

INVARIANT SUBSPACES OF COMPACT OPERATORS ON TOPOLOGICAL VECTOR SPACES

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Let (H, τ) be a topological vector space and let T be a compact linear operator mapping H into H (i.e., $T[V]$ is contained in a τ -compact set for some τ -neighborhood V of the zero vector in H). Sufficient conditions are given for (H, τ) so that T has a non-trivial, closed invariant linear subspace. In particular, it is shown that any complete, metrizable topological vector space with a Schauder basis satisfies the conditions stated in this paper. The proofs and conditions are stated within the framework of nonstandard analysis.

Introduction. This paper considers the following problem: given a compact operator T (Definition 2.11) on a topological vector space (H, τ) , does there exist a closed nontrivial linear subspace F of H such that $T[F] \subset F$? Aronszajn and Smith gave an affirmative answer to the above question when H is a Banach space (see [1]). Also it is easily shown that the Aronszajn and Smith result can be extended to locally convex spaces. However, it appears that other methods must be used for nonlocally convex spaces.

Sufficient conditions are given for a topological vector space so that a compact linear operator defined on the space has at least one nontrivial closed invariant linear subspace (Definitions 2.1 and 4.1, Theorems 3.2, 4.2 and 4.7). In particular, it is shown that a Fréchet space with a Schauder basis satisfies the conditions given in this paper (Theorem 5.6). The basic techniques are an outgrowth of Bernstein's and Robinson's methods in [2], [3] and [4]; consequently, the proofs and arguments of the main results are stated within the framework of nonstandard analysis. It will be assumed that the reader is familiar with Abraham Robinson's book on Nonstandard Analysis [11] or W. A. J. Luxemburg's paper on Monad Theory [10].

1. Nonstandard topological vector spaces. Let us briefly examine the basic concepts of nonstandard topological vector space theory that are used in this paper. A more detailed discussion of nonstandard topological vector spaces can be found in [6] and [7]. Let \mathbb{C} be the complex numbers, let H be a vector space over \mathbb{C} and let B_τ be the full set theoretical structure of $H \cup \mathbb{C}$. Throughout this paper (H, τ) will denote a *complex* topological vector space H with topology

τ . Also we will assume that the nonstandard structure $*B_\tau$ of $H \cup C$ is a higher order ultrapower enlargement of B_τ (see [6], Chapter 1, §1).

The nonnegative integers, positive integers and the real numbers are designated by \mathbf{N} , \mathbf{N}_+ and \mathbf{R} respectively. Moreover, the map $x \mapsto *x$ of $H \cup C$ into $*H \cup *C$ is the identity on \mathbf{N} , \mathbf{R} , C and H . As usual, the extensions to $*C$ of the algebraic operations $+$, \cdot and $| \cdot |$ on C are denoted by the same symbols. The same is true of the extension to $*\mathbf{R}$ of the ordering $<$ on \mathbf{R} .

An element $\lambda \in *C$ is called an *infinitesimal* if and only if $|\lambda| \leq \delta$ for each positive δ in \mathbf{R} and *finite* if and only if $|\lambda| \leq \delta$ for some positive $\delta \in \mathbf{R}$. If $\lambda \in *C$ is not finite then λ is called *infinite*. An element $\lambda \in *C$ is called *near-standard* if there is an element in C , denoted by ${}^\circ\lambda$, such that $\lambda - {}^\circ\lambda$ is an infinitesimal. It can be shown that ${}^\circ\lambda$ exists and is unique if and only if $\lambda \in *C$ is finite.

