B be a Boolean subalgebra of the center of A. A *chain base* of A is a finite sequence $0 = e_0 \leqq e_1 \leqq \cdots \leqq e_{n-1} = 1$ such that A is generated by $B \cup \{e_0, \cdots, e_{n-1}\}$. If A has a chain base then A is called a P_0-lattice [13], in which case every element $x \in A$ can be written in the form

$$x = \bigvee_{i=1}^{n-1} b_i e_i \,, \tag{1}$$

where $b_i \in B$. If $b_i \geqq b_{i+1}$ for all i, then (1) is called a *monotone representation* (abbreviated mon. rep.) of x. If $b_i b_j = 0$ for $i \neq j$, then (1) is called a *disjoint representation* (disj. rep.) of x. Every element in a P_0-lattice has both a mon. rep. and a disj. rep.

LEMMA 2.1. *If* (1) *is a mon. rep. of x and $y = \bigvee_i c_i e_i$ is a mon. rep., then $x \vee y = \bigvee_i (b_i \vee c_i) e_i$ and $xy = \bigvee_i b_i c_i e_i$ are mon. reps.*

Proof. This follows from the distributivity of A.

The following theorem shows that B must coincide with the center of the P_0-lattice A, and gives a method for constructing P_0-lattices.

THEOREM 2.2. *Let A be a bounded distributive lattice. Let*

B be a subalgebra of the center of A and let $0 = e_0 \leqq \cdots \leqq e_{n-1} = 1$. If A_0 is the sublattice generated by $B \cup \{e_0, \cdots, e_{n-1}\}$, and B_0 is the center of A_0, then $B_0 = B$.

Proof. Let $x = \bigvee_j b_j e_j$ be a disj. rep. of an element $x \in B_0$. For each i, $xb_i = b_i e_i$ is in B_0. Let $\bigvee_j c_j e_j = 0$ be a mon. rep. of the complement of $b_i e_i$. Then $b_i e_i \bigvee_j c_j e_j = 0$ implies $b_i c_i e_i = 0$, hence $b_i e_i \leqq b_i \bar{c}_i$. Also $1 = b_i e_i \vee \bigvee_j c_j e_j$ implies $1 \leqq e_i \vee c_i$, hence $b_i \bar{c}_i \leqq b_i e_i$. Thus $b_i e_i = b_i \bar{c}_i \in B$ for all i, and so $x \in B$.

DEFINITION 2.3. A P_0-lattice A is said to be of *order* n if n is the smallest integer such that A has a chain base with n terms.

LEMMA 2.4. *If $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_0-lattice, then $\langle A^d; e_{n-1}, \cdots, e_0\rangle$ is a P_0-lattice. A^d has the same order as A.*

Proof. This is obvious by inspection.

THEOREM 2.5. *If $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_0-lattice with center B and $A' = [e_i, e_j]$, where $i \leqq j$, then $\langle A'; e_i, \cdots, e_j\rangle$ is a P_0-lattice with center $B' = \{e_i \vee be_j : b \in B\}$. If $e_i = f_0 \leqq \cdots \leqq f_{r-1} = e_j$ is a chain base of A', then $e_0, \cdots, e_{i-1}, f_0, \cdots, f_{r-1}, e_{j+1}, \cdots, e_{n-1}$ is a chain base of A. If A has order n, then A' has order $j - i + 1$.*

Proof. Let $x = \bigvee_{k=1}^{n-1} b_k e_k$ be a mon. rep. of an element $x \in A'$. Then

$$x = (e_i \vee x)e_j = e_i \vee \bigvee_{k=i+1}^{j} b_k e_k = \bigvee_{k=i+1}^{j} (e_i \vee b_k e_j)e_k\ .$$

B' is clearly a subalgebra of the center of A'. Therefore by 2.2, B' is the center of the P_0-lattice $\langle A'; e_i, \cdots, e_j\rangle$. The remaining parts of the theorem hold because if $i \leqq k \leqq j$, then e_k is in the sublattice generated by $B' \cup \{f_0, \cdots, f_{r-1}\}$.

LEMMA 2.6. *Let A be a bounded distributive lattice with center B, and $x, y, z \in A$.*

(i) *If $x \rightarrow z$ and $y \rightarrow z$ exist, then $(x \vee y) \rightarrow z = (x \rightarrow z)(y \rightarrow z)$.*

(ii) *If $z \rightarrow x$ and $z \rightarrow y$ exist, then $z \rightarrow xy = (z \rightarrow x)(z \rightarrow y)$.*

(iii) *If $x \rightarrow y$ exists, $b \in B$ and $c \in B$, then $bx \rightarrow (c \vee y) = \bar{b} \vee c \vee (x \rightarrow y)$.*

(iv) *If $x \Rightarrow z$ and $y \Rightarrow z$ exist, then $(x \vee y) \Rightarrow z = (x \Rightarrow z)(y \Rightarrow z)$.*

(v) *If $z \Rightarrow x$ and $z \Rightarrow y$ exist, then $z \Rightarrow xy = (z \Rightarrow x)(z \Rightarrow y)$.*

(vi) *If $x \Rightarrow y$ exists, $b \in B$ and $c \in B$, then $bx \Rightarrow (c \vee y) = \bar{b} \vee c \vee (x \Rightarrow y)$.*

Proof. The proof is straightforward.

LEMMA 2.7. *If* $a_2 \leqq \cdots \leqq a_m$ *and* $b_1 \geqq \cdots \geqq b_{m-1}$ *are elements of a distributive lattice, then* $\bigvee_{j=1}^{m-1} a_{j+1}b_j = a_m b_1 \bigwedge_{j=2}^{m-1} (a_j \vee b_j)$.

Proof. This is easily proved by induction.

THEOREM 2.8. *Let* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *be a* P_0*-lattice with center* B. *Then the following are equivalent:*

(i) $e_i \Rightarrow 0$ *exists for all* i.

(ii) $\neg e_i$ *exists for all* i.

(iii) A *is pseudo-complemented.*

(iv) A *is a Stone lattice.*

(v) *Each* $x \in A$ *has a mon. rep.* $x = \bigvee_i b_i e_i$ *such that* $b_1 \leqq c_1$ *for every mon. rep.* $x = \bigvee_i c_i e_i$.

Proof. (i) implies (ii): Let $xe_i = 0$ and suppose $x = \bigvee_j b_j e_j$ is a mon. rep. of x. Then $b_j e_j = 0$ for $j \leqq i$, while if $j > i$, then $b_j e_i = 0$, so $b_j \leqq e_i \Rightarrow 0$. Hence $x \leqq e_i \Rightarrow 0$. Therefore, $\neg e_i$ exists and equals $e_i \Rightarrow 0$.

(ii) implies (iii): If $x = \bigvee_i b_i e_i$ is a mon. rep., then by 2.6(i) and 2.6(iii), $\neg x$ exists and equals $\bigwedge_i (\bar{b}_i \vee \neg e_i)$. If follows from 2.7 that

$$\neg x = \bigvee_{i=1}^{n-1} \bar{b}_i \neg e_{i-1} \,. \tag{2}$$

(iii) implies (iv): If $x, y \in A$, then by 2.1 and (2), $\neg(xy) = \neg x \vee \neg y$. This implies that A is a Stone lattice [8].

(iv) implies (v): If $x = \bigvee_i c_i e_i$ is any mon. rep., then $\bar{c}_1 x = 0$, so $\bar{c}_1 \leqq \neg x$, hence $x \leqq \neg\neg x \leqq c_1$. Therefore $x = \bigvee_i (c_i \neg\neg x) e_i$. If we set $b_i = c_i \neg\neg x$, we get a mon. rep. in which $b_1 = \neg\neg x$.

(v) implies (i): Let $e_i = \bigvee_j b_j e_j$ be a mon. rep. of e_i having the property stated in (v). Then $\bar{b}_1 e_i = 0$. If $b \in B$ and $be_i = 0$, then $e_i \leqq \bar{b}$, so $e_i = \bigvee_j \bar{b} b_j e_j$. By hypothesis, $\bar{b} b_1 \geqq b_1$. Therefore $b \leqq \bar{b}_1$, and so $e_i \Rightarrow 0 = \bar{b}_1$.

