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RAMSEY THEORY AND CHROMATIC NUMBERS

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Let $\chi(G)$ denote the chromatic number of a graph G. For positive integers $n_1, n_2, \cdots, n_k$ $(k \geqq 1)$ the chromatic Ramsey number $\chi(n_1, n_2, \cdots, n_k)$ is defined as the least positive integer p such that for any factorization $K_p = \bigcup_{i=1}^{k} G_i$, $\chi(G_i) \geqq n_i$ for at least one $i, 1 \leqq i \leqq k$. It is shown that $\chi(n_1, n_2, \cdots, n_k) = 1 + \prod_{i=1}^{k}(n_i - 1)$. The vertex-arboricity $a(G)$ of a graph G is the fewest number of subsets into which the vertex set of G can be partitioned so that each subset induces an acyclic graph. For positive integers $n_1, n_2, \cdots, n_k$ $(k \geqq 1)$ the vertex-arboricity Ramsey number $a(n_1, n_2, \cdots, n_k)$ is defined as the least positive integer p such that for any factorization $K_p = \bigcup_{i=1}^{k} G_i$, $a(G_i) \geqq n_i$ for at least one $i, 1 \leqq i \leqq k$. It is shown that $a(n_1, n_2, \cdots, n_k) = 1 + 2k\prod_{i=1}^{k}(n_i - 1)$.

Introduction. The classical *Ramsey number* $r(m, n)$, for positive integers m and n, is the least positive integer p such that for any graph G of order p, either G contains the complete graph K_m of order m as a subgraph or the complement $\bar{G}$ of G contains K_n as a subgraph. More generally, for $k(\geqq 1)$ positive integers $n_1, n_2, \cdots, n_k$, the *Ramsey number* $r(n_1, n_2, \cdots, n_k)$ is defined as the least positive integer p such that for any *factorization* $K_p = G_1 \cup G_2 \cup \cdots \cup G_k$ (i.e., the G_i are spanning, pairwise edge-disjoint, possibly empty subgraphs of K_p such that the union of the edge sets of the G_i equals the edge set of K_p), G_i contains K_{n_i} as a subgraph for at least one $i, 1 \leqq i \leqq k$. It is known (see [5]) that all such Ramsey numbers exist; however, the actual values of $r(n_1, n_2, \cdots, n_k)$, $k \geqq 1$, are known in only seven cases (see [2, 3]) for which $\min\{n_1, n_2, \cdots, n_k\} \geqq 3$.

A *clique* in a graph G is a maximal complete subgraph of G. The *clique number* $\omega(G)$ is the maximum order among the cliques of G. The Ramsey number $r(n_1, n_2, \cdots, n_k)$ may be alternatively defined as the least positive integer p such that for any factorization $K_p = G_1 \cup G_2 \cup \cdots \cup G_k$, $\omega(G_i) \geqq n_i$ for at least one $i, 1 \leqq i \leqq k$.

The foregoing observation suggests the following definition. Let f be a graphical parameter, and let $n_1, n_2, \cdots, n_k$, $k \geqq 1$ be positive integers. The *f-Ramsey number* $f(n_1, n_2, \cdots, n_k)$ is the least positive integer p such that for any factorization $K_p = G_1 \cup G_2 \cup \cdots \cup G_k$, $f(G_i) \geqq n_i$ for at least one $i, 1 \leqq i \leqq k$. Hence, $\omega(n_1, n_2, \cdots, n_k) = r(n_1, n_2, \cdots, n_k)$, i.e., the ω-Ramsey number is the Ramsey number.

The object of this paper is to investigate f-Ramsey numbers for two graphical parameters f, namely chromatic number and vertex-arboricity.

Chromatic Ramsey numbers. The *chromatic number* $\chi(G)$ of a graph G is the fewest number of colors which may be assigned to the vertices of G so that adjacent vertices are assigned different colors. For positive integers $n_1, n_2, \cdots, n_k$, the *chromatic Ramsey number* $\chi(n_1, n_2, \cdots, n_k)$ is the least positive integer p such that for any factorization $K_p = G_1 \cup G_2 \cup \cdots G_k$, $\chi(G_i) \geqq n_i$ for some i, $1 \leqq i \leqq k$. The existence of the numbers $\chi(n_1, n_2, \cdots, n_k)$ is guaranteed by the fact that $\chi(n_1, n_2, \cdots, n_k) \leqq r(n_1, n_2, \cdots, n_k)$. We are now prepared to present a formula for $\chi(n_1, n_2, \cdots, n_k)$. We begin with a lemma.

