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CONNECTOR THEORY

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Connector theory is a generalization of topology and uniformity. Each reflexive binary relation U of a space S induces a mapping from S to 2^S wherein $x \in S \rightarrow xU = \{y: (x, y) \in U\} \in 2^S$. This mapping is called a connector. A uniformity on S is a set of connectors which meets certain conditions. The results in this paper include a necessary-sufficient condition for a connector-set to induce a unique topology, generalizations of continuous mappings and uniformly continuous mappings and characterizations of the connector-sets which correspond to a specific type of topology, for instance, a compact topology, a pseudo-compact topology.

If $\mathfrak{A}$ is a connector-set on S, let $\mathfrak{A}^{\#}$ denote the connector-set, $\{U$: for each $x \in S$, there is $V(x) \in \mathfrak{A}$ such that $xU = xV(x)\}$. The following types of connector-sets are defined.

Cone: $U \leqq V$ and $U \in \mathfrak{A} \Rightarrow V \in \mathfrak{A}$.

Net: U, $V \in \mathfrak{A} \Rightarrow W \leqq U \cap V$ for some $W \in \mathfrak{A}$.

Filter: Cone and net.

Sharp: For each $U \in \mathfrak{A}^{\#}$, there is $V \in \mathfrak{A}$ such that $V \leqq U$.

Prenet: $\mathfrak{A}^{\#}$ is a net.

Topological: For each $U \in \mathfrak{A}$ and for each $x \in S$, there is $V \in \mathfrak{A}$ such that $xV^2 \subseteq xU$.

Topology: Topological sharp filter.

Pretopology: $\mathfrak{A}^{\#}$ is a topological net.

Uniform: For each $U \in \mathfrak{A}$, there is $V \in \mathfrak{A}$ such that $VV^{-1} \leqq U$.

Uniformity: Uniform filter.

Totally bounded: For each $U \in \mathfrak{A}$, there are $x_i \in S$, $i = 1, 2, 3, \cdots, m$ such that $S \subseteq \cup x_i U$.

Bounded: For each $U \in \mathfrak{A}$, there are $x_i \in S$, $i = 1, 2, 3, \cdots, m$ and a positive integer n such that $S \subseteq \cup x_i U^n$.

Absolutely bounded: For each $U \in \mathfrak{A}$ and for each $X \subseteq S$, there are $x_i \in X$ $(i = 1, 2, 3, \cdots, m)$, and a positive integer n such that $X \subseteq \cup x_i U^n$.

A prenet $\mathfrak{A}$ induces a topology $\mathscr{T}(\mathfrak{A})$ (in the usual sense) which will be called an *open*-topology. The interior theorem states that $\mathfrak{A}$ is a pretopology if and only if x is an interior point of xU for every $U \in \mathfrak{A}$. A topology corresponds uniquely to an open-topology and vice versa (The topology theorem). A topology $\mathfrak{T}$ induces a compact open-topology if and only if $\mathfrak{T}$ is totally bounded. The compact topology theorem states that compactness of $\mathscr{T}(\mathfrak{U})$ for a uniformity $\mathfrak{U}$ implies that $\mathfrak{U}$ is the strongest uniformity included in $\mathfrak{U}^{\#}$.

Suppose $\mathfrak{A}$ is a connector-set on S and $\mathfrak{D}$ is a connector set on R. Let $xM \in R$ denote the image of $x \in S$ by a mapping M. Each $U \in \mathfrak{D}$ induces the connector MUM^{-1} on S by $x \to \{z: (xM, zM) \in U\}$. M is $\mathfrak{A}$-continuous if every MUM^{-1}, $U \in \mathfrak{D}$ belongs to the sharp filter hull of $\mathfrak{A}$. M is uniformly $\mathfrak{A}$-continuous if they belong to the filter hull of $\mathfrak{A}$. The continuity theorem states that the definition is compatible with the continuity (in the usual sense) of M on the topological space $(S, \mathscr{T}(\mathfrak{A}))$ if $\mathfrak{A}$ is a prenet. The kernel theorem is: $\mathfrak{U}$ is the strongest uniformity included in a topology $\mathfrak{T}$ and $\mathfrak{D}$ is a uniform net, then $\mathfrak{T}$-continuity implies uniform $\mathfrak{U}$-continuity. The strongest uniformity included in a topology $\mathfrak{T}$ is bounded if and only if every $\mathfrak{T}$-continuous real-valued function is bounded (The pseudo-compact theorem).

1. Connector systems. Let S be a space on which we develop generalized structures of topology and uniformity. A mapping U from S to 2^S is called a *connector* on S if each x of S belongs to its image xU. The *inverse* U^{-1} of U is a mapping: $x \in S \to \{y: x \in yU\}$. Let U and V be connectors. We write $U \leqq V$ if $xU \subseteq xV$ for every x of S. The connector UV (the product of U and V) is defined by $x \in S \to \cup \{yV: y \in xU\}$. UV is denoted by U^2 if $U = V$. An intersection of connectors $\{U_\lambda: \lambda \in \Lambda\}$ is the connector defined by $x \in S \to \cap \{xU_\lambda: \lambda \in \Lambda\} \in 2^S$ and it is denoted by $\cap U_\lambda$. A non-empty set of connectors is called a *connector system*.

A connector system $\mathfrak{A}$ on S is called a *cone* if $V \in \mathfrak{A}$ whenever $U \leqq V$ and $U \in \mathfrak{A}$. Let $\mathfrak{A}^< = \{U: V \leqq U \text{ for some } V \in \mathfrak{A}\}$. $\mathfrak{A}^<$ is a cone, and $\mathfrak{A}^< \subseteq \mathfrak{B}$ if $\mathfrak{B}$ is a cone and $\mathfrak{A} \subseteq \mathfrak{B}$.

(1.1) $\mathfrak{A}$ is a cone if and only if $\mathfrak{A}^< = \mathfrak{A}$.

(1.2) $\mathfrak{A} \subseteq \mathfrak{A}^<$.

(1.3) $\mathfrak{A}^{\lessdot} = \mathfrak{A}^{<}$ $(\mathfrak{A}^{\lessdot} = (\mathfrak{A}^{<})^{<})$.
(1.4) $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A}^{<} \subseteq \mathfrak{B}^{<}$.
(1.5) $(\cup \mathfrak{A}_{\lambda})^{<} = \cup \mathfrak{A}_{\lambda}^{<}$.

A connector system $\mathfrak{A}$ is called a *net* if for U, $V \in \mathfrak{A}$, there is $W \in \mathfrak{A}$ such that $W \leqq U \cap V$. Let $\mathfrak{A}^{\times}$ denote the set of all finite intersections of connectors of $\mathfrak{A}$. Then $\mathfrak{A}^{\times}$ is a net.

(1.6) $\mathfrak{A} \subseteq \mathfrak{A}^{\times}$.
(1.7) $\mathfrak{A}^{\times\times} = \mathfrak{A}^{\times}$.
(1.8) $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A}^{\times} \subseteq \mathfrak{B}^{\times}$.
(1.9) $(\cup \mathfrak{A}_{\lambda})^{\times} = (\cup \mathfrak{A}_{\lambda}^{\times})^{\times}$.
(1.10) $\mathfrak{A}$ is a net if and only if $\mathfrak{A}^{\times} \subseteq \mathfrak{A}^{<}$.
(1.11) $\mathfrak{A}^{\times<} = \mathfrak{A}^{<\times}$.

Proof. It is obvious that $\mathfrak{A}^{\times<\times} \subseteq \mathfrak{A}^{\times<}$. Thus, by (1.6), (1.4) and (1.8), we obtain $\mathfrak{A}^{<\times} \subseteq (\mathfrak{A}^{\times})^{<\times} \subseteq \mathfrak{A}^{\times<}$. On the other hand, for each $U \in \mathfrak{A}^{\times<}$, there are $U_i \in \mathfrak{A}$, $i = 1, 2, 3, \cdots, n$ such that $\cap\{U_i : i = 1, 2, 3, \cdots n\} \leqq U$. Let V_i denote a connector: $x \in S \to (xU_i) \cup (xU)$. Then $U_i \leqq V_i$, $i = 1, 2, 3, \cdots n$ and $U = \cap\{V_i : i = 1, 2, 3, \cdots n\}$.

(1.12) $A^{\times<\times<} = \mathfrak{A}^{\times<}$.

Proof. Refer to (1.3), (1.7) and (1.11).

A connector system is called a *filter* if it is a cone and a net. Every finite intersection of connectors of a filter belongs to the filter. Therefore, a filter $\mathfrak{A}$ is a connector system which satisfies the following conditions.

(1) If $U \in \mathfrak{A}$ and $U \leqq V$ then $V \in \mathfrak{A}$.
(2) If $U, V \in \mathfrak{A}$ then $U \cap V \in \mathfrak{A}$.
(1.13) $\mathfrak{A}$ is a filter if and only if $\mathfrak{A} = \mathfrak{A}^{\times} = \mathfrak{A}^{<}$.
(1.14) $\mathfrak{A}$ is a filter if and only if $\mathfrak{A} = \mathfrak{A}^{\times<}$.
(1.15) $\mathfrak{A}^{<}$ is a filter if and only if $\mathfrak{A}$ is a net.

$\mathfrak{A}^{\times<}$ is a filter and $\mathfrak{B} \supseteq \mathfrak{A}^{\times<}$ if $\mathfrak{B}$ is a filter and $\mathfrak{B} \supseteq \mathfrak{A}$. Therefore, $\mathfrak{A}^{\times<}$ is called the *filter hull* of $\mathfrak{A}$.

(1.16) $\mathfrak{A}^{\times<} = \mathfrak{A}^{<}$ if and only if $\mathfrak{A}$ is a net.