LEMMA 2.9. *If* A *is a bounded distributive lattice, then* A^d *is a Stone lattice if and only if* A *is pseudo-supplemented and* $!(x \vee y) = !x \vee !y$ *for all* $x, y \in A$.

Proof. It is easily verified that the pseudo-complement of x in A^d is $\overline{!x}$ in this case.

THEOREM 2.10. *Let* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *be a* P_0*-lattice. Then the* *following are equivalent*:

(i) $!e_i$ *exists for all* i.

(ii) A *is pseudo-supplemented and* $!(x \vee y) = !x \vee !y$ *for all* $x, y \in A$.

(iii) *Each* $x \in A$ *has a mon. rep.* $\bigvee_i b_i e_i$ *such that* $b_{n-1} \geqq c_{n-1}$ *for every mon. rep.* $x = \bigvee_i c_i e_i$.

Proof. This is derived from 2.8 by using 2.4 and 2.9.

THEOREM 2.11. *Let* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *be a pseudo-complemented* P_0*-lattice. Then* A *has a chain base* $0 = f_0 \leqq f_1 \leqq \cdots \leqq f_{n-1} = 1$ *such that* f_1 *is the smallest dense element of* A. *If* $0 = g_0 \leqq \cdots \leqq g_{r-1} = 1$ *is any chain base of* A *such that* g_1 *is dense, then* $g_1 = f_1$ *and for any mon. rep.* $x = \bigvee_{j=1}^{r-1} b_j g_j$, *we have* $\neg x = \bar{b}_1$.

Proof. Let $f_0 = 0$, $f_1 = \bigvee_{i=1}^{n-1} (\neg e_{i-1}) e_i$, and $f_i = e_i \vee f_1$ for $i \geqq 2$. By (2), $\neg f_1 = \bigvee_i \neg\neg\, e_{i-1} \neg\, e_i = 0$. Also $f_i = e_i \vee \bigvee_{j>i} e_j \neg\, e_{j-1}$. Therefore $f_i \neg\neg\, e_i = e_i$, since $\neg\neg e_i \neg\, e_{j-1} \leqq \neg\neg\, e_i \neg\, e_i = 0$ for $j > i$. If $x = \bigvee_i b_i e_i$ is any element of A, then $x = \bigvee_i (b_i \neg\neg\, e_i) f_i$. Thus $f_0, \cdots, f_{n-1}$ is a chain base of A. Let $g_0, \cdots, g_{r-1}$ be a chain base of A such that g_1 is dense. If $x = \bigvee_j b_j g_j$ is a mon. rep., then $\neg x = \bar{b}_1$ by (2). So if x is dense, then $x \geqq g_1$. Thus $g_1 = f_1$ is the smallest dense element of A.

3. P_1-*lattices*.

THEOREM 3.1. *Let* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *be a* P_0*-lattice with center* B. *Then the following are equivalent*:

(i) $e_i \to e_j$ *exists for all* i, j.

(ii) A *is a Heyting algebra*.

(iii) A *is an* L *algebra*.

Proof. (i) implies (ii): If $x = \bigvee_i b_i e_i$ and $y = \bigvee_i c_i e_i$ are mon. reps., then by 2.7, $y = \bigwedge_{i=1}^{n-1} (c_i \vee e_{i-1})$. Therefore by 2.6, $x \to y$ exists and equals $\bigwedge_{i,j} (\bar{b}_i \vee c_j \vee (e_i \to e_j))$.

(ii) implies (iii): Let $x = \bigvee_i b_i e_i$, $y = \bigvee_i c_i e_i$ be mon. reps. Then $x \to y = \bigwedge_i (b_i e_i \to y) \geqq \bigwedge_i (\bar{b}_i \vee c_i)$. Therefore, $(x \to y) \vee (y \to x) \geqq \bigwedge_i (\bar{b}_i \vee c_i) \vee \bigwedge_i (b_i \vee \bar{c}_i) = \bigwedge_{i,j} (\bar{b}_i \vee b_j \vee c_i \vee \bar{c}_j) = 1$ since $\bar{b}_i \vee b_j = 1$ for $i \geqq j$, and $c_i \vee \bar{c}_j = 1$ for $i < j$.

(iii) implies (i): This is obvious.

DEFINITION 3.2. A P_1-lattice $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_0-lattice together with a chain base such that $e_{i+1} \to e_i = e_i$. It follows that $e_i \to e_j = e_j$ for $i > j$ and $e_i \to e_j = 1$ for $i \leqq j$, so that (i) of 3.1 holds.

THEOREM 3.3. *If $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_0-lattice and A is a Heyting algebra, then there exists a chain base $0 = f_0 \leqq \cdots \leqq f_{n-1} = 1$ such that $\langle A; f_0, \cdots, f_{n-1}\rangle$ is a P_1-lattice.*

Proof. This is obvious for $n = 1, 2$. Suppose $n > 2$ and the statement holds for $n - 1$. By 2.11, we may assume e_1 is dense. Let $A' = [e_1, 1]$. By 2.5, $\langle A'; e_1, \cdots, e_{n-1}\rangle$ is a P_0-lattice. If $x, y \in A'$, then $x \to y \in A'$. Therefore by the induction hypothesis, there exists a sequence $e_1 = f_1 \leqq \cdots \leqq f_{n-1} = 1$ such that $\langle A'; f_1, \cdots, f_{n-1}\rangle$ is a P_1-lattice. If we set $f_0 = 0$, then by 2.5, $\langle A; f_0, \cdots, f_{n-1}\rangle$ is a P_1-lattice.

THEOREM 3.4. *Let $\langle A; e_0, \cdots, e_{n-1}\rangle$ be a P_1-lattice. Then for some $m \geqq 1$, $0 = e_0 < e_1 < \cdots < e_{m-1} = e_m = \cdots = 1$. A has order m. For each i, e_{i+1} is the smallest dense element of $[e_i, 1]$. Thus $e_0, \cdots, e_{m-1}$ is the unique strictly increasing chain such that $\langle A; e_0, \cdots, e_{m-1}\rangle$ is a P_1-lattice. If $x = \bigvee_{i=1}^{n-1} b_i e_i$ is a mon. rep., then $e_i \vee b_{i+1} = (x \to e_i) \to e_i$, $0 \leqq i < n - 1$. If $x = \bigvee_{i=1}^{n-1} b_i e_i$ is a disj. rep., and $y = \bigvee_{i=1}^{n-1} c_i e_i$ is a mon. rep., then $x \to y = y \vee \bigvee_{i=0}^{n-1} b_i c_i$, where $b_0 = \bigwedge_{i=1}^{n-1} \bar{b}_i$, $c_0 = 1$.*

Proof. If m is the first integer such that $e_m = e_{m-1}$, then $e_{m-1} = e_m \to e_{m-1} = 1$. Since e_{i+1} is dense in $[e_i, 1]$ it follows from 2.5 and 2.11 that e_{i+1} is the smallest dense element of $[e_i, 1]$. Using 3.3, it follows that A has order m.

If $x = \bigvee_{i=1}^{n-1} b_i e_i$ is a mon. rep., then $x \vee e_i = \bigvee_{k=i+1}^{n-1} (e_i \vee b_k) e_k$. Applying 2.11 to $[e_i, 1]$, we find $(x \vee e_i) \to e_i = e_i \vee \bar{b}_{i+1}$. Since $(x \vee e_i) \to e_i = x \to e_i$, it follows by 2.6 that $(x \to e_i) \to e_i = e_i \vee b_{i+1}$.

To prove the last statement, we observe that

$$\begin{aligned} e_i \to y &= \bigvee_{j=1}^{n-1} (e_i \to c_j e_j) = \bigvee_{j=0}^{n-1} (e_i \to c_j)(e_i \to e_j) \\ &= \bigvee_{j=1}^{i-1} c_j e_j \vee \bigvee_{j=i}^{n-1} c_j = y \vee c_i \quad \text{for} \quad 1 \leqq i \leqq n-1 \,. \end{aligned}$$

Therefore,

$$\begin{aligned} x \to y &= \bigwedge_{i=1}^{n-1} (b_i e_i \to y) = \bigwedge_{i=1}^{n-1} (\bar{b}_i \vee c_i \vee y) = y \vee \bigwedge_{i-1}^{n-1} (\bar{b}_i \vee c_i) \\ &= y \vee \bigvee_{i=0}^{n-1} b_i c_i \,, \end{aligned}$$

where the last equality is easily proved by induction.

