PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 1, 1974

SEMI-GROUPS AND COLLECTIVELY COMPACT SETS OF LINEAR OPERATORS

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A set of linear operators from one Banach space to another is collectively compact if and only if the union of the images of the unit ball has compact closure. Semi-groups $S = \{T(t): t \geqq 0\}$ of bounded linear operators on a complex Banach space into itself and in which every operator $T(t)$, $t > 0$ is compact are considered. Since $T(t_1 + t_2) = T(t_1)T(t_2)$ for each operator in the semi-group, it would be expected that the theory of collectively compact sets of linear operators could be profitably applied to semi-groups.

1. Introduction. Let X be a complex Banach space with unit ball X_1 and let $[X, X]$ denote the space of all bounded linear operators on X equipped with the uniform operator topology. The semi-group definitions and terminology used are those of Hille and Phillips [6]. Let S be a semi-group of vector-valued functions $T: [0, \infty) \rightarrow [X, X]$. It is assumed that $T(t)$ is strongly continuous for $t \geqq 0$. If $\lim_{t\to t_0} \| T(t)x - T(t_0)x \| = 0$ for each $t_0 \geqq 0$, $x \in X$ and if there is a constant M such that the $\| T(t) \| \leqq M$ for each $t \geqq 0$, then $S = \{T(t): t \geqq 0\}$ is called an equicontinuous semi-group of class C_0. The infinitesimal generator A of the semi-group S is defined by

$$Ax = \lim_{S\to 0} \frac{1}{S}[T(s)x - x]$$

whenever the limit exists. The domain $D(A)$ of A is a dense subset of X consisting of just those elements x for which this limit exists. A is a closed linear operator having resolvents $R(\lambda)$ which, for each complex number λ with the real part of λ greater than zero, are given by the absolutely summable Riemann-Stieltjes integral

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x dt,\ x \in X . \tag{1}$$

It follows from (1) that

$$\| R(\lambda) \| \leqq \frac{M}{re(\lambda)},\ re(\lambda) > 0 . \tag{2}$$

In particular, sets of the type $\{R(\lambda): re(\lambda) \geqq \alpha > 0\}$ are equicontinuous subsets of $[X, X]$.

Results yielding the collective compactness of the resolvents of

A have recently been obtained independently by N. E. Joshi and M. V. Deshpande.

2. Semi-groups of compact operators. First, note that (1) states that the resolvents of A are Laplace transforms of the semi-group S. Consequently, there are many other important integral expressions involving the elements of the semi-group and the resolvents. In order to take advantage of these, we prove the following lemma, in which $|v|$ denotes the total variation of a complex measure v.

LEMMA 2.1. *Let Ω be a topological space and $\mathscr{M}$ a collection of complex-valued Borel measures on Ω. Suppose there exists a constant α for which $|v|\Omega \leqq \alpha$ for each $v \in \mathscr{M}$. Let $\mathscr{K}: \Omega \rightarrow [X, X]$ be an operator-valued function defined on Ω which is strongly measurable with respect to each $v \in M$* [6, page 74] *and suppose $\mathscr{K} = \{K(w): w \in \Omega\}$ is a bounded subset of $[X, X]$. For each $v \in \mathscr{M}$ and $x \in X$, let $F_v(x) = \int_\Omega K(w)xdv$, where the integral exists in the Bochner sense since $\int_\Omega \|K(w)x\| \, d|v| < \infty$* [6, page 80]. *Let $\mathscr{F} = \{F_v: v \in \mathscr{M}\}$. Whenever $\mathscr{K}(\mathscr{K}^*)$ is collectively compact, $\mathscr{F}(\mathscr{F}^*)$ is also collectively compact.*