A connector system $\mathfrak{A}$ is called *sharp* if for each connector system, $\{U(x) \in \mathfrak{A} : x \in S\}$, there is $V \in \mathfrak{A}$ such that $xV \subseteq xU(x)$ for every $x \in S$. Let $\mathfrak{A}^{\#} = \{U :$ for each $x \in S$, there is $V(x) \in \mathfrak{A}$ such that $xU = xV(x)\}$. The connector system $\mathfrak{A}^{\#}$ is sharp.

(1.17) $\mathfrak{A}$ is sharp if and only if $\mathfrak{A}^{\#} \subseteq \mathfrak{A}^{<}$.
(1.18) $\mathfrak{A}$ is sharp if and only if $\mathfrak{A}^{\#<} = \mathfrak{A}^{<}$.

Proof. (1.3) and (1.4) $\Rightarrow \mathfrak{A}^{\#} \subseteq \mathfrak{A}^{<}$ if and only if $\mathfrak{A}^{\#<} = \mathfrak{A}^{<}$.

(1.19) $\mathfrak{A} \subseteq \mathfrak{A}^{\#}$.
(1.20) $\mathfrak{A}^{\#\#} = \mathfrak{A}^{\#}$.
(1.21) $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A}^{\#} \subseteq \mathfrak{B}^{\#}$.
(1.22) $(\cup \mathfrak{A}_{\lambda})^{\#} = (\cup \mathfrak{A}_{\lambda}^{\#})^{\#}$.

A connector system is called a *prenet* if for U, $V \in \mathfrak{A}$ and for each $x \in S$, there is $W \in \mathfrak{A}$ such that $xW \subseteq xU \cap xV$. Every net is a prenet.
(1.23) $\mathfrak{A}^{\#}$ is a net if and only if $\mathfrak{A}$ is a prenet.
(1.24) Every sharp prenet is a net.

Proof. (1.10) and (1.23) $\Rightarrow \mathfrak{A}^{\times} \subseteq \mathfrak{A}^{\#\times} \subseteq \mathfrak{A}^{\#<}$ if $\mathfrak{A}$ is a prenet. (1.4) and (1.10) $\Rightarrow \mathfrak{A}^{\#<} \subseteq \mathfrak{A}^{\lessdot} = \mathfrak{A}^{<}$ if $\mathfrak{A}$ is sharp. Hence, $\mathfrak{A}$ is a net if $\mathfrak{A}$ is a sharp prenet.
(1.25) $\mathfrak{A}^{\#<} = \mathfrak{A}^{<\#}$.

Proof. Let $\{V(x): x \in S\}$ be a connector system of $\mathfrak{A}^{<}$. There is $W(x) \in \mathfrak{A}$ for each $V(x)$ such that $W(x) \leqq V(x)$. If W and U are defined respectively, by $W: x \in S \to xW(x)$ and $U: x \in S \to xV(x)$, then, $W \leqq U$ and $W \in \mathfrak{A}^{\#}$. Hence, U belongs to $\mathfrak{A}^{\#<}$ if $U \in \mathfrak{A}^{<\#}$. Conversely let U be a connector of $\mathfrak{A}^{\#<}$. The connector $U(x)$ is defined, for each $x \in S$, by $U(x)$: $y \in S \to yU(x) = xU$ if $y = x$ and $yU(x) = S$ otherwise. Then, $xU = xU(x)$ for every $x \in S$, and every $U(x)$, $(x \in S)$, belongs to $\mathfrak{A}^{<}$.
(1.26) $\mathfrak{A}^{\#\times} \subseteq \mathfrak{A}^{\times\#}$.

Proof. Each $U \in \mathfrak{A}^{\#\times}$ is a finite intersection of connectors $\{W_i: i = 1,2,3,\cdots,n\}$ of $\mathfrak{A}^{\#}$. There are $U_i \in \mathfrak{A}$ for each $x \in S$, such that $xW_i = xU_i$, $i = 1,2,\cdots,n$. $xU = \cap\{xW_i: i = 1,2,3,\cdots,n\} = \cap\{xU_i: i = 1,2,3,\cdots,n\} = x(\cap U_i)$, hence U belongs to $\mathfrak{A}^{\times\#}$.

(1.27) $\mathfrak{A}^{\#\times\#} = \mathfrak{A}^{\times\#}$.

Proof. Refer to (1.19), (1.8), (1.21), (1.26) and (1.20) in order.
(1.28) $\mathfrak{A}^{\times\#\times} = \mathfrak{A}^{\times\#}$.

Proof. Refer to (1.6), (1.26) and (1.7).
(1.29) $\mathfrak{A}^{\times\#\times\#} = \mathfrak{A}^{\#\times\#\times} = \mathfrak{A}^{\times\#}$.

Proof. Refer to (1.27) and (1.28).
(1.30) $\mathfrak{A}^{\times\#<} = \mathfrak{A}^{\times\#<\times\#<}$.

Proof. $\mathfrak{A}^{\times\#<\times\#<} = \mathfrak{A}^{\times\#\times<\#<}$ by (1.11), and $\mathfrak{A}^{\times\#\times<\#<} = \mathfrak{A}^{\times\#\times\#\lessdot}$ by

(1.25). (1.29) and (1.13) $\Rightarrow \mathfrak{A}^{\times\#\times\#<} = \mathfrak{A}^{\times\#<}$. Hence, $\mathfrak{A}^{\times\#<\times\#<} = \mathfrak{A}^{\times\#<}$.

(1.31) $\mathfrak{A}$ is a sharp filter if and only if $\mathfrak{A} = \mathfrak{A}^{\times\#<}$.

Proof. An implication of (1.13) and (1.18) is the following. $\mathfrak{A}$ is a sharp filter if and only if $\mathfrak{A} = \mathfrak{A}^{\times} = \mathfrak{A}^{<} = \mathfrak{A}^{\#}$. Therefore, $\mathfrak{A} = \mathfrak{A}^{\times\#<}$ if $\mathfrak{A}$ is a sharp filter. Conversely, $\mathfrak{A} = \mathfrak{A}^{\times\#<}$ implies $\mathfrak{A}^{\times} \subseteq \mathfrak{A}^{\times\#<} = \mathfrak{A} \subseteq \mathfrak{A}^{\times}$, thus $\mathfrak{A} = \mathfrak{A}^{\times}$. $\mathfrak{A} = \mathfrak{A}^{\#}$ and $\mathfrak{A} = \mathfrak{A}^{<}$ can be proved similarly.

$\mathfrak{A}^{\times\#<}$ is a sharp filter and $\mathfrak{A}^{\times\#<} \subseteq \mathfrak{B}$ if $\mathfrak{B}$ is a sharp filter and $\mathfrak{A} \subseteq \mathfrak{B}$. Therefore, $\mathfrak{A}^{\times\#<}$ is called the *sharp filter hull* of $\mathfrak{A}$.

(1.32) $\mathfrak{A}$ is a prenet if and only if $\mathfrak{A}^{\times\#<} = \mathfrak{A}^{\#<}$.

Proof. $\mathfrak{A}$ is a prenet if and only if $\mathfrak{A}^{\#}$ is a net. $\mathfrak{A}^{\#}$ is a net if and only if $\mathfrak{A}^{\#\times<} = \mathfrak{A}^{\#<}$. The last relation implies $\mathfrak{A}^{\#\times<\#} = \mathfrak{A}^{\#<\#} = \mathfrak{A}^{\#\#<} = \mathfrak{A}^{\#<}$. On the other hand, $\mathfrak{A}^{\#\times<\#} = \mathfrak{A}^{\#<}$ implies $\mathfrak{A}^{\#<} \subseteq \mathfrak{A}^{\#\times<} \subseteq \mathfrak{A}^{\#\times<\#} = \mathfrak{A}^{\#<}$, thus $\mathfrak{A}^{\#\times<} = \mathfrak{A}^{\#<}$. $\mathfrak{A}^{\#\times<\#} = \mathfrak{A}^{\times\#<}$ by (1.25) and (1.27).

(1.33) $\mathfrak{A}$ is a sharp prenet if and only if $\mathfrak{A}^{\times\#<} = \mathfrak{A}^{<}$.

Proof. If $\mathfrak{A}$ is a sharp prenet then, by (1.32) and (1.18), $\mathfrak{A}^{\times\#<} = \mathfrak{A}^{\#<} = \mathfrak{A}^{<}$. $\mathfrak{A}^{<} \subseteq \mathfrak{A}^{\#<} \subseteq \mathfrak{A}^{\times\#<}$, thus, $\mathfrak{A}^{\times\#<} = \mathfrak{A}^{<}$ implies $\mathfrak{A}^{<} = \mathfrak{A}^{\#<}$.

(1.34) $\mathfrak{A}^{\times\#<} = \mathfrak{A}^{\#}$ if $\mathfrak{A}$ is a filter.

Proof. Refer to (1.14), (1.25) and (1.31).

2. Base of filters. A connector system $\mathfrak{A}$ is called *stronger* than a connector system $\mathfrak{B}$ (or $\mathfrak{B}$ is *weaker* than $\mathfrak{A}$) if $\mathfrak{B} \subseteq \mathfrak{A}^{\times<}$.

(2.1) $\mathfrak{A}$ is a net. $\mathfrak{A}$ is stronger than $\mathfrak{B}$ if and only if $\mathfrak{B} \subseteq \mathfrak{A}^{<}$.

(2.2) $\mathfrak{A}$ is stronger and also weaker than $\mathfrak{B}$ if and only if $\mathfrak{A}^{\times<} = \mathfrak{B}^{\times<}$.

A connector system $\mathfrak{A}$ is called *finer* than a connector system $\mathfrak{B}$ if $\mathfrak{B} \subseteq \mathfrak{A}^{\times\#<}$. (1.28), (1.19), (1.32) and (1.33) imply (2.3), (2.4), (2.5) and (2.6) respectively.