DEFINITION 3.5. A *Stone lattice* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *of order* n is an L-algebra A in which there exists a chain $0 = e_0 < e_1 < \cdots < e_{n-1} = 1$ such that e_{i+1} is the smallest dense element of $[e_i, 1]$. If B_i is the center of $[e_i, 1]$, let $h_i\colon B_i \to B_{i+1}$ be the Boolean homomorphism defined by $h_i(x) = x \vee e_{i+1}$, with $B_0 = B$. These definitions are in [11].

THEOREM 3.6. *$\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_1-lattice of order n, if and only if $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a Stone lattice of order n such that h_i is onto B_{i+1} for each $i \geqq 0$.*

Proof. If $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_1-lattice of order n, then it is a Stone lattice of order n by 3.4, and h_i is onto by 2.5. Conversely, suppose $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a Stone lattice of order n and h_i is onto B_{i+1} for each i. Then $B_i = \{b \vee e_i\colon b \in B\}$ by 2.5. It was proved in [11, 3.4], that if $x \in A$, then $x = \bigwedge_{i=0}^{n-2} x_i$, where $x_i \in B_i$. Therefore $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_1-lattice.

THEOREM 3.7. *If A is a Heyting algebra with center B, $0=e_0 \leqq \cdots \leqq e_{n-1} = 1$, e_{i+1} is the smallest dense element of $[e_i, 1]$, and if whenever $i < j$, the center of $[e_i, e_j]$ is $\{e_i \vee be_j\colon b \in B\}$, then $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_1-lattice.*

Proof. The point of this theorem is that the condition that A is an L-algebra is replaced by the condition that A is a Heyting algebra such that the center of $[e_i, e_j]$ is $\{e_i \vee be_j\colon b \in B\}$, for all $i < j$. We omit details of proof since this theorem is not used in what follows.

4. *P_2-lattices.*

DEFINITION 4.1. $x = \bigvee_{i=1}^{n-1} b_i e_i$ is called the *highest* monotone representation (hi. mono. rep.) of x if for every mon. rep. $\bigvee_{i=1}^{n-1} c_i e_i$ of x, the relation $b_i \geqq c_i$ holds for all i. The *lowest* monotonic representation (lo. mon. rep.) is defined in a similar manner.

THEOREM 4.2. *Let $\langle A; e_0, \cdots, e_{n-1}\rangle$ be a P_0-lattice. Then the following are equivalent*:

(i) *$e_i \Rightarrow e_j$ exists for all i, j.*
(ii) *$e_i \rightarrow e_j$ and $!e_i$ exist for all i, j.*
(iii) *every $x \in A$ has a hi. mon. rep.*
(iv) *every $x \in A$ has a lo. mon. rep.*

(v) *A is a B-algebra.*

(vi) *A is a P-algebra.*

The hi. mon. rep. of x *is* $\bigvee_i (e_i \Rightarrow x)e_i$, *and the lo. mon. rep. of* x *is* $\bigvee_i (\overline{x \Rightarrow e_{i-1}})e_i$.

Proof. The equivalence of (i), (v), and (vi) is proved exactly as in the proof of 3.1. By [7], A is a P-algebra if and only if A is a pseudo-supplemented L-algebra in which $!(x \vee y) = !x \vee !y$ for all x, y. Therefore, by 3.1 and 2.10, (ii) is equivalent to (vi).

To prove (iii) implies (i), let $\bigvee_j b_j e_j$ be the hi. mon. rep. of e_i. Then $b_{i+1}e_{i+1} \leqq e_i$. Let $b \in B$, $be_{i+1} \leqq e_i$. Thus $e_1 \vee \cdots \vee e_i \vee be_{i+1}$ is a mon. rep. of e_i. Therefore $b_{i+1} \geqq b$, which proves $b_{i+1} = e_{i+1} \Rightarrow e_i$. Hence if $i > j$, $e_i \Rightarrow e_j = \bigwedge_{k=i}^{j-1} (e_k \Rightarrow e_{k+1})$, and for $i \leqq j$, $e_i \Rightarrow e_j = 1$.

To prove (vi) implies (iii), let $x = \bigvee_i b_i e_i$ be any mon. rep. Then $e_i \Rightarrow x \geqq e_i \Rightarrow b_i e_i \geqq b_i$. Also $e_i(e_i \Rightarrow x) \leqq x$. Therefore,

$$x \geqq \bigvee_i e_i(e_i \Rightarrow x) \geqq \bigvee_i b_i e_i = x \ .$$

Thus $\bigvee_i e_i(e_i \Rightarrow x)$ is the hi. mon. rep. of x.

The equivalence of (iv) and (vi) is a consequence of the equivalence of (iii) and (vi), since the dual of a P-algebra is a P-algebra. The formula for the lo. mon. rep. is obtained by duality, for if $x = \bigvee_i b_i e_i$ is a mon. rep., then $x = \bigwedge_i (b_i \vee e_{i-1})$.

DEFINITION 4.3. A P_2-lattice is a P_1-lattice $\langle A; e_0, \cdots, e_{n-1}\rangle$ such that $!e_i$ exists for all i.

Using 2.2, it is easy to construct a P_1-lattice which is not a P_2-lattice.

THEOREM 4.4. *If* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *is a* P_0*-lattice of order* n *and* A *is a B-algebra, then there exists a unique chain* $f_0, \cdots, f_{n-1}$ *such that* $\langle A; f_0, \cdots, f_{n-1}\rangle$ *is a* P_2*-lattice.*

Proof. This follows from 3.3, 3.4, and 4.2.

THEOREM 4.5. *Let* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *be a* P_2*-lattice. Then*

(i) *Every* $x \in A$ *has a unique mon. rep.* $\bigvee_i D_i(x)e_i$ *such that* $D_{n-1}(x) = !x$. *This representation is also the hi. mon. rep. of* x.

(ii) *Every* $x \in A$ *has a unique disj. rep.* $\bigvee_i C_i(x)e_i$ *such that* $C_{n-1}(x) = !x$.

(iii) $D_i(x) = e_i \Rightarrow x$, $C_i(x) = D_i(x) - D_{i+1}(x)$ *and for* $i < n-1$, $C_i(x) = (x \Rightarrow e_i)(e_i \Rightarrow x) - !(xe_i)$.

(iv) $D_i(x \vee y) = D_i(x) \vee D_i(y)$, $D_i(xy) = D_i(x)D_i(y)$.

(v) $x \Rightarrow y = \bigvee_{i=0}^{n-1} C_i(x)D_i(y)$, *where* $D_0(y) = 1$ *and* $C_0(x) = 1 - D_1(x)$.

Proof. (i) Let $x = \bigvee_j b_j e_j$ be a mon. rep. such that $b_{n-1} = !x$. If $i > j$, then $e_i \Rightarrow e_j = !(e_i \rightarrow e_j) = !e_j$. Therefore,

$$\begin{aligned} e_i \Rightarrow x &= \bigvee_j (e_i \Rightarrow b_j e_j) = \bigvee_j (e_i \Rightarrow b_j)(e_i \Rightarrow e_j) \\ &= \bigvee_{j<i} b_j !e_j \vee \bigvee_{j \geqq i} b_j = b_i \,, \end{aligned}$$

since $\bigvee_{j=1}^{n-1} b_j !e_j = !x \leqq b_i$. We set $D_i(x) = e_i \Rightarrow x$ for $0 \leqq i \leqq n-1$. By 4.2, the hi. mon. rep. of x is $\bigvee_i D_i(x)e_i$, and $D_{n-1}(x) = 1 \Rightarrow x = !x$.

(ii) Follows from (i), with $C_i(x) = D_i(x) - D_{i+1}(x)$.