Proof. Assume that $\mathscr{K}$ is collectively compact. Let $B = \{K(w)x: w \in \Omega, \|x\| \leqq 1\}$ and let C denote the balanced convex hull of B. Both B and C are totally bounded subsets of X. It suffices to show that $F_v(x) \in \alpha\bar{C}$ for any $F_v \in \mathscr{F}$ and x with $\|x\| \leqq 1$. Let $\varepsilon > 0$ and choose $\{K(w_1)x_1, \cdots, K(w_n)x_n\}$, an ε/α-net for B. For $i = 1, \cdots, n$, let $\Omega_i = \{w: \|K(w)x - K(w_i)x_i\| \leqq \varepsilon/\alpha\}$ and let $\Omega_i' = \Omega_j \backslash \bigcup_{j=1}^{i-1} \Omega_j$ be a decomposition of the Ω_i into pairwise disjoint sets. Then

$$\left\|F_v(x) - \sum_{i=1}^n K(w_i)x_i v(\Omega_i')\right\| \leqq \sum_{i=1}^n \int_{\Omega_i'} \|K(w)x - K(w_i)x_i\| \, d|v|(w)$$
$$\leqq (\varepsilon/\alpha)|v|(\Omega) \leqq \varepsilon .$$

Since $\sum_{i=1}^n |v(\Omega_i')| \leqq \alpha$, $\sum_{i=1}^n K(w_i)x_i v(\Omega_i')$ is an element of αC. It follows that $F_v(x) \in \alpha\bar{C}$ and so $\mathscr{F}$ is also collectively compact.

Now assume that $\mathscr{K}^*$ is collectively compact. Let V be any neighborhood of 0 in the norm topology of X. There exists an $\varepsilon > 0$ such that $U = \{x: \|x\| \leqq \varepsilon\} \subseteqq V$. Since $\mathscr{K}^*$ is collectively compact, [2, Theorem 2.11, part (c)] implies that there exists a weak neighborhood W of the origin with $\mathscr{K}(W \cap X_1) \subseteqq (1/\alpha)U$. For $F_v \in \mathscr{F}$ and $x \in W \cap X_1$, $\|F_v(x)\| \leqq \int_\Omega \|K(w)x\| \, d|v| \leqq (\varepsilon/\alpha)|v|(\Omega) \leqq$

ε. So $\mathscr{F}(W \cap X_1) \subseteqq V$. Again using [2, Theorem 2.1, part (c)], we see that $\mathscr{F}^*$ is also collectively compact.

The following is essentially a result of P. Lax [6, page 304]. Rephrased in the terminology of collectively compact sets of operators, it becomes quite transparent.

THEOREM 2.2. *Suppose that some* $T(t_0)$, $t_0 > 0$, *is a compact operator. Then* $\mathscr{K} = \{T(t): t \geqq t_0\}$ *is a totally bounded, collectively compact subset of* $[X, X]$. *Consequently,* $T(t)$ *is continuous in the uniform operator topology for* $t \geqq t_0$.

Proof. Since $T(t) = T(t - t_0)T(t_0) = T(t_0)T(t - t_0)$ for $t \geqq t_0$, it follows that $\mathscr{K} = T(t_0)\mathscr{S} = \mathscr{S}T(t_0)$. $T(t_0)$ is a compact operator and the collection $\mathscr{S}$ is equicontinuous. By Lemmas 2.1 and 2.3 of [2], both $\mathscr{K}$ and $\mathscr{K}^*$ are collectively compact. [2, Corollary 2.6] implies that $\mathscr{K}$ is a totally bounded subset of $[X, X]$. Since $T(t)$ is continuous in the strong operator topology, $T(t)$ is continuous in the uniform operator topology for $t \geqq t_0$.

COROLLARY 2.3. *Suppose every* $T(t)$, $t > 0$, *is a compact operator. Let* $\mathscr{F} = \{R(\lambda): re(\lambda) \geqq 1\}$ *be the collection of the resolvents of the infinitesimal generator* A *corresponding to the half-plane* $\{\lambda \in C: re(\lambda) \geqq 1\}$. *Then* $\mathscr{F}$ *is a totally bounded, collectively compact set of operators.*

It should be noted that for any $\alpha > 0$, the following arguments can be applied to $\{R(\lambda): re(\lambda) \geqq \alpha\}$. One particular half-plane is chosen simply to keep the notation as uncomplicated as possible.