(2.3) $\mathfrak{A}$ is finer than $\mathfrak{B}$ if and only if $\mathfrak{A}^{\times\#}$ is stronger than $\mathfrak{B}$.

(2.4) $\mathfrak{A}$ is finer than $\mathfrak{B}$ if $\mathfrak{A}$ is stronger than $\mathfrak{B}$.

(2.5) $\mathfrak{A}$ is a prenet. $\mathfrak{A}$ is finer than $\mathfrak{B}$ if and only if $\mathfrak{B} \subseteq \mathfrak{A}^{\#<}$.

(2.6) If $\mathfrak{A}$ is a sharp prenet and finer than $\mathfrak{B}$ then $\mathfrak{A}$ is stronger than $\mathfrak{B}$.

Let $\mathfrak{F}$ be a filter on S. A connector system $\mathfrak{A} \subseteq \mathfrak{F}$ is called a *basis* of $\mathfrak{F}$ if for each $U \in \mathfrak{F}$, there is $V \in \mathfrak{A}$ such that $V \leqq U$.

(2.7) $\mathfrak{A}$ is a basis of $\mathfrak{F}$ if and only if $\mathfrak{F} = \mathfrak{A}^{<}$.

(2.8) The following two statements are equivalent.

(1) $\mathfrak{A}$ is a basis of $\mathfrak{F}$.

(2) $\mathfrak{A}$ is a net, and $\mathfrak{A}$ is stronger and weaker than $\mathfrak{F}$.

Proof. The statement (2) is, by (1.16) and (2.27), equivalent to $\mathfrak{A}^{<} = \mathfrak{A}^{\times <} = \mathfrak{F}^{\times <}$. These relations follow if $\mathfrak{F} = \mathfrak{A}^{<}$ and $\mathfrak{F} = \mathfrak{F}^{\times <}$, hence, the statement (1) implies the statement (2). The converse is obvious.

(2.9) $\mathfrak{A}$ is a basis of $\mathfrak{A}^{<}$ if $\mathfrak{A}$ is a net.

Proof. Refer to (1.15) and (2.7).

(2.10) Every basis of a sharp filter is sharp.

Proof. If $\mathfrak{A}$ is a basis of a sharp filter $\mathfrak{F}$ then, since (1.31) and (2.7) imply $\mathfrak{F} = \mathfrak{F}^{\#}$ and $\mathfrak{A}^{<} = \mathfrak{F}$ respectively, $\mathfrak{A}^{\#<} = \mathfrak{A}^{<\#} = \mathfrak{F}^{\#} = \mathfrak{F} = \mathfrak{A}^{<}$. Hence, by (1.18), $\mathfrak{A}$ is sharp.

A connector system $\mathfrak{A}$ is called a *prebasis* of a filter $\mathfrak{F}$ if $\mathfrak{A}^{\#}$ is a basis of $\mathfrak{F}$.

(2.11) $\mathfrak{A}$ is a prebasis of $\mathfrak{F}$ if and only if $\mathfrak{A}^{\#<} = \mathfrak{F}$.

(2.12) A filter $\mathfrak{F}$ is sharp if there is a prebasis.

(2.13) Every basis of $\mathfrak{F}$ is a prebasis if $\mathfrak{F}$ is a sharp filter.

(2.14) Every prebasis is a prenet.

Proof. Refer to (1.23) and (2.8).

(2.15) $\mathfrak{B}$ is a prebasis of $\mathfrak{A}^{\times\#<}$ if and only if $\mathfrak{B}$ is a prenet and $\mathfrak{A}^{\times\#<} = \mathfrak{B}^{\times\#<}$.

Proof. An implication of (2.11) is the following. $\mathfrak{B}$ is a prebasis of $\mathfrak{A}^{\times\#<}$ if and only if $\mathfrak{A}^{\times\#<} = \mathfrak{B}^{\#<}$. (1.32) states that $\mathfrak{B}$ is a prenet if and only if $\mathfrak{B}^{\times\#<} = \mathfrak{B}^{\#<}$. $\mathfrak{A}^{\times\#<} = \mathfrak{B}^{\#<} \Rightarrow \mathfrak{A}^{\times\#<} = \mathfrak{B}^{\#<} \subseteq \mathfrak{B}^{\times\#<}$ and $\mathfrak{B}^{\times\#<} \subseteq \mathfrak{B}^{\#<\times\#<} = \mathfrak{A}^{\times\#<\times\#<} = \mathfrak{A}^{\times\#<}$. Therefore, $\mathfrak{A}^{\times\#<} = \mathfrak{B}^{\#<}$ if and only if $\mathfrak{B}^{\times\#<} = \mathfrak{B}^{\#<}$ and $\mathfrak{A}^{\times\#<} = \mathfrak{B}^{\times\#<}$.

(2.16) $\mathfrak{F}$ is a sharp filter. A connector system $\mathfrak{A} \subseteq \mathfrak{F}$ is a prebasis of $\mathfrak{F}$ if and only if $\mathfrak{A}$ is a prenet and finer than $\mathfrak{F}$.

The following two statements are equivalent if $\mathfrak{F}$ and $\mathfrak{G}$ are filters.

(1) $\mathfrak{F}$ is stronger than $\mathfrak{G}$.

(2) $\mathfrak{F} \supseteq \mathfrak{G}$.

An intersection of filters is a filter. Therefore, for each set of filters $\{\mathfrak{F}_{\lambda} : \lambda \in \Lambda\}$, there exists the strongest filter among the filters weaker than all $\mathfrak{F}_{\lambda}$, $\lambda \in \Lambda$. Likewise, there exists the weakest filter among the filters stronger than all $\mathfrak{F}_{\lambda}$, $\lambda \in \Lambda$. The two filters are respectively denoted by $\wedge \mathfrak{F}_{\lambda}$ and $\vee \mathfrak{F}_{\lambda}$.

(2.19) If $\mathfrak{A}_{\lambda}$ is a basis of $\mathfrak{F}_{\lambda}$ for each $\lambda \in \Lambda$ then $(\cup \mathfrak{A}_{\lambda})^{\times}$ is a basis of $\vee \mathfrak{F}_{\lambda}$.

Proof. The following relations are implied respectively by (1.3), (1.11), (1.5), (2.7) and (2.17). $(\cup \mathfrak{A}_{\lambda})^{\times <} = (\cup \mathfrak{A}_{\lambda})^{\times \lessdot} = (\cup \mathfrak{A}_{\lambda})^{<\times<} = (\cup \mathfrak{A}_{\lambda}^{<})^{\times <} = (\cup \mathfrak{F}_{\lambda})^{\times <} = \vee \mathfrak{F}_{\lambda}$. Hence, by (2.7), $(\cup \mathfrak{A}_{\lambda})^{\times}$ is a basis of $\vee \mathfrak{F}_{\lambda}$.

(2.20) $(\vee \mathfrak{F}_{\lambda})^{\#} = (\cup \mathfrak{F}_{\lambda})^{\times\#<}$.

(2.21) if $\mathfrak{A}_\lambda$ is a prebasis of $\mathfrak{F}_\lambda$ for each $\lambda \in \Lambda$, then $(\cup \mathfrak{A}_\lambda)^\times$ is a prebasis of $(\vee \mathfrak{F}_\lambda)^\#$.

Proof. $\mathfrak{F}_\lambda = \mathfrak{A}_\lambda^{\#<}$, $\lambda \in \Lambda$ and $(\vee \mathfrak{F}_\lambda)^\# = (\cup \mathfrak{F}_\lambda)^{\times\#<}$ by (2.11) and (2.20). (1.22) and (1.27) $\Rightarrow (\cup \mathfrak{A}_\lambda^\#)^{\times\#} = (\cup \mathfrak{A}_\lambda)^{\times\#}$. Therefore,

$$(\vee \mathfrak{F}_\lambda)^\# = (\cup \mathfrak{A}_\lambda^{\#<})^{\times\#<} = (\cup \mathfrak{A}_\lambda^\#)^{<\times\#<} = (\cup \mathfrak{A}_\lambda^\#)^{\times\#<} = (\cup \mathfrak{A}_\lambda^\#)^{\times\#<} = (\cup \mathfrak{A}_\lambda)^{\times\#<}.$$

(2.22) $(\vee \mathfrak{F}_\lambda)^\# = (\vee \mathfrak{F}_\lambda^\#)^\#$.

(2.22) is a generalization of Theorem 5 in §28 of **[1]**.

3. Topologies. A connector system $\mathfrak{A}$ is called *topological* if for each $U \in \mathfrak{A}$ and for each $x \in S$, there is $V \in \mathfrak{A}$ such that $xV^2 \subseteq xU$.

(3.1) If $\mathfrak{A}$ is topological then $\mathfrak{A}^<$, $\mathfrak{A}^\times$ and $\mathfrak{A}^\#$ are topological.

(3.2) If $\mathfrak{A}_\lambda$, $\lambda \in \Lambda$ are all topological then $\cup \mathfrak{A}_\lambda$ is topological.

A topological sharp filter is called a *topology*; i.e., a topology $\mathfrak{T}$ on S is a connector system which satisfies the following conditions:

(1) If $U, V \in \mathfrak{T}$ then $U \cap V \in \mathfrak{T}$.

(2) If $U \leqq V$ and $U \in \mathfrak{T}$ then $V \in \mathfrak{T}$.

(3) For each system $\{U(x) \in \mathfrak{T} \colon x \in S\}$, there exists $U \in \mathfrak{T}$ such that $xU = xU(x)$ for every $x \in S$.

(4) For each $U \in \mathfrak{T}$ and for each $x \in S$, there exists $V \in \mathfrak{T}$ such that $xV^2 \subseteq xU$.