(iii) For $0 \leqq i < n-1$,

$$\begin{aligned} x \Rightarrow e_i &= \bigwedge_j (D_j(x)e_j \Rightarrow e_i) = \bigwedge_j (\overline{D_j(x)} \vee (e_j \Rightarrow e_i)) \\ &= \bigwedge_{j>i} (\overline{D_j(x)} \vee !e_i) = \overline{D_{i+1}(x)} \vee !e_i \,. \end{aligned}$$

Therefore $(x \Rightarrow e_i)(e_i \Rightarrow x) = C_i(x) \vee D_i(x)!e_i$. Since $D_i(x)!e_i \leqq e_i(e_i \Rightarrow x) \leqq x$, we have $D_i(x)!e_i \leqq !x = D_{n-1}(x)$. Hence $D_i(x)!e_i = !x!e_i = !(xe_i)$. Also $C_i(x)!x = C_i(x)C_{n-1}(x) = 0$. Therefore,

$$C_i(x) = (x \Rightarrow e_i)(e_i \Rightarrow x) - !(xe_i) \,.$$

(iv) follows immediately from $D_i(x) = e_i \Rightarrow x$.

(v) By 3.4, $x \rightarrow y = y \vee \bigvee_{i=0}^{n-1} C_i(x)D_i(y)$. Therefore,

$$x \Rightarrow y = !y \vee \bigvee_{i=0}^{n-1} C_i(x)D_i(y) = \bigvee_{i=0}^{n-1} C_i(x)D_i(y) \,,$$

since $\bigvee_{i=0}^{n-1} C_i(x)D_i(y) \geqq D_{n-1}(y) \bigvee_{i=0}^{n-1} C_i(x) = !y$.

THEOREM 4.6. *The following are equivalent:*

(i) $\langle A; e_0, \cdots, e_{n-1}\rangle$ *is a P_2-lattice of order n.*

(ii) $\langle A; e_0, \cdots, e_{n-1}\rangle$ *is a Stone lattice of order n, the homomorphisms h_i of* 3.5 *are onto, and the kernel of h_i is a principal ideal for each i.*

(iii) $\langle A; e_0, \cdots, e_{n-1}\rangle$ *is a Stone lattice of order n and A^d is a Stone lattice.*

Proof. The equivalence of (i) and (ii) follows from 3.6 and 2.9, using the fact that the kernel of h_i is a principal ideal if and only if $!e_{i+1}$ exists. The equivalence of (ii) and (iii) was proved in [11].

The following is the dual of the definition given in [5].

DEFINITION 4.7. A Post algebra is a P_2-algebra $\langle A; e_0, \cdots, e_{n-1}\rangle$ such that $!e_{n-2} = 0$; that is, e_{n-2} is dense in A^d. Note that A has order n, unless $A = \{0\}$.

THEOREM 4.8. *If* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *is a* P_0*-lattice then the following are equivalent*:

(i) *A is a Post algebra.*

(ii) *every element* $x \in A$ *has a unique mon. rep.*

(iii) $e_i \Rightarrow e_{i-1} = 0$ *for all* $i > 0$.

Proof. This was proved in [13].

LEMMA 4.9. *If* $\langle A_j; e_{j0}, \cdots, e_{j(n_j-1)}\rangle$ *is a* P_r*-lattice for* $j \in J$, *where* $r = 0, 1$, *or* 2, $A = \prod_{j \in J} A_j$, $n = \max\{n_j\colon j \in J\} < \infty$, *and* e_{jk} *is defined to be* $e_{j(n_j-1)}$ *for* $k > n_j$, *then* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *is a* P_r*-lattice, where* $e_i = \langle e_{ji}\colon j \in J\rangle$.

Proof. This is obvious.

LEMMA 4.10. *If* $\langle A; e_0, \cdots, e_{n-1})$ *is a* P_0*-lattice, B is a distributive lattice and* $f\colon A \rightarrow B$ *is a lattice homomorphism onto, then* $\langle B; f(e_0), \cdots, f(e_{n-1})\rangle$ *is a* P_0*-lattice. If* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *is a* P_1*-lattice and* $f\colon A \rightarrow B$ *is a Heyting homomorphism onto, then* $\langle B; f(e_0), \cdots, f(e_{n-1})\rangle$ *is a* P_1*-lattice.*

Proof. This is easy to verify.

THEOREM 4.11. *Let A be a finite distributive lattice then the following are equivalent*:

(i) *A is a* P_0*-lattice.*

(ii) *A is a P-algebra.*

(iii) *A is a direct product of chains.*

(iv) *A has a chain base* $e_0, \cdots, e_{n-1}$ *such that* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *is a* P_2*-lattice.*

Proof. (i) implies (ii): Since A is finite, A is a pseudo-supplemented Heyting algebra. By 4.2, A is a P-algebra.

(ii) implies (iii) was proved in [7].

(iii) implies (iv): If A is a finite chain $0 = a_0 < \cdots < a_{n-1} = 1$, then $\langle A; a_0, \cdots, a_{n-1}\rangle$ is a P_2-lattice. Therefore (iv) follows by 4.9.

(iv) implies (i) is obvious.

A finite chain with n elements has exactly one chain base with n terms. If $\langle A; e_0, \cdots, e_{n-1}\rangle$ and $\langle B; f_0, \cdots, f_{m-1}\rangle$ are P_0-lattices of orders n and m respectively, where $n < m$, then $A \times B$ has more than one chain base. In addition to the chain base described in 4.9, there is also the chain base $(e_0, f_0), \cdots, (e_0, f_{m-n}), (e_1, f_{m-n+1}), (e_2, f_{m-n+2}), \cdots, (e_{n-1}, f_{m-1})$. These remarks lead to the next theorem.

THEOREM 4.12. *A distributive lattice A is a Post algebra of order n if and only if A has a unique n-term chain base.*

Proof. Let A be a Post algebra of order n, and let $e_0, \cdots, e_{n-1}$ be an n-term chain base. A is a subdirect power of an n element chain C. If $f_j = A \rightarrow C$ is the jth projection, then by 4.10, $f_j(e_0), \cdots, f_j(e_{n-1})$ is a chain base of C. This determines $f_j(e_i)$ uniquely for all i, j. Therefore $e_0, \cdots, e_{n-1}$ is unique.

Conversely, suppose A has a unique n-term chain base $e_0, \cdots, e_{n-1}$. We prove A is a Post algebra of order n by induction. This is obvious for $n = 1, 2$. Suppose $n > 2$ and the statement holds for $n - 1$. By 2.5, $[e_1, 1]$ has a unique chain base with $n - 1$ terms. Therefore, $[e_1, 1]$ is a Post algebra of order $n - 1$. This implies $e_{i+1} \Rightarrow e_i = 0$ in $[e_1, 1]$ for $i \geqq 1$. This implies $e_{i+1} \Rightarrow e_i = 0$ in A since the center of $[e_1, 1]$ is $\{b \vee e_1 : b \in B\}$, where B is the center of A. By 4.8, we need only show $e_1 \Rightarrow 0 = 0$. If not, there exists $b \in B$ with $be_1 = 0$, $b \neq 0$. Let $B_1 = \{0, b, \bar{b}, 1\}$, and let A_1 be the sublattice of A generated by $B_1 \cup \{e_0, \cdots, e_{n-1}\}$. By 2.2, A_1 has center B_1 and so every chain base of A_1 is a chain base of A. Thus A_1 is a finite lattice with a unique n-term chain base. By 4.11, A_1 is a direct product of finite chains. If all the chains have the same cardinal, then A_1 is a Post algebra with unique n-term chain base $e_0, \cdots, e_{n-1}$, and by 4.8, we have $e_1 \Rightarrow 0 = 0$, which contradicts $be_1 = 0$, $b \neq 0$. If two of the chains have different cardinal, then A_1 has more than one n-term chain base. This contradiction proves $e_1 \Rightarrow 0 = 0$.

THEOREM 4.13. *If $\langle A; e_0, \cdots, e_{n-1} \rangle$ is a P_1-lattice with center B, then there exists a P_2-lattice $\langle A'; e_0, \cdots, e_{n-1} \rangle$ with center B' such that B is a Boolean subalgebra of B' and A is the sublattice of A' generated by $B \cup \{e_0, \cdots, e_{n-1}\}$.*

Proof. By 3.1, A is an L-algebra. By [9], we may assume A is a Heyting subalgebra of a direct product of chains C_j, $j \in J$ and the projections $f_j : A \rightarrow C_j$ are onto. Then by 4.10, $\langle C_j; f_j(e_0), \cdots, f_j(e_{n-1}) \rangle$ is a P_1-lattice. Therefore, C_j has at most n elements and $\langle C_j; f_j(e_0), \cdots, f_j(e_{n-1}) \rangle$ is a P_2-lattice. Let $A' = \prod_{j \in J} C_j$. By 4.9, $\langle A; e_0, \cdots, e_{n-1} \rangle$ is a P_2-lattice. Since A is a sublattice of A' containing 0, 1, the center of A is a subalgebra of the center of A'.