Proof. It will suffice to show that for each $\varepsilon > 0$, there exists a totally bounded, collectively compact set of operators $\mathscr{K}$ such that for any $R(\lambda) \in \mathscr{F}$, there exists a $K \in \mathscr{K}$ with $\| R(\lambda) - K \| \leqq \varepsilon$. For this ε, choose $\delta > 0$ with $\int_0^\delta e^{-t}dt < \varepsilon/M$, where M is such that $\| T(\lambda) \| \leqq M$ for $t > 0$. Let λ be any complex number with $re(\lambda) \geqq 1$ and $x \in X$. Since $R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)xdt$, $\left\| R(\lambda)x - \int_\delta^\infty e^{-\lambda t}T(t)xdt \right\| \leqq \int_0^\delta e^{-\lambda t} \| T(t)x \| \, dt \leqq \int_0^\delta e^{-t}dtM \| x \| \leqq \varepsilon \| x \|$. Consequently, $\left\| R(\lambda) - \int_\delta^\infty e^{-\lambda t}T(t)dt \right\| \leqq \varepsilon$. Now $\mathscr{K} = \left\{\int_\delta^\infty e^{-\lambda t}T(t)dt: re(\lambda) \geqq 1\right\}$ is a totally bounded, collectively compact set of operators. To see this, note that $\sup \left\{\int_\delta^\infty | e^{-\lambda t} | \, dt: re(\lambda) \geqq 1\right\} \leqq 1$ and that both $\{T(t): t \geqq \delta\}$ and $\{T^*(t): t \geqq \delta\}$ are collectively compact. Lemma 2.1 implies that both

$\mathscr{K}$ and $\mathscr{K}^*$ are collectively compact. As before, [2, Corollary 2.6] implies that $\mathscr{K}$ is a totally bounded subset of $[X, X]$.

The following lemma will be useful in the next section. Since a quotable reference cannot be found, a brief proof is included.

LEMMA 2.4. *Let* $\mathscr{S}$ *be an equicontinuous semi-group of class* C_0. *Then* $R(\lambda)$ *converges to zero in the strong operator topology as* $|\lambda| \to \infty$, $re(\lambda) \geqq 1$. *Whenever* $\{R(\lambda): re(\lambda) \geqq 1\}$ *is a totally bounded subset of* $[X, X]$, *the* $R(\lambda)$ *converge to zero in the uniform operator topology as* $|\lambda| \to \infty$, $re(\lambda) \geqq 1$.

Proof. The second assertion follows immediately from the first.

Let $x \in D(A)$, the domain of the infinitesimal generator A. Since $R(\lambda)(\lambda - A)x = x$, we have the identity

$$R(\lambda)x = \frac{1}{\lambda}[x + R(\lambda)Ax] .$$

By (2) of § 1, $\{R(\lambda)Ax: re(\lambda) \geqq 1\}$ is a bounded subset of X. It follows that $\|R(\lambda)x\| \to 0$ as $|\lambda| \to \infty$, $re(\lambda) \geqq 1$, for each $x \in D(A)$. Since $D(A)$ is dense in X, the Banach-Steinhaus theorem implies that this type of convergence holds for each $x \in X$. We see that the first assertion of this lemma holds also.

3. **Semi-groups with compact resolvents.** Suppose that the domain of the infinitesimal generator of a semi-group can be given a topology τ such that the topological space $\langle D(A), \tau\rangle$ is a Banach space and the natural injection $i: \langle D(A), \tau\rangle \to X$ is a compact operator. In such cases, it might be possible to prove that certain sets of the resolvents of A are equicontinuous subsets of $[X, \langle D(A), \tau\rangle]$, i.e., collectively compact subsets of $[X, X]$. A specific example is the case in which X is some L^p space and A is the negative of a uniformly strongly elliptic differential operator defined on a Sobolev space $H = \langle D(A), \tau\rangle$. The so-called "a priori inequalities" [4, Theorems 18.2 and 19.2, pages 69 and 77] imply that, after a suitable translation, $\{R(\lambda): re(\lambda) \geqq 1\}$ is an equicontinuous subset of $[L^p, H]$. Since the injection $i: H \to L^p$ is a compact operator [4, Theorem 11.2, page 31], $\{R(\lambda): re(\lambda) \geqq 1\}$ is a collectively compact subset of $[L^p, L^p]$. The obvious question is what are the implications of such assumptions for a general semi-group $\mathscr{S}$.

We first consider the case in which A has one compact resolvent. Of course, the first resolvent equation,

$$R(\lambda_1) - R(\lambda_2) = (\lambda_2 - \lambda_1)R(\lambda_1)R(\lambda_2) ,$$

then implies that all resolvents of A are compact operators.