The use of the word "topology" is not conventional. Compatibility of the terminology is cleared in the following section. Topologies are filters, hence, a topology is stronger than another if and only if the former includes the latter. A *topology hull* of a connector system $\mathfrak{A}$ is the weakest topology among the topologies stronger than $\mathfrak{A}$. A topology hull is unique if it exists.

(3.3) $\mathfrak{A}^{\times\#<}$ is the topology hull of $\mathfrak{A}$ if $\mathfrak{A}$ is topological.

Proof. Refer to (1.31) and (3.1).

(3.4) If one of $\mathfrak{A}$, $\mathfrak{A}^\times$, $\mathfrak{A}^<$ and $\mathfrak{A}^\#$ has a topology hull, then all of them have the same topology hull.

Proof. If $\mathfrak{T}$ is a topology and $\mathfrak{T}$ includes $\mathfrak{A}$ then $\mathfrak{T}$ includes all $\mathfrak{A}^<$, $\mathfrak{A}^\times$ and $\mathfrak{A}^\#$ because $\mathfrak{T} = \mathfrak{T}^{\times\#<}$. If $\mathfrak{T}$ includes one of $\mathfrak{A}^<$, $\mathfrak{A}^\times$ and $\mathfrak{A}^\#$ then $\mathfrak{T}$ includes $\mathfrak{A}$, thus $\mathfrak{T}$ includes the other two.

(3.5) $\mathfrak{A}^\#$ is the topology hull of $\mathfrak{A}$ if $\mathfrak{A}$ is a topological filter.

Proof. Refer to (1.34) and (3.3).

(3.6) $\mathfrak{A}^<$ is the topology hull of $\mathfrak{A}$ if and only if $\mathfrak{A}$ is topological sharp prenet.

Proof. Refer to (3.3) and (1.33).

(3.7) If $\mathfrak{A}$ is a topological prenet, then $\mathfrak{A}^{\#<}$ is the topology hull of $\mathfrak{A}$.

Proof. Refer to (3.3) and (1.32).

(3.8) Every basis of a topology $\mathfrak{T}$ is a topological sharp prenet, and $\mathfrak{T}$ is its topology hull.

Proof. $\mathfrak{T} = \mathfrak{A}^{<} \Rightarrow \mathfrak{A}^{\times\#<} = \mathfrak{A}^{<\times\#<} = \mathfrak{T}^{\times\#<} = \mathfrak{T} = \mathfrak{A}^{<}$. A basis of a topology is topological, hence, $\mathfrak{T}$ is the topology hull of $\mathfrak{A}$.

(3.9) A topology $\mathfrak{T}$ is the topology hull of every prebasis of $\mathfrak{T}$.

(3.10) If $\mathfrak{T}_\lambda$ is the topology hull of $\mathfrak{A}_\lambda$ for each $\lambda \in \Lambda$ then $(\vee \mathfrak{T}_\lambda)^{\#}$ is the topology hull of $\cup \mathfrak{A}_\lambda$.

Proof. $(\vee \mathfrak{T}_\lambda)^{\#}$ is the topology hull of $\cup \mathfrak{T}_\lambda$ by (3.2), (3.3) and (2.20). If $\mathfrak{T}$ is a topology which includes all $\mathfrak{A}_\lambda$, $\lambda \in \Lambda$ then $\mathfrak{T}$ includes all $\mathfrak{T}_\lambda$, $\lambda \in \Lambda$. Hence the both unions have the same topology hull.

A connector system $\mathfrak{A}$ is called a *pretopology* if $\mathfrak{A}^{\#}$ is a topological net. Every topology is, by (3.1), a pretopology.

(3.11) Every pretopology is a prenet.

(3.12) Every topological prenet is a pretopology.

(3.13) If $\mathfrak{A}$ is a pretopology then $\mathfrak{A}^{\#<}$ is the topology hull of $\mathfrak{A}$ and $\mathfrak{A}$ is a prebasis of $\mathfrak{A}^{\#<}$.

Proof. (1.27) and (3.1) $\Rightarrow \mathfrak{A}^{\times\#<}$ is topological if $\mathfrak{A}$ is a pretopology. Then, $\mathfrak{A}^{\times\#<}$ is the topology hull of $\mathfrak{A}$. $\mathfrak{A}^{\times\#<} = \mathfrak{A}^{\#<}$ by (3.11) and (1.32), hence, $\mathfrak{A}^{\#<}$ is the topology hull of $\mathfrak{A}$.

(3.14) If every $\mathfrak{A}_\lambda$, $\lambda \in \Lambda$ is a pretopology then $(\cup \mathfrak{A}_\lambda)^{\times}$ is a pretopology.

Proof. (1.27) and (1.22) $\Rightarrow (\cup \mathfrak{A}_\lambda)^{\times\#} = (\cup \mathfrak{A}_\lambda^{\#})^{\#\times\#}$, and (3.1) and (3.2) $\Rightarrow (\cup \mathfrak{A}_\lambda^{\#})^{\#\times\#}$ is topological if $\mathfrak{A}_\lambda^{\#}$, $\lambda \in \Lambda$ are topological. $(\cup \mathfrak{A}_\lambda)^{\times\#}$ is a net because $(\cup \mathfrak{A}_\lambda)^{\times\#} = (\cup \mathfrak{A}_\lambda)^{\times\#\times}$. Hence, $(\cup \mathfrak{A}_\lambda)^{\times}$ is a pretopology.

4. Open sets. The word 'topology' has been already used for a connector system. To avoid confusion, a topology in usual sense is called an *open*-topology. An open-topology $\mathscr{T}$ is a family of subsets of S and

(1) $X_\lambda \in \mathscr{T}$, $\lambda \in \Lambda \Rightarrow \cup X_\lambda \in \mathscr{T}$,

(2) $X, Y \in \mathscr{T} \Rightarrow X \cap Y \in \mathscr{T}$,

(3) $\varnothing, S \in \mathscr{T}$.

Each prenet $\mathfrak{A}$ on S induces an open-topology in the following way. A set X of S is an open set if for each $x \in X$, there is $U \in \mathfrak{A}$ such that $xU \subseteq X$. This open-topology is denoted by $\mathscr{T}(\mathfrak{A})$.

(4.1) $\mathfrak{A}$ and $\mathfrak{B}$ are prenets. Then $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$ if $\mathfrak{A}$ is finer than $\mathfrak{B}$, and $\mathcal{T}(\mathfrak{A}) = \mathcal{T}(\mathfrak{B})$ if $\mathfrak{A}^{\#<} = \mathfrak{B}^{\#<}$.

Proof. (2.5) states that $\mathfrak{B} \subseteq \mathfrak{A}^{\#<}$ if a prenet $\mathfrak{A}$ is finer than $\mathfrak{B}$. Thus, for each $U \in \mathfrak{B}$ and for each $x \in S$, *there is* $V \in \mathfrak{A}$ *such that* $xV \subseteq xU$. Hence, $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$. If $\mathfrak{A}^{\#<} = \mathfrak{B}^{\#<}$ then $\mathfrak{A} \subseteq \mathfrak{B}^{\#<}$ and $\mathfrak{B} \subseteq \mathfrak{A}^{\#<}$, thus, $\mathcal{T}(\mathfrak{A}) = \mathcal{T}(\mathfrak{B})$.

(4.2) $\mathcal{T}(\mathfrak{A}) = \mathcal{T}(\mathfrak{A}^{<}) = \mathcal{T}(\mathfrak{A}^{\times}) = \mathcal{T}(\mathfrak{A}^{\#})$.

A connector U is called $\mathfrak{A}$-*open* if $xU \in \mathcal{T}(\mathfrak{A})$ for every $x \in S$.

(4.3) If a connector is $\mathfrak{A}$-open then the connector belongs to $\mathfrak{A}^{\#<}$.

$\mathfrak{A}$-Int X denotes the interior of a set $X \subseteq S$, relative to the open-topology $\mathcal{T}(\mathfrak{A})$, i.e., A-Int $X = \cup\{Y \in \mathcal{T}(\mathfrak{A}) \colon Y \subseteq X\}$.

(4.4) $\mathfrak{A}$-Int $X = \mathfrak{B}$-Int X for every $X \subseteq S$ if and only if $\mathcal{T}(A) = \mathcal{T}(\mathfrak{B})$.

INTERIOR THEOREM. The following statements are equivalent.

(1) $\mathfrak{A}$ is a pretopology.

(2) $\mathfrak{A}$-Int $X = \{x; xU \subseteq X$ for some $U \in \mathfrak{A}\}$ for every subset $X \subseteq S$.

(3) If $U \in \mathfrak{A}$ then x belongs to the $\mathfrak{A}$-interior of xU for every $x \in S$.

Proof. (1) $\Rightarrow$ (2). Let $X \subseteq S$ and let $Y = \{x \in S\colon xV \subseteq X$ for some $V \in \mathfrak{A}^{\#}\}$. $xV \subseteq X$ for some $V \in \mathfrak{A}^{\#}$ if and only if $xU \subseteq X$ for some $U \in \mathfrak{A}$. Therefore, $Y = \{x \in S\colon xU \subseteq X$ for some $U \in \mathfrak{A}\}$. We show Y is the $\mathfrak{A}^{\#}$-interior of X. If $xV \subseteq X$ for some $V \in \mathfrak{A}^{\#}$, then, since $\mathfrak{A}^{\#}$ is topological, there is $W \in \mathfrak{A}^{\#}$ such that $yW \subseteq xV \subseteq X$ for every $y \in xW$. Therefore $xW \subseteq Y$ for some $W \in \mathfrak{A}^{\#}$ if $x \in Y$. This implies $Y \in \mathcal{T}(\mathfrak{A}^{\#})$. If $Z \subseteq X$ and $Z \in \mathcal{T}(\mathfrak{A}^{\#})$ then $Z \subseteq Y$, hence, Y is the $\mathfrak{A}^{\#}$-interior of X. $\mathfrak{A}$-Int $X = \mathfrak{A}^{\#}$-Int X by (4.2) and (4.4), thus $\mathfrak{A}$-Int $X = \{x \in S\colon xU \subseteq X$ for some $U \in \mathfrak{A}\}$. (2) obviously implies (3). (3) $\Rightarrow$ (1): If $\mathfrak{A}$ is a prenet then $\mathfrak{A}^{\#}$ is a net. Therefore, it is sufficient to show that $\mathfrak{A}^{\#}$ is topological. If $x \in S$ and $U \in \mathfrak{A}^{\#}$ then, since $x \in \mathfrak{A}$-Int (xU), there exists $Y \in \mathcal{T}(\mathfrak{A})$ such that $x \in Y \subseteq xU$. For each $y \in Y$, there is $U(y) \in \mathfrak{A}$ such that $yU(y) \subseteq Y$. Define V by $yV = yU(y)$ if $y \in Y$ and $yV = yU$ otherwise. Then $V \in \mathfrak{A}^{\#}$ and $xV^2 \subseteq xU$. Hence $\mathfrak{A}^{\#}$ is topological.