LEMMA. *If* $G = G_1 \cup G_2 \cup \cdots \cup G_k$, *then*

$$\chi(G) \leqq \sum_{i=1}^{k} \chi(G_i) .$$

Proof. For $i = 1, 2, \cdots, k$, let a $\chi(G_i)$-coloring be given for G_i. We assign to a vertex v of G the color $(c_1, c_2, \cdots, c_k)$, where c_i is the color assigned to v in G_i. This produces a coloring of G using at most $\prod_{i=1}^{k} \chi(G_i)$ colors; hence, $\chi(G) \leqq \prod_{i=1}^{k} \chi(G_i)$.

THEOREM 1. *For positive integers* $n_1, n_2, \cdots, n_k$,

$$\chi(n_1, n_2, \cdots, n_k) = 1 + \prod_{i=1}^{k} (n_i - 1) .$$

Proof. The result is immediate if $n_i = 1$ for some i; hence, we assume that $n_i \geqq 2$ for all i, $1 \leqq i \leqq k$. First, we verify that

$$\chi(n_1, n_2, \cdots, n_k) \leqq 1 + \prod_{i=1}^{k} (n_i - 1) .$$

Let $p = 1 + \prod_{i=1}^{k} (n_i - 1)$, and assume there exists a factorization $K_p = G_1 \cup G_2 \cup \cdots \cup G_k$ such that $\chi(G_i) \leqq n_i - 1$ for each $i = 1, 2, \cdots, k$. Then by the Lemma, it follows that

$$1 + \prod_{i=1}^{k} (n_i - 1) = \chi(K_p) \leqq \prod_{i=1}^{k} \chi(G_i) \leqq \prod_{i=1}^{k} (n_i - 1) ,$$

which produces a contradiction. Thus, in any factorization $K_p = G_1 \cup G_2 \cup \cdots \cup G_k$ for $p = 1 + \prod_{i=1}^{k} (n_i - 1)$, we have $\chi(G_i) \geqq n_i$ for at least one i, $1 \leqq i \leqq k$.

In order to show that

$$\chi(n_1, n_2, \cdots, n_k) \geqq 1 + \prod_{i=1}^{k} (n_i - 1) ,$$

we exhibit a factorization $K_{N_k} = G_1 \cup G_2 \cup \cdots \cup G_k$, where $N_k =$

$\prod_{i=1}^{k}(n_i - 1)$ and $\chi(G_i) \leqq n_i - 1$ for $i = 1, 2, \cdots, k$. The factorization is accomplished by employing induction on k. For $k = 1$, we simply observe that $\chi(K_{N_1}) = \chi(K_{n_1-1}) = n_1 - 1$. Assume there exists a factorization $K_{N_{k-1}} = H_1 \cup H_2 \cup \cdots \cup H_{k-1}$ such that $\chi(H_i) \leqq n_i - 1$ for $i = 1, 2, \cdots, k - 1$. Let F denote $n_k - 1$ (pairwise disjoint) copies of $K_{N_{k-1}}$ and define G_k by $G_k = \bar{F}$. Thus, $\bar{G}_k$ contains $n_k - 1$ pairwise disjoint copies of H_i for $i = 1, 2, \cdots, k - 1$, which we denote by G_i. Hence, $K_{N_k} = G_1 \cup G_2 \cup \cdots \cup G_k$, where $\chi(G_i) \leqq n_i - 1$ for each i, $1 \leqq i \leqq k$, which produces the desired result.

Vertex-arboricity Ramsey numbers. The *vertex-arboricity* $a(G)$ of a graph G is the minimum number of subsets into which the vertex set of G may be partitioned so that each subset induces an acyclic subgraph. As with the chromatic number, the vertex-arboricity may be considered a coloring number since $a(G)$ is the least number of colors which may be assigned to the vertices of G so that no cycle of G has all of its vertices assigned the same color.

Our next result will establish a formula for the *vertex-arboricity Ramsey number* $a(n_1, n_2, \cdots, n_k)$, defined as the least positive integer p such that for every factorization $K_p = G_1 \cup G_2 \cup \cdots \cup G_k$, $a(G_i) \geqq n_i$ for some i, $1 \leqq i \leqq k$. Since $a(K_n) = \{n/2\}$, it follows that $a(n_1, n_2, \cdots, n_k) \leqq r(2n_1 - 1, 2n_2 - 1, \cdots, 2n_k - 1)$. In the proof of the following result, we shall make use of the *(edge) arboricity* $a_1(G)$ of a graph, which is the minimum number of subsets into which the edge set of G may be partitioned so that the subgraph induced by each subset is acyclic. It is known (see [1, 4]) that $a_1(K_n) = \{n/2\}$.