LEMMA 3.1. *Suppose A has one compact resolvent. Let Ω be a compact subset of $\{\lambda: re(\lambda) > 0\}$. Then $\{R(\lambda): \lambda \in \Omega\}$ is collectively compact.*

Proof. Since $R(\lambda)$ is a holomorphic function in the right half-plane, $\{R(\lambda): \lambda \in \Omega\}$ is a totally bounded subset of $[X, X]$. Each element in this collection is a compact operator. So [2, Corollary 2.7] implies that $\{R(\lambda): \lambda \in \Omega\}$ is collectively compact.

The following is a partial converse of Theorem 2.2.

PROPOSITION 3.2. *Suppose A has compact resolvents. Let $t_0 > 0$. If $T(t)$ is continuous in the uniform operator topology for $t \in [t_0, \infty)$, then $T(t_0)$ is a compact operator.*

Proof. Since the resolvents are Laplace transforms of $\{T(t): t \geqq 0\}$, we may use the formula based upon fractional integration of order two [6, page 220] which states that

$$\int_0^s (s - t)T(t)dt = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{e^{\lambda s}}{\lambda^2}R(\lambda)d\lambda,\ s > 0 .$$

For $\varepsilon > 0$, choose N such that

$$\int_{1-i\infty}^{1-iN} + \int_{1+iN}^{1+i\infty} \frac{1}{|\lambda^2|}\| e^{\lambda s}R(\lambda) \| \, d\,|\lambda| < \varepsilon .$$

Then

$$\left\|\int_0^s (s - t)T(t)dt - \frac{1}{2\pi i}\int_{1-iN}^{1+iN} \frac{e^{\lambda s}}{\lambda^2}R(\lambda)d\lambda\right\| < \varepsilon .$$

By Lemmas 3.1 and 2.1, the integral of $(e^{\lambda s}/\lambda^2)R(\lambda)$ over the finite segment of the vertical line is a compact operator. It follows that for each $s \geqq 0$, $\int_0^s (s - t)T(t)dt$ is a compact operator.

Consider the function

$$F(s) = \int_0^s (s - t)T(t)dt,\ s \geqq 0 .$$

Each value of F is a compact operator. Elementary calculations show that F is differentiable in the uniform operator topology. Consequently, each

$$F'(s) = \int_0^s T(t)dt, \; s \geqq 0 \, ,$$

is the limit in the uniform operator topology of a sequence of compact operators. Hence, each $F'(s)$, $s \geqq 0$, is a compact operator. In taking derivatives again, we see that for $h > 0$,

$$\left\|\frac{1}{h}\int_{t_0}^{t_0+h} T(t)dt - T(t_0)\right\| \leqq \sup \{\| T(t_0 + \alpha) - T(t_0) \|: 0 \leqq \alpha \leqq h\} \, .$$

If $T(t_0 + \alpha)$ is continuous in the uniform operator topology for $\alpha \geqq 0$, then

$$T(t_0) = \text{uniform} - \lim_{h \to 0+} \frac{1}{h}\int_{t_0}^{t_0+h} T(t)dt \, .$$

It follows that $T(t_0)$ is a compact operator.

See [6, page 537] for a discussion of the following example.

EXAMPLE 3.3. Consider the semi-group $\mathscr{S}$ of left translations on the space $C_0[0, 1]$ consisting of continuous functions $x(u)$ vanishing at 1, where the norm $\| x \| = \sup \{| x(u) |: 0 \leqq u \leqq 1\}$. Let $[T(t)x](u) = x(u + t)$, for $0 \leqq u \leqq \max \{0, 1 - t\}$, and 0 for $\max \{0, 1 - t\} \leqq u \leqq 1$. The infinitesimal generator of $\mathscr{S}$ is the operator of differentiation $d/(du)$ with domain

$$D\left(\frac{d}{du}\right) = \{x: x' \in C_0[0, 1]\} \, .$$

The compact resolvents are given by

$$[R(\lambda)x](u) = \int_0^{1-u} e^{-\lambda t}x(u + t)dt, \; \lambda \in \boldsymbol{C} \, .$$

For $t \geqq 1$, $T(t)$ is the compact operator 0 while for $t, s < 1$, $\| T(t) - T(s) \| = 2$. This can easily be seen by evaluating $T(t) - T(s)$ at a function $x \in C_0[0, 1]$ with $\| x \| \leqq 1$ and $x(t) = 1$, $x(s) = -1$. So $T(t)$ is continuous in the uniform operator topology only for $t \geqq 1$.