(4.5) If $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$ and $\mathfrak{B}$ is a pretopology then $\mathfrak{A}$ is finer than $\mathfrak{B}$.

Proof. If $U \in \mathfrak{B}$ then, by the interior theorem, the connector $V\colon x \to \mathfrak{B}$-Int$(xU)$ is a $\mathfrak{B}$-open connector. V is also $\mathfrak{A}$-open because $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$. $V \subseteq U$ and $V \in \mathfrak{A}^{\#<}$, hence $U \in \mathfrak{A}^{\#<}$.

BASIS THEOREM. $\mathfrak{T}$ is a topology. Then, the set of all $\mathfrak{T}$-open connectors is a basis of $\mathfrak{T}$.

Proof. A mapping $V: x \in S \to \mathfrak{T}\text{-Int}(xU)$ is a connector if $U \in \mathfrak{T}$. V is $\mathfrak{T}$-open and $V \leqq U$. (4.3) implies $V \in \mathfrak{T}$. Hence, the set of all $\mathfrak{T}$-open connectors is a basis of $\mathfrak{T}$.

COMPARISON THEOREM. *$\mathfrak{A}$ and $\mathfrak{B}$ are topologies. $\mathfrak{B} \subseteq \mathfrak{A}$ if and only if $\mathcal{T}(\mathfrak{B}) \subseteq \mathcal{T}(\mathfrak{A})$.*

Proof. $\mathfrak{A}$ is finer than $\mathfrak{B}$ if and only if $\mathfrak{B} \subseteq \mathfrak{A}$. Hence, (4.1) implies the one direction and (4.5) implies the other.

TOPOLOGY THEOREM. $\mathcal{T}$ is an open topology on S. There is a unique topology $\mathfrak{T}$ such that $\mathcal{T} = \mathcal{T}(\mathfrak{T})$.

Proof. Let $\mathfrak{A} = \{U; xU \in \mathcal{T}$ for every $x \in S\}$. U is a connector and $x \in S$. Define V by $yV = xU$ if $y \in xU$ and $yV = S$ otherwise. Then, $xV^2 = xU$ and V belongs to $\mathfrak{A}$ if U does. Therefore, $\mathfrak{A}$ is topological. $\mathfrak{A}$ is a sharp net because $\mathfrak{A} = \mathfrak{A}^\times = \mathfrak{A}^\#$. Let $\mathfrak{T} = \mathfrak{A}^<$. Then $\mathcal{T}(\mathfrak{T}) = \mathcal{T}(\mathfrak{A})$ and $\mathfrak{T}$ is, by (3.6), the topology hull of $\mathfrak{A}$. $\mathcal{T} = \mathcal{T}(\mathfrak{A})$ is clear by the definition of $\mathfrak{A}$, hence, $\mathcal{T} = \mathcal{T}(\mathfrak{T})$. The comparison theorem implies the uniqueness.

5. Uniformities. A connector system $\mathfrak{A}$ is called *uniform* if for each $U \in \mathfrak{A}$, there is $V \in \mathfrak{A}$ such that $VV^{-1} \leqq U$. If $\mathfrak{A}$ is uniform, then for each $U \in \mathfrak{A}$, there are $V, W \in \mathfrak{A}$ such that $WW^{-1} \leqq V$ and $VV^{-1} \leqq U$. $W^{-1} \leqq V$ implies $W \leqq V^{-1}$, thus $W^2 \leqq VV^{-1} \leqq U$.

(5.1) Every uniform system is topological.

(5.2) If $\mathfrak{A}$ is uniform, then $\mathfrak{A}^<$ and $\mathfrak{A}^\times$ are uniform.

(5.3) If $\mathfrak{A}_\lambda$, $\lambda \in \Lambda$ are uniform then $\cup \mathfrak{A}_\lambda$ is uniform.

A uniformity is a filter which is uniform. ([1], [2]). If $\mathfrak{U}$ is the weakest among the uniformities stronger than $\mathfrak{A}$, then $\mathfrak{U}$ is called the *uniformity hull* of $\mathfrak{A}$.

(5.4) The filter hull $\mathfrak{A}^{\times<}$ is the uniformity hull of $\mathfrak{A}$ if $\mathfrak{A}$ is uniform.

(5.5) $\mathfrak{U}$ is a uniformity. Then, a basis $\mathfrak{B}$ of $\mathfrak{U}$ is a uniform net and $\mathfrak{U}$ is the uniformity hull of $\mathfrak{B}$.

(5.6) If $\mathfrak{A}$ is a uniform net then $\mathfrak{A}^<$ is the uniformity hull and $\mathfrak{A}$ is a basis of $\mathfrak{A}^<$.

Proof. Refer to (1.16), (5.4) and (2.7).

(5.7) If $\mathfrak{U}_\lambda$ is the uniformity hull of $\mathfrak{A}_\lambda$, $\lambda \in \Lambda$, then $\vee \mathfrak{U}_\lambda$ is the uniformity-hull of $\cup \mathfrak{A}_\lambda$.

Proof. (5.3) and (5.4) $\Rightarrow$ $\vee \mathfrak{U}_\lambda$ is a uniformity. If $\mathfrak{U}$ is a uniformity and $\mathfrak{U}$ includes all $\mathfrak{A}_\lambda$, $\lambda \in \Lambda$ then $\mathfrak{U}$ includes all $\mathfrak{U}_\lambda$, $\lambda \in \Lambda$ and $\mathfrak{U} = \mathfrak{U}^{\times <}$. Hence, $\mathfrak{U}$ includes $(\cup \mathfrak{U}_\lambda)^{\times <} = \vee \mathfrak{U}_\lambda$.

$\mathfrak{A}$ is a connector system. If there is a uniformity which is the strongest among the uniformities weaker than $\mathfrak{A}$ then the uniformity is called the *uniformity kernel* of $\mathfrak{A}$. A uniformity kernel is unique if it exists.

(5.8) Every filter has a uniformity kernel.

Proof. Every filter includes the uniformity which contains the only connector, $x \in S \to S$ (for every $x \in S$). Let $\{\mathfrak{U}: \mathfrak{U} \subseteq \mathfrak{F}\}$ denote the collection of uniformities weaker than a filter $\mathfrak{F}$. (5.7) $\Rightarrow$ $\vee\{\mathfrak{U}: \mathfrak{U} \subseteq \mathfrak{F}\}$ is a uniformity. (2.17) states $\vee\{\mathfrak{U}: \mathfrak{U} \subseteq \mathfrak{F}\} = (\cup\{\mathfrak{U}: \mathfrak{U} \subseteq \mathfrak{F}\})^{\times <} \subseteq \mathfrak{F}^{\times <} = \mathfrak{F}$. Therefore, the uniformity is included in $\mathfrak{F}$ and the strongest among $\{\mathfrak{U}: \mathfrak{U} \subseteq \mathfrak{F}\}$.

The topology hull of a uniformity $\mathfrak{U}$ is $\mathfrak{U}^{\#}$ because $\mathfrak{U}$ is a topological filter. $\mathcal{T}(\mathfrak{U}^{\#})$ is a uniform topology (called the induced topology in **[1]** and **[2]**). Therefore, a topology $\mathfrak{T}$ corresponds to a uniform topology if and only if there is a uniformity $\mathfrak{U}$ such that $\mathfrak{T} = \mathfrak{U}^{\#}$.

(5.9) $\mathfrak{T}$ is a topology. There is a uniformity $\mathfrak{U}$ such that $\mathfrak{T} = \mathfrak{U}^{\#}$ if and only if $\mathfrak{T}$ is the topology hull of its uniformity kernel.

Proof. Since a topology is a filter, a topology $\mathfrak{T}$ has the uniform kernel $\mathfrak{V}$. If $\mathfrak{T} = \mathfrak{U}^{\#}$ for some uniformity $\mathfrak{U}$ then $\mathfrak{U} \subseteq \mathfrak{V} \subseteq \mathfrak{T}$. Hence $\mathfrak{T} = \mathfrak{V}^{\#}$.

(5.10) $\mathfrak{T}_\lambda, \lambda \in \Lambda$ are topologies and $\mathfrak{U}_\lambda, \lambda \in \Lambda$ are uniformities. If $\mathfrak{T}_\lambda = \mathfrak{U}_\lambda^{\#}$ for each $\lambda \in \Lambda$, then there is a uniformity $\mathfrak{U}$ such that $(\vee \mathfrak{T}_\lambda)^{\#} = \mathfrak{U}^{\#}$.

Proof. $(\vee \mathfrak{U}_\lambda)^{\#} = (\vee \mathfrak{U}_\lambda^{\#})^{\#} = (\vee \mathfrak{T}_\lambda)^{\#}$ by (2.22). $\vee \mathfrak{U}_\lambda$ is a uniformity by (5.7).