The extension to $*H$ of vector addition, $+$, on H is again denoted by the same symbol and the scalar multiplication operation on $*C \times *H$ takes (λ, z) to λz . For each x in H , $\mu_\tau(x)$ is defined by

$$(1.1) \quad \mu_\tau(x) = \cap \{ *V \mid V \in \mathcal{N}_\tau(x) \}$$

where $\mathcal{N}_\tau(x)$ is the filter of τ -neighborhoods of x . Since $\mathcal{N}_\tau(x) = \{ V + x \mid V \in \mathcal{N}_\tau(0) \}$ it follows that

$$(1.2) \quad \mu_\tau(x) = \mu_\tau(0) + x$$

for every x in H . It is easy to see that $\mu_\tau(0) + \mu_\tau(0) = \mu_\tau(0)$ and $\lambda \mu_\tau(0) \subset \mu_\tau(0)$ for finite λ in $*C$ ([6], Proposition I.6).

An element z of $*H$ is called *τ -near standard* if and only if $z \in \mu_\tau(x)$ for some x in H which is equivalent to $z - x \in \mu_\tau(0)$ for some x in H . The set of all τ -near-standard points of $*H$ will be denoted by $ns_\tau(*H)$. It is easily shown that $ns_\tau(*H)$ is closed under addition and multiplication by finite elements of $*C$. If τ is Hausdorff then for each $z \in ns_\tau(*H)$ there exists a unique point in H , denoted by ${}^\circ z$, such that $z - {}^\circ z \in \mu_\tau(0)$. Note that $H \subset ns_\tau(*H)$.

For $A \subset *H$, the τ -standard part of A , $st_\tau(A)$, is defined by

$$(1.3) \quad st_\tau(A) = \{ x \in H \mid z \in \mu_\tau(x) \text{ for some } z \in A \}.$$

Consequently, $st_\tau(A) \subset H$ for $A \subset *H$ and it can be shown that

$$(1.4) \quad st_\tau(*A) = \bar{A}$$

for any $A \subset H$ ([6], Proposition I.2). Observe that $st_\tau(\{z\}) = \{ {}^\circ z \}$ for each $z \in ns_\tau(*H)$ if τ is Hausdorff.

Another interesting concept, although not pertinent to the main objective, is the notion of a finite point. The set of τ finite points, $\text{fin}_\tau(*H)$, is defined by

$$(1.5) \text{fin}_\tau(*H) = \cap \{ \alpha \mu_\tau(0) \mid \alpha \in *R \text{ is positive and infinite} \}.$$

It can be shown that $z \in \text{fin}_\tau(*H)$ if and only if for $V \in \mathcal{N}_\tau(0)$ there exists an $n \in \mathbf{N}_+$ such that $z \in *(nV)$ ([6], Corollary I.15). Furthermore, $ns_\tau(*H) \subset \text{fin}_\tau(*H)$ and $\text{fin}_\tau(*H)$ is closed under addition and multiplication by finite elements of $*C$ (see [6], Proposition I.16 or [7], Theorem 1.2).

Henson and Moore first explored the relationship between τ -bounded subsets of H and $\text{fin}_\tau(*H)$ by establishing the fact that $A \subset H$ is τ -bounded if and only if $*A \subset \text{fin}_\tau(*H)$ ([7], Theorem 2.1). This idea can be generalized still further.

PROPOSITION 1.6. *If $A \subset *H$ is internal and $A \subset \text{fin}_\tau(*H)$ then $\text{st}_\tau(A)$ is τ -bounded.*

Proof. Let $A \subset *H$ be internal (i.e., $A \in *\mathcal{P}(H)$) and assume $A \subset \text{fin}_\tau(*H)$. Let $V \in \mathcal{N}_\tau(0)$ be closed and balanced and let $\phi(\mathcal{P}(H), \mathbf{R}, V)$ designate the following sentence.

“If $X \in \mathcal{P}(H)$ such that $X \subset \delta V$ for some positive $\delta \in \mathbf{R}$ then either $X \subset \delta V$ for all positive $\delta \in \mathbf{R}$ or there exists a positive $\delta_0 \in \mathbf{R}$ such that $X \subset \delta V$ for $\delta_0 < \delta$ and $X \not\subset \delta V$ for $0 < \delta < \delta_0$.”