THEOREM 4.14. *Let $\langle A; e_0, \cdots, e_{n-1} \rangle$ be a P_2-lattice of order n with center B. Then A is order isomorphic with a direct product of Post algebras of maximum order n.*

Proof. Let $u_k = !e_k - !e_{k-1}$, $1 \leqq k \leqq n - 1$. Then $u_j u_k = 0$ for

$j \neq k$, and $u_1 \vee \cdots \vee u_{n-1} = 1$. Let $P_k = [0, u_k]$. Clearly the center of P_k is $B \cap P_k$. Let $e_{ki} = e_i u_k$, $0 \leqq i \leqq k$. If $x = \bigvee_{i=1}^{n-1} b_i e_i$ is a mon. rep. of any $x \in P_k$, then

$$x - xu_k = \bigvee_{i=1}^{n-1} b_i e_i u_k = \bigvee_{i=1}^{k-1} b_i e_i u_k \vee \bigvee_{i=k}^{n-1} b_i u_k = \bigvee_{i=1}^{k} (b_i u_k) e_{ki} .$$

Therefore $\langle P_k; e_{k0}, \cdots, e_{kk}\rangle$ is a P_0-lattice. If $b \in P_k \cap B$, $be_{ki} \leqq e_{k(i-1)}$, $0 < i \leqq k$, then $be_i \leqq e_{i-1}$. Therefore $b \leqq e_i \Rightarrow e_{i-1} = !e_{i-1}$. This implies $b = 0$, since $b \leqq u_k$. Thus by 4.8, P_k is a Post algebra of order $k+1$, or else $P_k = \{0\}$. Define $f\colon A \to \prod_{k=1}^{n-1} P_k$ by $f(x) = (xu_1, \cdots, xu_{n-1})$. f is onto since if $z_i \in P_i$, then $f(z_1 \vee \cdots \vee z_{n-1}) = (z_1, \cdots, z_{n-1})$. If $x \leqq y$ then $f(x) \leqq f(y)$, and $f(x) \leqq f(y)$ implies

$$x = \bigvee_{i=1}^{n-1} xu_i \leqq \bigvee_{i=1}^{n-1} yu_i = y .$$

Therefore f is an order isomorphism. Finally, P_{n-1} has order n since $u_{n-1} \neq 0$.

Theorem 4.14 may be used to apply known results on Post algebras to P_2-lattices. For example, since every Post algebra is isomorphic with the set of all continuous functions on a Boolean space to a finite discrete chain, it follows that every P_2-lattice is isomorphic with the set of all such functions which are $\leqq$ some fixed continuous function. In other words, a P_2-lattice is a principal ideal in a Post algebra. It also follows from 4.14 that a P_2-lattice is complete if and only if its center is complete, and that the normal completion of a P_2-lattice A is a P_2-lattice whose center is the normal completion of the center of A. Also every P_2-algebra is isomorphic with its dual. This isomorphism is given explicitly in the following theorem.

THEOREM 4.15. *Let $\langle A; e_0, \cdots, e_{n-1}\rangle$ be a P_2-lattice. Let $f_i = \bigvee_{k=1}^{n-1-i} e_k \overline{!e_{k+i-1}}$, $0 \leqq i < n-1$ and $f_{n-1} = 0$. Then A is isomorphic with A^d under the normal involution*

$$\beta(x) = \bigvee_{j=1}^{n-1} \overline{D_j(x)} f_{j-1} = \bigwedge_{j=1}^{n-1} (\overline{D_j(x)} \vee f_j) .$$

Proof. We have $f_0 \geqq \bigvee_{k=1}^{n-1} (!e_k - !e_{k-1}) = 1$. For $0 < i < n-1$, $!f_i = 0$, so that by 4.5(i), $D_k(f_i) = \overline{!e_{k+i-1}}$ for $1 \leqq k \leqq n-1-i$, and $D_k(f_i) = 0$ for $k \geqq n-i$.

If $1 \leqq i \leqq n-2$,

$$\begin{aligned}\beta(f_i) &= \bigvee_{j=1}^{n-1} f_{j-1}\overline{D_j(f_i)} = \bigvee_{j=1}^{n-1-i} f_{j-1} !e_{j+i-1} \vee \bigvee_{j=n-i}^{n-1} f_{j-1} \\ &= \bigvee_{j=1}^{n-1} f_{j-1} !e_{j+i-1} = \bigvee_{j=1}^{n-i} !e_{j+i-1} \bigvee_{k=1}^{n-j} e_k \overline{!e_{j+k-2}} \\ &= \bigvee_{k=1}^{i} e_k \bigvee_{j=1}^{n-i} (!e_{j+i-1} - !e_{j+k-2}) \vee \bigvee_{k=i+1}^{n-1} e_k \bigvee_{j=1}^{n-k} (!e_{j+i-1} - !e_{j+k-2}) .\end{aligned}$$

But $\bigvee_{j=1}^{n-k}(!e_{j+i-1} - !e_{j+k-2}) = 0$ if $k > i$, and by 2.7, if $k \leqq i$,

$$\bigvee_{j=1}^{n-i}(!e_{j+i-1} - !e_{j+k-2}) = \overline{!e_{k-1}} \bigwedge_{j=2}^{n-i}(!e_{j+i-2} \vee \overline{!e_{j+k-2}}) = \overline{!e_{k-1}} .$$

Therefore, $\beta(f_i) = \bigvee_{k=1}^{i}(e_k - !e_{k-1}) = e_i$.

Now $x \leqq y$ implies $\beta(x) \leqq \beta(y)$ and

$$\beta(\beta(x)) = \bigvee_{j=1}^{n-1} f_{j-1} \bigvee_{i=1}^{n-1} D_i(x)\overline{D_j(f_i)} = \bigvee_{i=1}^{n-1} \beta(f_i)D_i(x) = \bigvee_{i=1}^{n=1} D_i(x)e_i = x .$$

This implies that $\beta: A \to A^d$ is an isomorphism. The proof that β is a normal involution—that is, that $x\beta(x) \leqq y \vee \beta(y)$ for all $x, y \in A$—is omitted since this fact will not be used here [10].

5. Axioms and P_2-functions. P_2-algebras $\langle A; e_0, \cdots, e_{n-1}\rangle$ of order $\leqq n$ may be regarded as algebras $\langle A; \vee, \wedge, C_0, \cdots, C_{n-1}, e_0, \cdots, e_{n-1}\rangle$ with two binary operations, n binary operations, and n distinguished constants. This class of algebras can be characterized by the following set of equational axioms, in which $x \leqq y$ is used as an abbreviation for $x \wedge y = x$.

H1. Identities characterizing $\langle A; \vee, \wedge\rangle$ as a distributive lattice [8, pp. 5, 35].

H2. (a) $e_0 \leqq x$

(b) $e_i \leqq e_j$ for $0 \leqq i \leqq j \leqq n-1$

(c) $x \leqq e_{n-1}$

H3. (a) $C_i(x) \wedge C_j(x) = e_0$ for $i \neq j$

(b) $C_0(x) \vee C_1(x) \vee \cdots \vee C_{n-1}(x) = e_{n-1}$

H4. (a) $C_i(x \wedge y) = (C_i(x) \wedge \bigvee_{k=i}^{n-1} C_k(y)) \vee (C_i(y) \wedge \bigvee_{k=i}^{n-1} C_k(x))$

(b) $C_{n-1}(x \vee y) = C_{n-1}(x) \vee C_{n-1}(y)$

H5. (a) $C_i(e_j) = e_0$ for $j \neq i$ and $i < n-1$

(b) $C_{n-1}(e_0) = e_0$

H6. $x = (C_1(x) \wedge e_1) \vee \cdots \vee (C_{n-1}(x) \wedge e_{n-1})$.