6. Mappings. Let R be a space and let U be a connector on R. M is a mapping from a space S to R. xM and $(xM)U$ denote the image of x by M and the image of $xM \in R$ by U respectively. Define a connector on S by corresponding x of S to the set $\{y \in S: yM \in (xM)U\}$. This connector is denoted by MUM^{-1}. $x(MUM^{-1})$ is the image of x by MUM^{-1}. The following formulas are found in **[2]**.

(6.1) $M(U \cap V)M^{-1} = MUM^{-1} \cap MVM^{-1}$.

(6.2) $U \leqq V$ implies $MUM^{-1} \leqq MVM^{-1}$.

(6.3) $(MUM^{-1})^{-1} = M(U^{-1})M^{-1}$.

(6.4) $(MUM^{-1})\,(MVM^{-1}) \leqq MUVM^{-1}$.

Each connector system $\mathfrak{D}$ on R can be used to construct a connector system on S by a transfer of each $U \in \mathfrak{D}$ to the connector MUM^{-1} on S. $M\mathfrak{D}M^{-1}$ denotes the connector system $\{MUM^{-1}: U \in \mathfrak{D}\}$. The following propositions are proved by referring to (6.1), (6.2), (6.3) and (6.4).

(6.5) $M\mathfrak{D}^{<}M^{-1} \subseteq (M\mathfrak{D}M^{-1})^{<}$ and $(M\mathfrak{D}^{<}M^{-1})^{<} = (M\mathfrak{D}M^{-1})^{<}$.

(6.6) $M\mathfrak{D}^{\#}M^{-1} \subseteq (M\mathfrak{D}M^{-1})^{\#}$ and $(M\mathfrak{D}^{\#}M^{-1})^{\#} = (M\mathfrak{D}M^{-1})^{\#}$.

(6.7) $M\mathfrak{D}^{\times}M^{-1} = (M\mathfrak{D}M^{-1})^{\times}$.

(6.8) If $\mathfrak{D}$ is a net, then $M\mathfrak{D}M^{-1}$ is a net.

(6.9) If $\mathfrak{D}$ is a prenet, then $M\mathfrak{D}M^{-1}$ is a prenet.

Proof. Refer to (6.6), (1.23) and (6.8).

(6.10) If $\mathfrak{D}$ is topological then $M\mathfrak{D}M^{-1}$ is topological.

(6.11) If $\mathfrak{D}$ is uniform then $M\mathfrak{D}M^{-1}$ is uniform.

(6.12) If $\mathfrak{D}$ is a pretopology then $M\mathfrak{D}M^{-1}$ is a pretopology.

Proof. $(M\mathfrak{D}^{\#}M^{-1})^{\#}$ is a topological net by (6.8), (6.10), (1.23) and (3.1), if $\mathfrak{D}$ is a pretopology. Hence, by (6.6), $(M\mathfrak{D}M^{-1})^{\#}$ is a topological net.

(6.13) $\mathfrak{D}$ and $\mathfrak{E}$ are connector systems on R. $M\mathfrak{D}M^{-1}$ is stronger than $M\mathfrak{E}M^{-1}$ if $\mathfrak{D}$ is stronger than $\mathfrak{E}$.

(6.14) $M\mathfrak{D}M^{-1}$ is finer than $M\mathfrak{E}M^{-1}$ if $\mathfrak{D}$ is finer than $\mathfrak{E}$.

YM^{-1} denotes the inverse image of a set $Y \subseteq R$ by a mapping M.

(6.15) $\mathfrak{D}$ is a prenet on R and $\mathcal{T}(\mathfrak{D})$ is the open-topology induced by $\mathfrak{D}$. $\mathcal{T}(M\mathfrak{D}M^{-1})$ is the open-topology on S induced by $M\mathfrak{D}M^{-1}$. Then, $\{YM^{-1}: Y \in \mathcal{T}(\mathfrak{D})\} \subseteq \mathcal{T}(M\mathfrak{D}M^{-1})$.

(6.16) $\{YM^{-1}: Y \in \mathcal{T}(D)\} = \mathcal{T}(M\mathfrak{D}M^{-1})$ if $\mathfrak{D}$ is a pretopology.

Proof. If $X \in \mathcal{T}(M\mathfrak{D}M^{-1})$ and $x \in X$, then there is $U(x) \in \mathfrak{D}$ such that $x(MU(x)M^{-1}) \subseteq X$. According to the interior theorem, since $\mathfrak{D}$ is a pretopology, $y = xM$ belongs to $\mathfrak{D}\text{-Int}(yU(x)) \in \mathcal{T}(\mathfrak{D})$, and $\mathfrak{D}\text{-Int}(yU(x)) \subseteq yU(x)$. Let $Y = \cup\{\mathfrak{D}\text{-Int}(yU(x)): y = xM, x \in X\}$. Then $Y \in \mathcal{T}(\mathfrak{D})$ and $X \subseteq YM^{-1} \subseteq \cup\{(yU(x))M^{-1}: y = xM, x \in X\} \subseteq X$. Hence, $X = YM^{-1}$.

$\mathfrak{A}$ is a connector system on S and $\mathfrak{D}$ is a connector system on R. A mapping M from S to R is called *$\mathfrak{A}$-continuous* w.r.t. $\mathfrak{D}$ if $\mathfrak{A}$ is finer than $M\mathfrak{D}M^{-1}$. This is a generalization of continuity. Neither connector system needs to be a topology.

(6.17) M is $\mathfrak{A}$-continuous w.r.t. $\mathfrak{D}$ if and only if for each $U \in \mathfrak{D}$ and for each $x \in S$, there are $V_i \in \mathfrak{A}$, $i = 1, 2, 3, \cdots, n$ such that $\cap\{xV_i: i = 1, 2, 3, \cdots, n\} \subseteq x(MUM^{-1})$.

(6.18) $\mathfrak{A}$ is a prenet. M is $\mathfrak{A}$-continuous w.r.t. $\mathfrak{D}$ if and only if for each $U \in \mathfrak{D}$ and for each $x \in S$, there is $V \in \mathfrak{A}$ such that $(xV)M \subseteq (xU)M$.

CONTINUITY THEOREM. $\mathfrak{A}$ and $\mathfrak{D}$ are connector systems on S and R respectively. $\mathfrak{A}$ is a prenet and $\mathfrak{D}$ is a pretopology. M is a mapping from S to R. M is $\mathfrak{A}$-continuous w.r.t. $\mathfrak{D}$ if and only if $\{YM^{-1}: Y \in \mathcal{T}(\mathfrak{D})\} \subseteq \mathcal{T}(\mathfrak{A})$.

Proof. $M\mathfrak{D}M^{-1}$ is a pretopology if $\mathfrak{D}$ is a pretopology. (4.1) and (4.5) $\Rightarrow$ a prenet $\mathfrak{A}$ is finer than a pretopology $M\mathfrak{D}M^{-1}$ if and only if $\mathcal{T}(M\mathfrak{D}M^{-1}) \subseteq \mathcal{T}(\mathfrak{A})$. Hence, by (6.16), $\mathfrak{A}$ is finer than $M\mathfrak{D}M^{-1}$ if and only if $\{YM^{-1}: Y \in \mathcal{T}(\mathfrak{D})\} \subseteq \mathcal{T}(\mathfrak{A})$.

Comment on the continuity theorem: An implication of the theorem is a compatibility of continuous mappings and continuous functions from a topological space $(S, \mathcal{T}(\mathfrak{A}))$ to a topological space $(R, \mathcal{T}(\mathfrak{D}))$. The theorem states that if $\mathfrak{A}$ is a prenet and $\mathfrak{D}$ is a pretopology, then a continuous mapping is topologically continuous and vice versa.

$\mathfrak{A}$ is a connector system on S and $\mathfrak{D}$ is a connector system on R. A mapping M from S to R is called *uniformly* $\mathfrak{A}$*-continuous* w.r.t. $\mathfrak{D}$ if $\mathfrak{A}$ is stronger than $M\mathfrak{D}M^{-1}$. $\mathfrak{A}$ is stronger, then $\mathfrak{A}$ is finer, therefore, M is $\mathfrak{A}$-continuous if M is uniformly $\mathfrak{A}$-continuous. (2.6) implies the converse if $\mathfrak{A}$ is a sharp prenet.

(6.19) M is uniformly $\mathfrak{A}$-continuous w.r.t. $\mathfrak{D}$ if and only if for each $U \in \mathfrak{D}$, there are $V_i \in \mathfrak{A}$, $i = 1, 2, 3, \cdots, n$, such that $\cap V_i \leqq MUM^{-1}$.

(6.20) $\mathfrak{A}$ is a sharp prenet. M is $\mathfrak{A}$-continuous w.r.t. $\mathfrak{D}$ if and only if M is uniformly $\mathfrak{A}$-continuous w.r.t. $\mathfrak{D}$.

KERNEL THEOREM. $\mathfrak{T}$ is a topology on S, $\mathfrak{U}$ is the uniformity kernel of $\mathfrak{T}$ and $\mathfrak{D}$ is a uniform net on R. If a mapping M from S to R is $\mathfrak{T}$-continuous w.r.t. $\mathfrak{D}$, then M is uniformly $\mathfrak{U}$-continuous w.r.t. $\mathfrak{D}$.

Proof. The continunity of M implies $M\mathfrak{D}M^{-1} \subseteq \mathfrak{T}$. $M\mathfrak{D}M^{-1}$ is a uniform net by (6.8) and (6.11), if $\mathfrak{D}$ is a uniform net. Then, $(M\mathfrak{D}M^{-1})^{<}$ is the uniformity hull of $M\mathfrak{D}M^{-1}$, and $M\mathfrak{D}M^{-1} \subseteq (M\mathfrak{D}M^{-1})^{<} \subseteq \mathfrak{U} \subseteq \mathfrak{T}$. Hence, M is uniformly $\mathfrak{U}$-continuous w.r.t. $\mathfrak{D}$.