Since V is balanced it follows that $\phi(\mathcal{P}(H), \mathbf{R}, V)$ is true in B_Γ and consequently $\phi(*\mathcal{P}(H), *R, *V)$ is true in $*B_\Gamma$. Since $A \subset \text{fin}_\tau(*H)$, we have $A \subset \lambda *V$ for positive, infinite $\lambda \in *R$. If $A \subset \lambda *V$ for each positive $\lambda \in *R$ then $A \subset *V$. If there exists a positive $\lambda_0 \in *R$ such that $A \subset \lambda *V$ for $\lambda_0 < \lambda$ and $A \not\subset \lambda *V$ for $0 < \lambda < \lambda_0$ then λ_0 is finite. Indeed, if λ_0 were infinite then $2^{-1}\lambda_0$ is infinite and $2^{-1}\lambda_0 < \lambda_0$ which would imply $A \not\subset 2^{-1}\lambda_0 *V$. But $A \subset \text{fin}_\tau(*H) \subset 2^{-1}\lambda_0 *V$. In either case there exists a positive $\delta \in \mathbf{R}$ such that $A \subseteq *(\delta V)$. Therefore, $\text{st}_\tau(A)$ is τ -bounded since $\text{st}_\tau(A) \subset \text{st}_\tau(*(\delta V)) = \delta V$ by (1.4).

Finally, we need to consider the relationship between internal linear subspaces of $*H$ and linear subspaces of H . Let $\mathcal{F}\mathcal{S}(H)$ denote the collection of all finite dimensional linear subspaces of H . We will refer to elements of $*\mathcal{F}\mathcal{S}(H)$ as $*$ -finite dimensional subspaces of $*H$. Also, let us symbolize ${}^\circ F = \text{st}_\tau(F)$ for $F \in *\mathcal{F}\mathcal{S}(H)$.

PROPOSITION 1.7. *If (H, τ) is a metrizable topological vector space and if $F \in {}^*\mathcal{FS}(H)$ then ${}^\circ F$ is a τ -closed linear subspace of H .*

By definition of $\mathcal{FS}(H)$, there exists $d: \mathcal{FS}(H) \rightarrow \mathbf{N}$ such that $d(F) \in \mathbf{N}$ is the dimension of F for $F \in \mathcal{FS}(H)$. Therefore $*d: {}^*\mathcal{FS}(H) \rightarrow {}^*\mathbf{N}$ and $*d(F)$ is called the **-finite dimension* of F for $F \in {}^*\mathcal{FS}(H)$.

PROPOSITION 1.8. *Let (H, τ) be a Hausdorff topological vector space and let $F, F_1 \in {}^*\mathcal{FS}(H)$ such that $F \subset F_1$. If $*d(F_1) = *d(F) + 1$ then ${}^\circ F \subset {}^\circ F_1$ and any two points of ${}^\circ F_1$ are linearly dependent modulo ${}^\circ F$.*

The proofs of the above propositions may be found in [6], Chapter I, §8.

2. Property 1. We now have the machinery needed to give sufficient conditions for a compact operator on the complex topological vector space (H, τ) to have a nontrivial τ -closed invariant linear subspace.

Let $\mathcal{L}(H)$ denote the vector space of all linear transformations of H into H . We will refer to elements of $*\mathcal{L}(H)$ as internal linear transformation of $*H$ into $*H$.

DEFINITION 2.1. A topological vector space (H, τ) is said to satisfy Property 1 if and only if there exist $P \in *\mathcal{L}(H)$ and $H_\eta \in {}^*\mathcal{FS}(H)$ such that the following conditions hold.

- (1) The internal linear transformation P maps $*H$ into H_η .
- (2) For $V \in \mathcal{N}_\tau(0)$ there exists $W \in \mathcal{N}_\tau(0)$ such that $P[*W] \subset *V$.
- (3) If $x \in H$ then $P(x) - x \in \mu_\tau(0)$.