THEOREM 2. *For positive integers* $n_1, n_2, \cdots, n_k$,

$$a(n_1, n_2, \cdots, n_k) = 1 + 2k \prod_{i=1}^{k} (n_i - 1) .$$

Proof. In order to show that

$$a(n_1, n_2, \cdots, n_k) \leqq 1 + 2k \prod_{i=1}^{k} (n_i - 1) ,$$

we let $p = 1 + 2k \prod_{i=1}^{k}(n_i - 1)$ and assume there exists a factorization $K_p = G_1 \cup G_2 \cup \cdots \cup G_k$ such that $a(G_i) \leqq n_i - 1$ for each $i = 1, 2, \cdots, k$. For each $i = 1, 2, \cdots, k$, there is a partition $\{U_{i,1}, U_{i,2}, \cdots, U_{i,n_i-1}\}$ of the vertex set $V(G_i)$ of G_i such that the subgraph $\langle U_{i,j}\rangle$ of G_i induced by $U_{i,j}$ is acyclic, $j = 1, 2, \cdots, n_i - 1$. At least one of the sets $U_{1,1}, U_{1,2}, \cdots, U_{1,n_1-1}$, say U_{1,m_1}, contains at least $1 + 2k \prod_{i=2}^{k}(n_i - 1)$ vertices. Thus, at least one of the sets $U_{2,1}, U_{2,2}, \cdots,$

U_{2,n_2-1}, say U_{2,m_2}, contains at least $1 + 2k\prod_{i=3}^{k}(n_i - 1)$ vertices of U_{1,m_1}. Proceeding inductively, we arrive at subsets $U_{1,m_1}, U_{2,m_2}, \cdots, U_{k,m_k}$ such that $\bigcap_{i=1}^{t} U_{i,m_i}$ contains at least $1 + 2k\prod_{i=t+1}^{k}(n_i - 1)$ vertices, $1 \leqq t \leqq k-1$. In particular, $\bigcap_{i=1}^{k} U_{i,m_i}$, contains a set U having $1 + 2k$ vertices. For each $i = 1, 2, \cdots, k$, $\langle U \rangle$ is an acyclic subgraph of the graph $\langle U_{i,m_i} \rangle$. This implies that $a_1(K_{1+2k}) \leqq k$, which is contradictory. Therefore, $a(G_i) \geqq n_i$ for at least one i, $1 \leqq i \leqq k$.

The proof will be complete once we have verified that

$$a(n_1, n_2, \cdots, n_k) \geqq 1 + 2k \prod_{i=1}^{k} (n_i - 1) \, .$$

Let $r = \prod_{i=1}^{k}(n_i - 1)$. We shall exhibit a factorization $K_{2kr} = G_1 \cup G_2 \cup \cdots \cup G_k$ such that $a(G_i) \leqq n_i - 1$ for $i = 1, 2, \cdots, k$. We begin with r pairwise disjoint copies of K_{2k}, labeled $K_{2k}^1, K_{2k}^2, \cdots, K_{2k}^r$. Since $a_1(K_{2k}) = k$, it follows that $K_{2k} = \bigcup_{i=1}^{k} F_i$, where each F_i is an acyclic graph. We introduce the notation F_{il} to denote the F_i contained in K_{2k}^l, $l = 1, 2, \cdots, r$ and $i = 1, 2, \cdots, k$. With each of the r k-tuples $(c_1, c_2, \cdots, c_k)$, $c_j = 1, 2, \cdots, n_j - 1$ and $j = 1, 2, \cdots, k$, we identify a complete graph K_{2k}^l, $l = 1, 2, \cdots, r$, in such a way that the identification is one-to-one. Then, for each $i = 1, 2, \cdots, k$ and $l = 1, 2, \cdots, r$, we associate with F_{il} the k-tuple identified with K_{2k}^l. Define the graph G_i, $i = 1, 2, \cdots, k$, to consist of the graphs $F_{i1}, F_{i2}, \cdots, F_{ir}$; in addition, each vertex of F_{is} is adjacent to each vertex of F_{it}, $s, t = 1, 2, \cdots, r$, provided the ith coordinate is the first coordinate in which their associated k-tuples differ (otherwise, there are no edges between F_{is} and F_{it}). It is then seen that $K_{2kr} = \bigcup_{i=1}^{k} G_i$. For each $i = 1, 2, \cdots, k$, define $V_{i,j}$ to be the set of all vertices v such that v is a vertex of an F_{il} whose associated k-tuple $(c_1, c_2, \cdots, c_k)$ has $c_i = j$; $j = 1, 2, \cdots, n_i - 1$. Then $\{V_{i,1}, V_{i,2}, \cdots, V_{i,n_i-1}\}$ is a partition of $V(G_i)$ for which the subgraph $\langle V_{i,j} \rangle$ consists of $r/(n_i - 1)$ pairwise disjoint copies of F_i, $j = 1, 2, \cdots, n_i - 1$. Thus, $\langle V_{i,j} \rangle$ is an acyclic graph for each such j. Hence, $a(G_i) \leqq n_i - 1$, $i = 1, 2, \cdots, k$.

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