Comment on the kernel theorem: A uniform net is a pretopology by (3.12) and (5.1), and a topology is a prenet. Therefore, by the continuity theorem, the kernel theorem can be applied to a continuous function from a topological space $(S, \mathcal{T}(\mathfrak{T}))$ to a topological space $(R, \mathcal{T}(\mathfrak{D}))$. (5.6) states that $\mathfrak{D}^{<}$ is the uniformity hull of a uniform net $\mathfrak{D}$. Since $\mathcal{T}(\mathfrak{D}) = \mathcal{T}(\mathfrak{D}^{<})$, $\mathcal{T}(\mathfrak{D})$ is a topology induced by a uniformity. The kernel theorem presents an answer to the following question in a generalized form. If S is a topological space and R is a uniform-topological space, then what is a uniformity on S, for which every continuous function from S to R is uniformly continuous?

7. Mappings. Let R_λ, $\lambda \in \Lambda$ be a system of spaces and let $\mathfrak{D}_\lambda$ be a connector system on R_λ for each $\lambda \in \Lambda$. If a topology hull $\mathfrak{T}$ of $\cup \{M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1} : \lambda \in \Lambda\}$ exists, then $\mathfrak{T}$ is called the weak topology on S by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\{\mathfrak{D}_\lambda : \lambda \in \Lambda\}$. $\mathfrak{T}$ is the weakest topology among those for which M_λ, $\lambda \in \Lambda$ are continuous.

(7.1) $\mathfrak{T}$ is the weak topology by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\{\mathfrak{D}_\lambda : \lambda \in \Lambda\}$. If $\mathfrak{D}_\lambda \subseteq \mathfrak{E}_\lambda \subseteq \mathfrak{D}_\lambda^{\times\#<}$ for every $\lambda \in \Lambda$, then $\mathfrak{T}$ is the weak topology by those mappings w.r.t. $\{\mathfrak{E}_\lambda : \lambda \in \Lambda\}$.

Proof. Refer to (1.5), (1.9), (1.22), (6.5), (6.6) and (6.7).

(7.2) If $\mathfrak{D}_\lambda$, $\lambda \in \Lambda$ are all topological or all of them are pre-topologies, then there exists a weak topology by every system of mappings, w.r.t. those connector systems.

Proof. Refer to (6.10), (6.12), (3.3) and (3.13).

WEAK TOPOLOGY THEOREM. $\mathfrak{T}$ is a topology on S. $\mathfrak{D}_\lambda$ is a connector system on R_λ and M_λ is a $\mathfrak{T}$-continuous mapping, w.r.t. $\mathfrak{D}_\lambda$ from S to R_λ, for each $\lambda \in \Lambda$. $\mathfrak{T}$ is the weak topology on S by those mappings, w.r.t. the given connector systems if for each $U \in \mathfrak{T}$ and for each $x \in S$, there is $\lambda \in \Lambda$ and $V \in \mathfrak{D}_\lambda$ such that $x(M_\lambda V M_\lambda^{-1}) \subseteq xU$.

Proof. The hypotheses are: $M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1} \subseteq \mathfrak{T}$ for every $\lambda \in \Lambda$ and

$$\mathfrak{T} \subseteq (\cup M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1})^{\#<}.$$

$$(\cup M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1})^{\#<} \subseteq (\cup M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1})^{\times\#<} \subseteq \mathfrak{T}^{\times\#<} = \mathfrak{T}.$$

Hence, $\mathfrak{T}$ is the topology hull of $\cup M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1}$.

If there exists a uniformity hull of $\cup M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1}$, then the uniformity hull is called the *weak uniformity* by those mappings, w.r.t. the given connector systems. A weak uniformity is the weakest uniformity among those for which each M_λ is uniformly continuous w.r.t. $\mathfrak{D}_\lambda$.

(7.3) $\mathfrak{U}$ is a weak uniformity w.r.t. $\{\mathfrak{D}_\lambda : \lambda \in \Lambda\}$. If $\mathfrak{D}_\lambda \subseteq \mathfrak{E}_\lambda \subseteq \mathfrak{D}_\lambda^{\times<}$ for each $\lambda \in \Lambda$, then $\mathfrak{U}$ is the weak uniformity by the same mappings, w.r.t. $\{\mathfrak{E}_\lambda : \lambda \in \Lambda\}$.

WEAK UNIFORMITY THEOREM. **$\mathfrak{D}_\lambda$ is a unifrom connector system on R_λ and $M\lambda$ is a mapping from S to R_λ for each $\lambda \in \Lambda$. Then there exists a weak uniformity $\mathfrak{U}$ by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\{\mathfrak{D}_\lambda : \lambda \in \Lambda\}$, and the topology hull of $\mathfrak{U}$ is the weak topology by those mappings, w.r.t. the given connector systems.**

Proof. Let $\mathfrak{V} = \cup M_\lambda \mathfrak{D}_\lambda M_\lambda^{-1}$ and let $\mathfrak{U} = \mathfrak{V}^{\times<}$. $\mathfrak{V}$ is uniform and $\mathfrak{U}$ is the uniformity hull by (6.11), (5.3) and (5.4). $\mathfrak{U}^{\times\#<}$ is the topology hull of $\mathfrak{V}$ by (3.3) and (3.4), as well as that of $\mathfrak{U}$. Hence the topology hull of $\mathfrak{U}$ is the weak topology by $\{M_\lambda : \lambda \in \Lambda\}$.

Let M_λ, $\lambda \in \Lambda$ be mapping from S to R and let U be a connector on R. Let $\cap M_\lambda U M_\lambda^{-1}$ denote the connector, $x \in S \to \cap \{(yU)M_\lambda^{-1}: y = xM_\lambda, \lambda \in \Lambda\}$.

(7.4) *U and V are connectors on R and W is the product of $\cap M_\lambda U M_\lambda^{-1}$ and $\cap M_\lambda V M_\lambda^{-1}$. Then $W \leqq \cap M_\lambda UV M_\lambda^{-1}$.*

(7.5) $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}^<\} \subseteq \{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}^<$.

(7.6) $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}^\times\} = \{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}^\times$.

(7.7) *If $\mathfrak{D}$ is topological then $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}$ is topological.*

(7.8) *If $\mathfrak{D}$ is uniform then $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}$ is uniform.*

$\mathfrak{A}$ is a connector system on S and $\mathfrak{D}$ is a connector system on R. A system of mappings M_λ, $\lambda \in \Lambda$ from S to R is called *$\mathfrak{A}$-equi-continuous* w.r.t. $\mathfrak{D}$ if $\mathfrak{A}$ is finer than $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}$, i.e., the latter is included in $\mathfrak{A}^{\times\#<}$. The mappings are called *uniformly $\mathfrak{A}$-equi-continuous* if $\mathfrak{A}$ is stronger than $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}$, i.e., the latter is included in $\mathfrak{A}^{\times<}$. The uniformly equi-continuity implies uniform continuity of each mapping in the system. If there exists a topology hull of $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}$ then the topology hull is called the equi-topology by $\{M_\lambda : \lambda \in \Lambda\}$.

(7.9) *$\mathfrak{D}$ and $\mathfrak{E}$ are connector systems on R and $\mathfrak{D} \subseteq \mathfrak{E} \subseteq \mathfrak{D}^{\times<}$. If $\mathfrak{T}$ is an equi-topology by a system of mappings to R, w.r.t. $\mathfrak{D}$ then $\mathfrak{T}$ is the equi-topology by the same mappings, w.r.t. $\mathfrak{E}$.*

(7.10) *$\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}^{\times\#<}$ is the equi-topology by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\mathfrak{D}$ if $\mathfrak{D}$ is topological.*

Proof. Refer to (7.7) and (3.3).

If there exists a uniformity hull of $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}$ then the uniformity hull is called the *equi-uniformity* by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\mathfrak{D}$, i.e., it is the weakest uniformity for which the mappings are uniformly equi-continuous.

(7.11) *$\mathfrak{D}$ and $\mathfrak{E}$ are connector systems on R and $\mathfrak{D} \subseteq \mathfrak{E} \subseteq \mathfrak{D}^{\times<}$. If $\mathfrak{U}$ is an equi-uniformity by a system of mappings w.r.t. $\mathfrak{D}$, then $\mathfrak{U}$ is the equi-uniformity by the same mappings w.r.t. $\mathfrak{E}$.*

(7.12) *If $\mathfrak{D}$ is uniform then $\{\cap M_\lambda U M_\lambda^{-1}: U \in \mathfrak{D}\}^{\times<}$ is the equi-uniformity by $\{M_\lambda : \lambda \in \Lambda\}$, w.r.t. $\mathfrak{D}$ and the equi-topology is the topology hull of the equi-uniformity.*

Proof. Refer to (7.8), (5.4), (3.5) and (7.10).

8. Uniformalizable topologies. The set of all real numbers is denoted by $(-\infty, \infty)$. For each positive ϵ, let $U(\epsilon)$ denote the connector on $(-\infty, \infty)$ such that $xU(\epsilon) = \{y: |x-y| \leq \epsilon\}$ for every $x \in (-\infty, \infty)$. Then the connector system $\{U(\epsilon): 0 < \epsilon\}$ is a uniform net. This is the only connector system, in this paper, we deal with for the real numbers. $\mathfrak{U}[a, b]$ denotes the connector system $\mathfrak{U}(-\infty, \infty)$ if each connector is restricted on a closed interval [a.b].

f is a function from a space S to $(-\infty, \infty)$. We write $a \leqq f \leqq b$ if the image of f is bounded by a and b. A topology $\mathfrak{T}$ on S is called *completely regular* if for each $U \in \mathfrak{T}$ and for each $x \in S$, there is a $\mathfrak{T}$-continuous function f from S to $[0, 1]$, w.r.t. $\mathfrak{U}[0, 1]$ such that $xf = 1$ and $yf = 0$ if y does not belong to xU.