Choose a monotonically increasing sequence of positive functions $\{y_n\} \subseteqq C_0[0, 1]$ such that $\lim_n y_n(u) = 1$ for each $u < 1$. For $t < 1$, $\{T(t)y_n\}$ is a sequence of functions having no subsequence which can converge uniformly. So $T(t)$, $t < 1$, is not a compact operator.

For $\lambda = \sigma + i\tau$, let $x_n(u) = e^{i\tau u}y_n(u)$ in the definition of $R(\lambda)$. We see that

$$[R(\lambda)x_n](0) = \int_0^1 e^{-\sigma t}y_n(t)dt \, .$$

Since $\|x_n\| = 1$ for each n,

$$\|R(\lambda)\| \geqq \sup_n |[R(\lambda)x_n](0)| = \int_0^1 e^{-\sigma t}dt\ .$$

It follows immediately from the definition of $R(\lambda)$ that the reverse inequality holds also. Consequently, $\|R(\lambda)\| = \int_0^1 e^{-\sigma t}dt$. In particular, $\lim_{|\tau|\to\infty}\|R(\sigma + i\tau)\| \neq 0$. This serves to distinguish this differential operator from the class of infinitesimal generators which we consider next.

LEMMA 3.4. *Suppose $\mathscr{S}$ is a semi-group such that the set of resolvents $\{R(\lambda): re(\lambda) = 1\}$ corresponding to the vertical line $re(\lambda) = 1$ is collectively compact. Then $\{R(\lambda): re(\lambda) \geqq 1\}$ is also collectively compact.*

Proof. For each $x \in X$, $R(\lambda)x$ is a holomorphic and bounded function of λ, $re(\lambda) > 1/2$. So $R(\lambda)x$ admits Poisson's integral representation [6, page 229]

$$R(\sigma + i\tau)x = \frac{\sigma - 1}{\pi}\int_{-\infty}^{\infty}\frac{R(1 + i\beta)x}{(\sigma - 1)^2 + (\tau - \beta)^2}d\beta$$

for $\sigma > 1$, $x \in X$. Since $\{R(1 + i\beta): -\infty < \beta < \infty\}$ is collectively compact and the integral of the Poisson kernel over $-\infty < \beta < \infty$ is identically one, Lemma 2.1 implies that $\{R(\lambda): re(\lambda) > 1\}$ is collectively compact. Taking the union of this set and $\{R(\lambda): re(\lambda) = 1\}$, one obtains the desired result.

For $x \in X$ and $x^* \in X^*$,

$$\langle x^*, R(\sigma + i\tau)x\rangle = \int_0^\infty e^{-i\tau t}(e^{-\sigma t}\langle x^*, T(t)x\rangle)dt\ .$$

This is this Fourier transform of the absolutely summable function $e^{-\sigma t}\langle x^*, T(t)x\rangle$, $t \geqq 0$. The convergence of

$$\|R(\sigma + i\tau)\| = \sup\{|\langle x^*, R(\sigma + i\tau)x\rangle| : \|x\|, \|x^*\| \leqq 1\}$$

to 0 as $|\sigma|$ and $|\tau|$ approach infinity can be viewed as a "uniform" Riemann-Lebesgue lemma.

THEOREM 3.5. *If $\mathscr{F} = \{R(\lambda): re(\lambda) \geqq 1\}$ is collectively compact, then $\|R(\lambda)\|$ converges to 0 as $|\lambda|$ approaches ∞, $re(\lambda) \geqq 1$.*

Proof. Throughout the following proof, we assume that $re(\lambda) \geqq 1$.