We shall show that compact operators (Definition 2.11) on complex spaces which satisfy property 1 have nontrivial closed invariant linear subspaces.

Before discussing the proof of the above statement, let us first examine the influence of property 1 on a τ -continuous linear transformation T , i.e., a linear operator, defined on H . The idea is to define $T' = P*TP$, where P satisfies the condition of Property 1, and show that if $F \subset H_\eta$ is a **-finite dimensional subspace* such that $T'[F] \subset F$ then $T[{}^\circ F] \subset {}^\circ F$. Thus we will have a means of generating invariant subspaces for T .

Assume (H, τ) satisfies Property 1 and let $P \in *\mathcal{L}(H)$ and $H_\eta \in {}^*\mathcal{FS}(H)$ be determined by Property 1. Also throughout this section, it will be assumed that (H, τ) is Hausdorff. We see immediately

$$(2.2) \quad H = \text{st}_\tau(H_\eta) = {}^\circ H_\eta \quad \text{and} \quad x = {}^\circ[P(x)]$$

for $x \in H$. Also, Condition 2 of Definition 2.1 implies

$$(2.3) \quad P[\mu_\tau(0)] \subset \mu_\tau(0).$$

PROPOSITION 2.4. *If $z \in \text{ns}_\tau(*H)$ then $P(z) \in \text{ns}_\tau(*H)$, ${}^\circ[P(z)] = {}^\circ z$ and $z - P(z) \in \mu_\tau(0)$.*

(See [6], Proposition II.2.)

Let T be a linear operator on H , i.e., $T \in \mathcal{L}(H)$ and T is continuous, and consider $*T$. It can be shown that

$$(2.5) \quad *T[\mu_\tau(0)] \subset \mu_\tau(0)$$

which implies

$$(2.6) \quad {}^\circ[*T(z)] = T({}^\circ z)$$

for $z \in \text{ns}_\tau(*H)$ (see [11], Theorem 4.2.7 or [6], Propositions I.4 and II.3). By defining

$$(2.7) \quad T' = P*TP$$

We obtain an internal linear transformation that maps $*H$ into H_η . Furthermore

$$(2.8) \quad T'[\mu_\tau(0)] \subset \mu_\tau(0)$$

by (2.3) and (2.5). If a momentary abuse of language is permitted then we can say that the next proposition shows how well T' approximates T on near-standard points.

PROPOSITION 2.9. *Let $T \in \mathcal{L}(H)$ be τ -continuous. If $z \in \text{ns}_\tau(*H)$ then $T'(z) \in \text{ns}_\tau(*H)$ and ${}^\circ[T'(z)] = T({}^\circ z)$.*

(See [6], Proposition II.5.)

As promised, the following proposition gives a useful way of generating invariant linear subspaces for τ -continuous $T \in \mathcal{L}(H)$.

PROPOSITION 2.10. *Let $F \in *F\mathcal{S}(H)$ such that $F \subset H_\eta$ and let $T \in \mathcal{L}(H)$ be τ -continuous. If $T'[F] \subset F$ then $T[{}^\circ F] \subset {}^\circ F$.*

Proof. If $x \in {}^\circ F$ then $z - x \in \mu_\tau(0)$ for some $z \in F$ which implies ${}^\circ z = x$. We infer $T(x) = T({}^\circ z) = {}^\circ[T'(z)]$ by Proposition 2.9. Therefore, $T'[F] \subset F$ implies $T'(z) \in F$ which implies $T(x) \in {}^\circ F$. Consequently $T[{}^\circ F] \subset {}^\circ F$.

We are particularly concerned with the relationship between T and T' when T is compact. So let us now formally define compact operators.

DEFINITION 2.11. Let (H, τ) be a topological vector space. A map $T \in \mathcal{L}(H)$ is called a compact operator if and only if there exists a $V \in \mathcal{N}_\tau(0)$ such that $T[V]$ is contained in a τ -compact set.