The above definition is obviously compatible with the definition of a completely regular topology on the topological space $(S, \mathcal{T}(\mathfrak{T}))$.

(8.1) If $\mathfrak{T}$ is completely regular then $\mathfrak{T}$ is a weak topology by a system of functions.

Proof. Refer to the weak topology theorem in §7.

(8.2) A weak topology by functions, w.r.t. $\mathfrak{U}(-\infty, \infty)$ is completely regular.

UNIFORMALIZATION THEOREM. A topology is the topology hull of a uniformity if and only if it is completely regular.

Proof. If $\mathfrak{T}$ is completely regular then $\mathfrak{T}$ is a weak topology and by the weak uniformity theorem in §7, $\mathfrak{T}$ is the topology hull of the weak uniformity by the same functions which induce the weak topology. Conversely, if $\mathfrak{T}$ is the topology hull of a uniformity $\mathfrak{U}$ then, since $\mathfrak{T} = \mathfrak{U}^{\#}$, for each $U \in \mathfrak{T}$ and for each $x \in S$, there is $V \in \mathfrak{U}$ such that $xU = xV$. Theorem 5 in §31 of [1] and Theorem 19.1 of [2] state that there is a uniformly continuous function f such that $0 \leqq f \leqq 1$, $xf = 1$ and $yf = 0$ if y does not belong to xV.

9. Bounded connectors. A connector U on a space S is called *bounded* if there are $x_i \in S$, $i = 1, 2, 3, \cdots, n$ such that $S = \cup\{x_iU: i = 1, 2, 3, \cdots, n\}$. A connector U is called *absolutely bounded* if for each nonempty set $X \subseteq S$, there are $x_i \in X$, $i = 1, 2, 3, \cdots, n$ such that $X \subseteq \cup\{x_iU: i = 1, 2, 3, \cdots, n\}$. If $U \leqq V$ and U is bounded or absolutely bounded, then V is bounded or absolutely bounded respectively.

A connector system $\mathfrak{A}$ on S is called *totally bounded* if every connector in $\mathfrak{A}$ is bounded. This is a generalization of totally bounded uniformities.

(9.1) If a connector system $\mathfrak{A}$ is uniform and totally bounded then every connector in $\mathfrak{A}$ is absolutely bounded.

Proof. For each $U \in \mathfrak{A}$, since $\mathfrak{A}$ is uniform, there is $V \in \mathfrak{A}$ such that $V^{-1}V \leqq U$. Let X be a nonempty subset of S. There are $x_i \in S$, $i = 1,2,3,\cdots,n$ such that X is included in the union of x_iV, $i = 1,2,3,\cdots,n$ and $X \cap x_iV$ is nonempty for every $i = 1,2,3,\cdots,n$. If $y_i \in X \cap x_iV$, $i = 1,2,3,\cdots,n$, then, $x_iV \subseteq y_iV^{-1}V \subseteq y_iU$ thus X is included in the union of y_iU, $i = 1,2,3,\cdots,n$ and each y_i belongs to X. Hence, $\mathfrak{U}$ is absolutely bounded.

A topology $\mathfrak{T}$ on S is called *compact* if the open-topology $\mathscr{T}(\mathfrak{T})$ is compact (in usual sense), i.e., every open covering of S has a finite subcovering.

(9.2) A topology $\mathfrak{T}$ is compact if and only if $\mathfrak{T}$ is totally bounded.

Proof. The basis theorem in §4 states that there is an open connector $V \in \mathfrak{T}$ for each $U \in \mathfrak{T}$, such that $V \leqq U$. If $\mathfrak{T}$ is compact then, since all xV are $\mathfrak{T}$-open, V is bounded. Hence, $\mathfrak{T}$ is totally bounded if $\mathfrak{T}$ is compact. Conversly, suppose $S = \cup\{X_\lambda : \lambda \in \Lambda\}$ for some $X_\lambda \in \mathscr{T}(\mathfrak{T})$, $\lambda \in \Lambda$. Correspond each $x \in S$ to one of these X_λ which contains X_λ which contains x and define a connector U. Then U is $\mathfrak{T}$-open and $U \in \mathfrak{T}^{\#<} = \mathfrak{T}$. U is bounded if $\mathfrak{T}$ is totally bounded. Therefore, there are $x_i \in S$, $i = 1,2,3,\cdots,n$ such that S is the union of x_iU, $i = 1,2,3,\cdots,n$, which is a finite union of some X_λ, $\lambda \in \Lambda$.

Comment on (9.2). (9.2) is a generalization of a theorem on uniform spaces. The open-topology $\mathscr{T}(\mathfrak{T})$ in (9.2) is not necessarily a uniform topology. In fact, the theorem is a characterization of the compact topologies because each open-topology is induced by a topology.

COMPACT TOPOLOGY THEOREM. If the topology hull $\mathfrak{U}^\#$ of a uniformity $\mathfrak{U}$ is compact then $\mathfrak{U}$ is the uniformity kernel of $\mathfrak{U}^\#$.

Proof. First, we prove that a real valued $\mathfrak{U}$-continuous function on S is uniformly continuous if $\mathfrak{U}^\#$ is compact. By the definition of continuity, for each positive ϵ and $x \in S$, there is $U \in \mathfrak{U}$ such that $|xf - yf| < \epsilon/2$ if $y \in xU$. There is $V \in U$ such that $V^2 \leqq U$, and $|xf - yf| < \epsilon/2$ if $y \in xV^2$. By corresponding xV to x, we obtain a connector $W \in \mathfrak{U}^\#$. Since $\mathfrak{U}^\#$ is totally bounded by (9.2), there are $x_i \in S$, $i = 1,2,3,\cdots,n$ such that S is included in the union of x_iW, $i = 1,2,3,\cdots,n$. By the definition of W, there are corresponding $V_i \in \mathfrak{U}$, $i = 1,2,3,\cdots,n$ such that $x_iW = x_iV_i$, and $|x_if - yf| < \epsilon/2$ if

$y \in x_i V_i^2$. Let $V = \cap V_i$. Then $|xf - yf| < \epsilon$ whenever $y \in xV$. Hence, f is uniformly $\mathfrak{U}$-continuous. Let $\mathfrak{V}$ be the uniformity kernel of $\mathfrak{U}^{\#}$. $\mathfrak{U}^{\#}$ is totally bounded and $\mathfrak{V} \subseteq \mathfrak{U}^{\#}$, thus $\mathfrak{V}$ is totally bounded. Every uniformly $\mathfrak{V}$-continuous function is $\mathfrak{U}$-continuous, thus it is uniformly $\mathfrak{U}$-continuous. Therefore, $\mathfrak{V} \subseteq \mathfrak{U}$ by Theorem 6 in §33 of **[1]**. $\mathfrak{U} \subseteq \mathfrak{V}$ always holds because $\mathfrak{V}$ is the strongest uniformity included in $\mathfrak{U}^{\#}$. Hence, $\mathfrak{U}$ is the uniformity kernel of $\mathfrak{U}^{\#}$.

10. Semi-bounded connectors. A connector U on a space S is called *semi-bounded* if there is a positive integer n such that U^n is bounded. U^n is defined by induction, i.e., $U^n = U^{n-1}U$. A connector U is called *absolutely semi-bounded* if for each nonempty set $X \subseteq S$, there are $x_i \in X$, $i = 1, 2, 3, \cdots, m$ and a positive integer n such that $X \subseteq \cup \{x_i U^n : i = 1, 2, 3, \cdots, m\}$. If $U \leqq V$ and U is semi-bounded or absolutely semi-bounded, then V is semi-bounded or absolutely semi-bounded respectively.

A connector system $\mathfrak{A}$ is called bounded if every connector in $\mathfrak{A}$ is semi-bounded.

(10.1) A uniformity $\mathfrak{U}$ is bounded if and only if every uniformly $\mathfrak{U}$-continuous function is bounded.

Proof. Refer to Theorem 2 in §32 of **[1]**.

A connector system $\mathfrak{A}$ is called *absolutely bounded* if every connector in $\mathfrak{A}$ is absolutely semi-bounded.

(10.2) A uniformity $\mathfrak{U}$ is absolutely bounded if and only if $\mathfrak{U}$ is totally bounded.

Proof. Refer to Theorem 2 in §33 of **[1]**.

A topology $\mathfrak{T}$ on a space S is called *pseudo-compact* if every $\mathfrak{T}$-continuous function (real-valued) is bounded.

PSEUDO-COMPACT TOPOLOGY THEOREM. A topology $\mathfrak{T}$ is pseudo-compact if and only if the uniformity kernel $\mathfrak{U}$ of $\mathfrak{T}$ is bounded.

Proof. If $\mathfrak{U}$ is bounded then, since every $\mathfrak{T}$-continuous function is uniformly $\mathfrak{U}$-continuous by the kernel theorem in §6, $\mathfrak{T}$ is pseudo-compact. On the other hand, every uniformly $\mathfrak{U}$-continuous function is $\mathfrak{T}$-continuous because $\mathfrak{U} \subseteq \mathfrak{T}$. Hence, every uniformly $\mathfrak{U}$-continuous function is bounded if $\mathfrak{T}$ is pseudo-compact. By (10.1), $\mathfrak{U}$ is bounded.

REFERENCES

1. H. Nakano, *Topology and linear topological spaces*, Tokyo, 1951.
2. ———, *Uniform spaces and transformation groups*, Wayne State University Press, 1968.

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