Note that in every P_2-lattice H4 holds by 4.5(iv) and H5 holds by 4.5(ii). Conversely, if A satisfies the axioms then one proves $C_{n-1}(1) = 1$, $C_0(0) = 0$, $C_0(x) = \neg x$ and $C_{n-1}(x) = !x$. Then using H4 and H5, it can be proved that $xe_i = e_{i-1}$ implies $x = e_{i-1}$. This shows that $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_2-lattice.

The class of Post algebras of order n (together with the trivial lattice $\{0\}$) can be characterized by adding the axiom $C_{n-1}(e_{n-2}) = 0$ (see also [6]).

We may also characterize P_2-lattices equationally as the class of all algebras $\langle A; \vee, \wedge, \Rightarrow, e_0, \cdots, e_{n-1}\rangle$ with 3 binary operations and n constants which satisfy the following axioms.

K1. Identities characterizing $\langle A; \vee, \wedge, \Rightarrow, e_0, e_{n-1}\rangle$ as a P-algebra (see [7]).

K2. $e_i \leqq e_j$ for $i \leqq j$

K3. $e_{i+1}(e_i \Rightarrow e_j) \leqq e_j$ for $j < i < n-1$

K4. $x = \bigvee_{j=1}^{n-1} (e_i \wedge (e_i \Rightarrow x))$.

Indeed, if we set $D_i(x) = e_i \Rightarrow x$ for $0 \leqq i \leqq n-1$ and let $C_i(x) = D_i(x) - D_{i+1}(x)$ for $i < n$, then H1–3, H5(b), and H6 are obvious. By properties of P-algebras, $D_i(x \vee y) = D_i(x) \vee D_i(y)$ and $D_i(x \wedge y) = D_i(x) \wedge D_i(y)$. This proves H4. H5(a) is equivalent to $e_i \Rightarrow e_j = e_{i+1} \Rightarrow e_j$ for $j \neq i$, $i < n-1$. This is obvious for $i < j$, and follows from K3 for $i > j$.

P_2-lattices may also be characterized equationally as algebras $\langle A; \vee, \wedge, \rightarrow, !, e_0, \cdots, e_{n-1}\rangle$, since $x \Rightarrow y = !(x \rightarrow y)$, $x \rightarrow y = y \vee (x \Rightarrow y)$ and $!x = 1 \Rightarrow x$.

A P_2-function of order n in m variables is a function built from the identity functions $I_j(x_1, \cdots, x_m) = x_j$, $1 \leqq j \leqq m$, and the operations in any of the fundamental sets of operations described above. A normal form for such functions is given in the next theorem.

THEOREM 5.1. *If h is a P_2-function of order n in m variables, then*

$$h(x_1, \cdots, x_m) = \bigvee_{0 \leqq i_k \leqq n-1} h(e_{i_1}, \cdots, e_{i_m}) C_{i_1}(x_1) \cdots C_{i_m}(x_m)\,.$$

Proof. The n^m terms $C_{i_1}(x_1) \cdots C_{i_m}(x_m)$ are pairwise disjoint and have join 1, by axiom H3. By H6, the statement holds when h is one of the identity functions. If the statement holds for h_1 and h_2, then it holds for $h_1 \vee h_2$ and $h_1 \wedge h_2$. If it holds for h, then it holds for $D_i(h)$ by 4.5(iv). From this it follows that the statement holds for $C_j(h)$.

The normal form in 5.1 was proved for Post algebras in [5], and gives a truth table approach to Post functions. However, in a P_2-lattice, $h(e_{i_1}, \cdots, e_{i_m})$ is not necessarily in $\{e_0, \cdots, e_{n-1}\}$, as is the case for Post algebras.

6. Applications. P_2-lattices are of interest in computer science. They can be applied to the theory of machines with m_i-stable devices, $2 \leqq m_i \leqq n$, and to the analysis of machines with 2-stable devices Q_i (flip-flops) whose outputs are discretized as signals in transition $0 = e_0 < e_1 < \cdots < e_{n-1} = 1$. The case $n = 3$ is of special interest and is studied in [2] and [3]. P_2-lattices provide the complete multiple-valued logics for these applications.

P_2-lattices which admit operations of ring addition and multiplication are of interest in information processing. It is known that if R is a ring with unit element which satisfies the identities $x^p = x$ and $px = 0$, where p is a prime (so-called p-rings [12]), then lattice operations can be defined as polynomials in such a way that R becomes a Post algebra of order p. Conversely in any Post algebra of order p, ring operations can be defined in terms of the Post operations so that we obtain a p-ring. Therefore, direct products of finitely many p-rings are P_2-lattices. Such direct products can be characterized equationally. Indeed one can show that a ring R with unit element is a direct product of rings $R_1, \cdots, R_t$, where R_i is a p_i-ring and $p_i \neq p_j$ for $i \neq j$, if and only if R satisfies the following set of identities:

(1) $x^m = x$, where $m = 1 + \text{l.c.m.}\ (p_1 - 1, \cdots, p_t - 1)$.

(2) $p_1 \cdots p_t x = 0$.

(3) $(\prod_{j \neq i} p_j)(x^{p_i} - x) = 0,\ 1 \leqq i \leqq t$.

7. Prime ideals.

DEFINITION 7.1. Let $\mathscr{P}(A)$ be the set of prime ideals of A. Let $\langle A; e_0, \cdots, e_{n-1}\rangle$ be a P_0-lattice with center B. If $Q \in \mathscr{P}(B)$ and $1 \leqq k \leqq n-1$, let $P_k(Q) = \{x \in A$: x has a mon. rep. $\bigvee_i b_i e_i$ such that $b_k \in Q\}$. It was proved in [13, Th. 1.5] that either $P_k(Q) \in \mathscr{P}(A)$ or $P_k(Q) = A$ (the latter possibility was not mentioned). If $P_k(Q) \neq A$, then $P_k(Q) \cap B = Q$ since if $b \in Q$ then $b = \bigvee_i be_i \in P_k(Q)$ and prime ideals in B are maximal ideals. If $P \in \mathscr{P}(A)$, then P is said to be of *type* k if k is the smallest integer such that $e_k \notin P$. Since $e_{k-1} = e_1 \vee \cdots \vee e_{k-1} \in P_k(Q)$, $P_k(Q)$ is of type $\geqq k$.

LEMMA 7.2. *If P is a prime ideal of type k in A and $Q = P \cap B$, then*

$$P = P_k(Q) = \{x\text{: for every mon. rep. } \textstyle\bigvee_i b_i e_i \text{ of } x,\ b_k \in Q\}\ .$$

Proof. If $x \in A$ has a mon. rep. $\bigvee_i b_i e_i$ with $b_k \in Q$ then $x \leqq e_{k-1} \vee b_k \in P$. If $x \in P$ and $\bigvee_i b_i e_i$ is any mon. rep. of x, then $b_k e_k \in P$ and $e_k \notin P$, so that $b_k \in Q$.

THEOREM 7.3. *The prime ideals of a P_0-lattice $\langle A; e_0, \cdots, e_{n-1}\rangle$ lie in disjoint maximal chains with at most $n-1$ members.*

Proof. By 7.1, each prime ideal of A is of the form $P_k(Q)$. If $P_k(Q_1) \subseteqq P_j(Q_2)$, then $Q_1 = P_k(Q_1) \cap B \subseteqq P_j(Q_2) \cap B = Q_2$, so that $Q_1 = Q_2$. It is obvious that $P_k(Q) \subseteqq P_{k+1}(Q)$. This proves the theorem.

LEMMA 7.4. *If $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_0-lattice with center B, and $Q \in \mathscr{P}(B)$, then $P_k(Q) = \{x$: for some $b \in Q$, $x \leqq e_{k-1} \vee b\}$. Also*

$P_{k+1}(Q) = P_k(Q)$ *if and only if* $e_k \in P_k(Q)$.

Proof. If $x \in P_k(Q)$, there exists a mon. rep. $\bigvee_i b_i e_i$ of x such that $b_k \in Q$. Also $x \leqq e_{k-1} \vee b_k$. If $x \leqq e_{k-1} \vee b$ and $b \in Q$ then $x \in P_k(Q)$ since $e_{k-1} \vee b = \bigvee_{j=1}^{k-1} e_j \vee \bigvee_{j=k}^{n-1} be_j$. Suppose $e_k \in P_k(Q)$. If $x \in P_{k+1}(Q)$, then $x \leqq e_k \vee b$ for some $b \in Q$, hence $x \in P_k(Q)$. Thus $P_{k+1}(Q) = P_k(Q)$. Conversely if $P_{k+1}(Q) = P_k(Q)$, then $e_k \in P_k(Q)$ since $e_k \in P_{k+1}(Q)$.