Let $\varepsilon > 0$ be given and choose real β so large that $1 + \beta \geqq M/\varepsilon$, where M is the constant in § 1 which bounds the operator norms of elements of $\mathscr{S}$. By (2),

$$\| R(\lambda + \beta) \| \leqq \frac{M}{re(\lambda) + \beta} \leqq \frac{M}{1 + \beta} \leqq \varepsilon .$$

In view of Lemma 2.4, $\mathscr{F}$ is an equicontinuous collection with $R(\lambda)$ converging to zero as $|\lambda| \to \infty$ pointwise on the relatively compact set $\mathscr{F}(X_1)$. Therefore, $\| R(\lambda)F \| \to 0$ as $|\lambda| \to \infty$ uniformly for $F \in \mathscr{F}$. Choose N such that $|\lambda| \geqq N$ implies that

$$\| R(\lambda)R(\lambda + \beta) \| \leqq \varepsilon/\beta .$$

The first resolvent equation states that

$$R(\lambda) - R(\lambda + \beta) = (\lambda + \beta - \lambda)R(\lambda)R(\lambda + \beta) .$$

So, for $|\lambda| \geqq N$,

$$\| R(\lambda) \| \leqq \| \beta R(\lambda)R(\lambda + \beta) \| + \| R(\lambda + \beta) \| \leqq 2\varepsilon .$$

Note that we have used the fact that $\mathscr{F}$ contains those resolvents $R(\lambda)$ with $re(\lambda)$ arbitrarily large in an essential way.

COROLLARY 3.6. *Let $\mathscr{S}$ be any semi-group whose infinitesimal generator A has compact resolvents, i.e., each $R(\lambda)$, $re(\lambda) > 0$, is a compact operator. Then $\mathscr{F} = \{R(\lambda): re(\lambda) \geqq 1\}$ is collectively compact if and only if $\| R(\lambda) \| \to 0$ as $|\lambda| \to \infty$, $re(\lambda) \geqq 1$.*

Proof. The assumption that $\| R(\lambda) \| \to 0$ as $|\lambda| \to \infty$, $re(\lambda) \geqq 1$, simply implies that $R(\lambda)$ can be extended to a continuous function on the compactification of the half-plane $\{\lambda: re(\lambda) \geqq 1\}$. Consequently, if A has compact resolvents, $\mathscr{F}$ is a totally bounded set of compact operators. [2, Corollary 2.7] implies that $\mathscr{F}$ is collectively compact.

The converse is simply Theorem 3.5.

The behavior of the holomorphic function $R(\lambda)$ on the vertical line $re(\lambda) = 1$ is of fundamental importance. For example, if $d(\lambda)$ denotes the distance of the complex number λ from the spectrum of A, then [3, page 566]

$$d(1 + i\tau) \geqq \frac{1}{\| R(1 + i\tau) \|} .$$

We see that the spectrum of A must be bounded on the right by the curve

$$\gamma(\tau) = 1 - \frac{1}{\|R(1 + i\tau)\|} + i\tau, \ -\infty < \tau < \infty .$$

In particular, it follows from Theorem 3.5 and Lemma 3.4 that when $\{R(\lambda): re(\lambda) = 1\}$ is collectively compact, the spectrum of A is severely restricted.

The usual methods of inverting Fourier transforms can be typified by the use of $(C, 1)$ means. In [5, page 350], it is shown that for each $t > 0$

$$T(t) = \lim_{w\to\infty} \frac{1}{2\pi} \int_{-w}^{w} \left(1 - \frac{|\tau|}{w}\right) e^{(1+i\tau)t} R(1 + i\tau) d\tau .$$

However, the measures involved no longer satisfy the requirements of Lemma 2.1. As this situation is typical, we are not able to prove that if $\{R(\lambda): re(\lambda) = 1\}$ is collectively compact, then each $T(t) \in \mathscr{S}$, $t > 0$, is a compact operator.

REFERENCES

1. P. M. Anselone and R. H. Moore, *Approximate solutions of integral and operator equations*, J. Math. Anal. Appl., **9** (1964), 268-277.
2. J. D. DePree and H. S. Klein, *Characterizations of collectively compact set, of linear operators*, Pacific J. Math., **55** (1974), 45-54.
3. N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, New York: Interscience, 1958.
4. A. Friedman, *Partial Differential Equations*, New York: Holt, Rinehart and Winston, 1969.
5. A. Grothendieck, *Sur les Applications Lineaires Faiblement Compactes D'Epaces Du Type C(K)*, Canad. J. Math., **5** (1953), 129-173.
6. E. Hille and R. Phillips, *Functional Analysis and Semi-Groups*, Providence, R. I.: American Mathematical Society, 1957.

Received April 4, 1973.

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