THEOREM 7.5. *Let* $\langle A; e_0, \cdots, e_{n-1}\rangle$ *be a* P_0*-lattice with center* B, *and let* I_k *be the ideal* $\{b \in B: be_k \leqq e_{k-1}\}$ *in* B, $1 \leqq k \leqq n-1$. *Then the following are equivalent*:

(i) *Every chain in* $\mathscr{P}(A)$ *has fewer than* $n-1$ *elements.*

(ii) *For every* $Q \in \mathscr{P}(B)$, *there exists* $b \in Q$ *and an integer* $k \geqq 1$ *such that* $e_k \leqq e_{k-1} \vee b$.

(iii) $I_1 \vee \cdots \vee I_{n-1} = B$.

(iv) *A has a chain base with fewer than n elements.*

Proof. (i) implies (ii): If $Q \in \mathscr{P}(B)$, then either $P_{n-1}(Q) = A$ or $P_k(Q) = P_{k+1}(Q)$ for some $k < n-1$. Hence by 7.4, $e_k \in P_k(Q)$ for some k, $1 \leqq k \leqq n-1$, and therefore there exists $b \in Q$ such that $e_k \leqq e_{k-1} \vee b$.

(ii) implies (iii): If $I_1 \vee \cdots \vee I_{n-1} \neq B$, there exists $Q \in \mathscr{P}(B)$ such that $Q \supseteqq I_1 \vee \cdots \vee I_{n-1}$. There exists $b \in Q$ and k such that $e_k \leqq e_{k-1} \vee b$. But then $\bar{b} \in I_k \subseteqq Q$, which is impossible.

(iii) implies (iv): There exist elements $b_k \in I_k$ such that $1 = b_1 \vee \cdots \vee b_{n-1}$. By replacing the b_k by smaller elements, we may assume the b_k are pairwise disjoint. Let $f_0 = 0$ and

$$f_k = e_k \vee e_{k+1} \bigvee_{j=1}^{k} b_j\ , \qquad 1 \leqq k \leqq n-2\ .$$

Then $f_n \leqq f_{n+1}$ and $f_{n-2} = 1$, since $b_1 \vee \cdots \vee b_{n-2} = \bar{b}_{n-1}$ and $b_{n-1} \leqq e_{n-2}$. Now $f_{k-1} \vee f_k \bigvee_{j=k+1}^{n-1} b_j = e_{k-1} \vee \bigvee_{j \neq k} b_j$. Therefore,

$$e_k = f_{k-1} \vee f_k \bigvee_{j=k+1}^{n-1} b_j\ ,$$

and so $f_0, \cdots, f_{n-2}$ is a chain base of A.

(iv) implies (i) by 7.3.

THEOREM 7.6. *Let A be a* P_0*-lattice. Then A is of order n if and only if the maximum number of elements in a chain in* $\mathscr{P}(A)$ *in* $n-1$.

Proof. This follows from 7.3 and the equivalence of 7.5(i) and 7.5(iv).

DEFINITION 7.7. Let $\mathscr{P}_0(A) = \phi$, and let $\mathscr{P}_{k+1}(A)$ be the set of minimal elements of $\mathscr{P}(A) - \mathscr{P}_k(A)$.

THEOREM 7.8. *Let $\langle A; e_0, \cdots, e_{n-1}\rangle$ be a P_1-lattice with center B. Then for $0 \leqq i \leqq n-2$,*

$$e_i \in \bigcap \mathscr{P}_{i+1}(A) - \bigcup_{j \leqq i} \bigcup \mathscr{P}_j(A) .$$

Proof. By 7.4, $P_1(Q) \neq A$ for all $Q \in \mathscr{P}(B)$. Hence $\mathscr{P}_1(A) = \{P_1(Q)\colon Q \in \mathscr{P}(B)\}$. If $1 \leqq i \leqq n-2$, then $e_i \in P_i(Q)$ if and only if $e_i \leqq e_{i-1} \vee b$ for some $b \in Q$. This in turn is equivalent to $\bar{b} \leqq e_i \rightarrow e_{i-1}$ which is equivalent to $\bar{b} \leqq e_{i-1}$, or $1 \leqq e_{i-1} \vee b$. By 7.4, this is equivalent to $P_i(Q) = A$. Also, $P_i(Q) = P_{i+1}(Q)$ if and only if $e_i \in P_i(Q)$. Therefore $\mathscr{P}_i(A) = \{P_i(Q)\colon P_i(Q) \neq A\}$, and $e_i \notin P_i(Q)$ for all $P_i(Q) \in \mathscr{P}_i(A)$. Since $e_i \in P_{i+1}(Q)$ for all $Q \in \mathscr{P}(B)$, the proof is complete.

LEMMA 7.9. *Let A be a bounded distributive lattice. Suppose $\mathscr{P}(A)$ is a union of disjoint maximal chains and there exists an element $e \in \bigcap (\mathscr{P}(A) - \mathscr{P}_1(A)) - \bigcup \mathscr{P}_1(A)$. Let $A_1 = [e, 1]$. Then $\mathscr{P}_i(A_1) = \{P \cap A_1\colon P \in \mathscr{P}_{i+1}(A)\}$ for each $i \geqq 1$.*

Proof. If $P \in \mathscr{P}(A) - \mathscr{P}_1(A)$, let $\varphi(P) = P \cap A_1$. Then $\varphi(P) \in \mathscr{P}(A_1)$. If $Q \in \mathscr{P}(A_1)$, let $\psi(Q) = \{x \in A\colon x \geqq$ an element of $Q\}$. Then $\psi(Q) \in \mathscr{P}(A) - \mathscr{P}_1(A)$ and $\psi\varphi(P) = P$. Thus $\varphi\colon \mathscr{P}(A) - \mathscr{P}_1(A) \rightarrow \mathscr{P}(A_1)$ is an order isomorphism.

LEMMA 7.10. *Under the hypotheses of 7.9, let B and B_1 be the centers of A and A_1 respectively. Then $B_1 = \{b \vee e\colon b \in B\}$. If $x \in A$, then there exists $b \in B$ such that $x = b(e \vee x)$.*

Proof. Let $\{D_i\colon i \in S\}$ be the set of maximal chains in $\mathscr{P}(A)$. The intersection and union of any nonempty subset of D_i is in D_i. Let P_i and Q_i be respectively the smallest and largest member of D_i. Let $V = \{i\colon P_i \neq Q_i\}$. For $i \in V$, let $R_i = \bigcap \{P \in D_i\colon e \in P\}$. R_i is the immediate successor of P_i in D_i. We divide the proof of the lemma into several parts.

(a) If $x \in P_i$, there exists y such that $xy = 0$ and $y \notin Q_i$.

Indeed for each j such that $x \notin P_j$, choose $y_j \in P_j - Q_i$. Then every prime ideal in A contains a member of $\{x\} \cup \{y_j\colon x \notin P_j\}$. Therefore, the filter generated by this set is not proper and so there exists a finite meet y of the y_j such that $xy = 0$. Clearly $y \notin Q_i$.

(b) If $x \notin Q_i$, there exists $y \in P_i$ such that $x \vee y = 1$.

For each j such that $x \in Q_j$ choose $y_j \in P_i - Q_j$. The ideal generated by $\{x\} \vee \{y_j\colon x \in Q_j\}$ is not proper. Therefore, a finite join

y of the y_j satisfies the requirements.

(c) If $x \notin Q_i$ there exists $y \leqq x$ such that $y \notin Q_i$ and $y \in P_j$ whenever $x \in Q_j$.

By (b) there exists $z \in P_i$ such that $x \vee z = 1$. By (a) there exists $y \notin Q_i$ such that $yz = 0$. If $x \in Q_j$, then $z \notin Q_j$, hence $z \notin P_j$ and so $y \in P_j$. If P is any prime ideal containing x then $P \in D_j$ for some j, and so $x \in Q_j$. This implies $y \in P_j \subseteqq P$. Thus $y \leqq x$.

(d) If $x \notin P_i$, there exists $y \notin Q_i$ such that $ey \leqq x$.

For each j such that $x \in P_j$, choose $y_j \in P_j - Q_i$. If P is a prime ideal containing x but not e, then $P = P_j$ for some j and so $y_j \in P$. This implies that x belongs to the filter generated by $\{e\} \vee \{y_j : x \in P_j\}$. The desired y will be the meet of a finite number of y_j.

(e) If $x \in R_i$, there exists $y \notin Q_i$ such that $xy \leqq e$.

For each j such that $x \notin R_j$ choose $y_j \in P_j - Q_i$. If P is a prime ideal containing e but not x, then $P \supseteqq R_j$ for some j and since $x \notin R_j$, $y_j \in P_j \subseteqq P$. This implies that e belongs to the filter generated by $\{x\} \vee \{y_j : x \notin R_j\}$. The desired y is the meet of a finite number of y_j.

(f) If $x \in B_1$ then for all i, either $x \in R_i$ or $x \notin Q_i$.

Let y be the complement of x in A_1. If $x \in Q_i$ then $y \notin Q_i$ since $x \vee y = 1$. Therefore $y \notin R_i$, hence $x_i \in R_i$ since $xy = e \in R_i$.

(g) If for all i, $x \in P_i$ or $x \notin Q_i$, then $x \in B$.

By (a), for each i such that $x \in P_i$, there exists $y_i \notin Q_i$ such that $xy_i = 0$. No prime ideal contains x and $\{y_i : x \in P_i\}$. There exists a finite join y of the y_i such that $x \vee y = 1$ and clearly $xy = 0$.

(h) If $x \in A$, there exists $y \in B$ such that $x = y(e \vee x)$.

Let $T = \{j : x \in P_j\}$. If $T = S$ then $x = 0$ and $y = 0$. If $T = \phi$ then $x \geqq e$ and $y = 1$ will do. Suppose $T \neq S$, $T \neq \phi$. By (d), for each $i \in S - T$, there exists $y_i \notin Q_i$ such that $ey_i \leqq x$. By (a), for each $j \in T$ there exists $z_j \notin Q_j$ such that $xz_j = 0$. No prime ideal contains $\{y_i : i \in S - T\} \cup \{z_j : j \in T\}$. Therefore, there exist y, z such that $y \vee z = 1$, $ey \leqq x$, and $xz = 0$. This implies $x = xy = xy \vee ey = x(y \vee e)$. If $j \in T$, then $ey \leqq x \in P_j$, $e \notin P_j$ so that $y \in P_j$. If $i \in S - T$. then $z \in P_i$ since $x \notin P_i$ and $xz = 0$. Thus $yz \in P_i$ for all $i \in S$, and so $yz = 0$. Hence $y \in B$.

(i) If $x \in B_1$, there exists $y \in B$ such that $x = y \vee e$.

Let $W = \{j \in V : x \in R_j\}$. If $W = V$, then $x = e$ and $y = 0$. If $W = \phi$ then by (f), $x = 1$ and $y = 1$. Suppose $W \neq V$, $W \neq \phi$. By (c), for each $i \in S - W$ there exists $y_i \leqq x$ such that $y_i \notin Q_i$ and $y_i \in P_j$ for all $j \in W$. By (e), for each $j \in W$, there exists $z_j \notin Q_j$ such that $xz_j \leqq e$. No prime ideal contains $\{y_i : i \in S - W\} \vee \{z_j : j \in W\}$. Therefore, there exist y, z such that $1 = y \vee z$, $xz \leqq e$, $y \leqq x$ and $y \in P_j$ for all $j \in W$. If $i \in V - W$ then $x \notin R_i$ and $e \in R_i$, hence $z \in R_i$ and so $y \notin R_i$. If $i \in S - V$, then $y \in P_i$ or $y \notin Q_i$ since $P_i = Q_i$. Therefore by (g), $y \in B$. Finally $y \vee e \leqq x = xy \vee xz \leqq y \vee e$, and so $x = y \vee e$.

(h) and (i) yield the lemma since it is obvious that $\{b \vee e: b \in B\} \subseteqq B_1$.

THEOREM 7.11. *Let A be a bounded distributive lattice. Suppose $\mathscr{P}(A)$ is a union of disjoint maximal chains with maximum number of elements equal to $n-1$, and for each i, $0 \leqq i \leqq n-2$, there exists an element $e_i \in \bigcap \mathscr{P}_{i+1}(A) - \bigcup_{j \leqq i} \bigcup \mathscr{P}_j(A)$. If we set $e_{n-1} = 1$, then $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_1-lattice.*

Proof. Clearly $0 = e_0 < e_1 < \cdots < e_{n-2} < 1$. If $n = 2$, then A is a Boolean algebra by Nachbin's theorem [8, p. 76], and the theorem holds. Assume $n > 2$ and the theorem holds for $n-1$. Let $A_1 = [e_1, 1]$. By 7.9, A_1 satisfies the hypothesis for $n-1$. Therefore $\langle A_1; e_1, \cdots, e_{n-1}\rangle$ is a P_1-lattice. Let x be any member of A. By 7.10, $x \vee e_1 = \bigvee_{i=2}^{n-1} (e_1 \vee b_i)e_i$, where $b_i \in B$. Again by 7.10, there exists $b \in B$ such that $x = b(x \vee e_1)$. Therefore $x = be_1 \vee \bigvee_{i=2}^{n-1} bb_ie_i$. Clearly $e_{i+1} \rightarrow e_i = e_i$ in A, for $i \geqq 1$. It remains to show $e_1 \rightarrow 0 = 0$. Suppose $ye_1 = 0$ and $y \neq 0$. There exists a maximal filter F containing y. But $A - F \in \mathscr{P}_1(A)$, and so $e_1 \in F$. Thus $0 \in F$, a contradiction.

THEOREM 7.12. *Let A be a bounded distributive lattice. Then there exists a sequence $e_0, \cdots, e_{n-1}$ such that $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_1-lattice of order n if and only if*

(i) *$\mathscr{P}(A)$ is a union of disjoint maximal chains with maximum number of elements equal to $n-1$, and*

(ii) $\bigcap \mathscr{P}_{i+1}(A) - \bigcup_{j \leqq i} \bigcup \mathscr{P}_j(A) \neq \phi$.

Proof. This follows from 7.6, 7.8, and 7.11.

THEOREM 7.13. *Let A be a bounded distributive lattice. Then there exists a sequence $e_0, \cdots, e_{n-1}$ such that $\langle A; e_0, \cdots, e_{n-1}\rangle$ is a P_2-lattice of order n if and only if conditions* (i) *and* (ii) *of Theorem* 7.12 *hold as well as*

(iii) *There exists an element $c \in A$ such that for all $P \in \mathscr{P}(A)$, $c \in P$ if and only if P is a maximal ideal.*

Proof. By the equivalence of (i) and (iii) in Theorem 4.6, this is a consequence of [11, 4.9].

A characterization of Post algebras A by properties of $\mathscr{P}(A)$ is known [4]. However, we know no such characterization of P_0-lattices. We give an example of a P-algebra A such that $\mathscr{P}(A)$ consists of disjoint maximal chains with at most 2 elements but A is not a P_0-lattice. Let $C = \{0, e, 1\}$ be a 3 element chain and let A be the set of all functions f on an infinite set I to C such that $\{C: f(i) = e\}$ is finite. Since A is a P-subalgebra of a Post algebra of order 3, each chain of prime ideals of A has length at most 2, [7, Th. 7.1]. If $0 = f_0 < f_1 < \cdots < f_{n-1} = 1$ and $S_k = \{i: f_k(i) = e\}$, and if $f = \bigvee_{i=1}^{n-1} b_i f_i$, where b_i are in the center of A, then $\{i: f(i) = e\} \subsetneqq S_1 \cup \cdots \cup S_{n-1}$. Therefore, A does not have a chain base.

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Received March 15, 1974. This work was supported in part by the Office of Naval Research under Contract N00014-67-A-0269-0019, NR 048-299.

University of Southern California,
Indiana University and
University of California, Los Angeles