Compact operators are continuous, a fact easily derived from the boundedness of compact sets. Equivalently $T \in \mathcal{L}(H)$ is compact if and only if the τ -closure of $T[V]$ is compact for some $V \in \mathcal{N}_\tau(0)$ since (H, τ) is *regular*. Moreover, using Robinson's characterization of compact sets, we arrive at still another equivalent form of Definition 2.11, i.e., $T \in \mathcal{L}(H)$ is compact if and only if $*T[*V] \subset \text{ns}_\tau(*H)$ for some $V \in \mathcal{N}_\tau(0)$ (see [11], Theorem 4.1.13). The next proposition shows that this idea is transferable to T' .

PROPOSITION 2.12. *If $T \in \mathcal{L}(H)$ is compact then there exists a $W \in \mathcal{N}_\tau(0)$ such that $T'[*W] \subset \text{ns}_\tau(*H)$.*

Proof. If $\overline{T[V]}$ is compact for some $V \in \mathcal{N}_\tau(0)$ then

$$*T[*V] \subset *\overline{T[V]} \subset \text{ns}_\tau(*H)$$

(see [11], Theorem 4.1.13 or [6], Proposition I.1). By Condition 2 of Definition 2.1, there exists $W \in \mathcal{N}_\tau(0)$ such that $P[*W] \subset *V$ which implies $*TP[*W] \subset \text{ns}_\tau(*H)$. Therefore,

$$T'[*W] = P[*TP[*W]] \subset \text{ns}_\tau(*H)$$

by Proposition 2.4.

3. Invariant subspaces. As stated earlier, we want to use Property 1 (Definition 2.1) to produce a nontrivial, closed invariant linear subspace for compact operator T on H . In the previous section, we established some interesting relationships between operators and Property 1; however, to give some motivation for defining Property 1, let's examine the following outline of a nonstandard proof of the

existence of a nontrivial, closed invariant linear subspace for a compact operator T when we assume the underlying space (H, τ) is a Hilbert space (see [4]).

First a chain of closed invariant linear subspaces of T is generated mainly by the use of an internal orthogonal projection P of $*H$ onto H_ω where H_ω is an internal, ω -dimensional subspace of $*H$, $\omega \in *N - N$, with the property that any standard point $x \in H$ is *infinitesimally* close to some point of H_ω . Next, it is shown that some member of the chain is nontrivial by using the compactness of T and the properties of *-finite sequences of internal orthogonal projections on $*H$.

Although there is a heavy dependence on the existences of orthogonal projections on H , this dependence is not uniform in the two main components of the above procedure. In the first part, i.e., generating the chain of closed invariant linear subspaces for T , the three essential properties of projections that were needed are: (1) projections are linear, (2) the norm of the projection of a point x in H is dominated by the norm of x , and (3) the projection of a point $x \in H$ is the *nearest* point, in the subspace, to x . To establish, in analogous manner, a chain of closed invariant linear subspaces for a compact operator T when the underlying space (H, τ) is a topological vector space, we need at least one internal linear transformation P on $*H$ that captures the effects of the three properties stated above. Definition 2.1 gives the conditions for such a transformation and, as we have seen earlier, this definition does give us a means of generating invariant linear subspaces for T (see Proposition 2.10).

In the second part, i.e., showing that some member of the chain is nontrivial, only the compactness of T and the *nearest point* property of projections are used. In other words, it is not necessary to have an internal *linear* map of $*H$ into F that captures the effects of a nearest point for every internal linear subspace F of $*H$; rather, we need only an internal *function* that maps $*H$ into F and maintains the essence of a nearest point for each *-finite dimensional subspace F of $*H$. Indeed, it is possible to show that such functions exist for a significant class of spaces.

In the following theorem, $\mathcal{F}[H]$ denotes the collection of all functions mapping H into H .

THEOREM 3.1. *If (H, τ) is a metrizable topological vector space then there exists a function $\nabla: \mathcal{FS}(H) \rightarrow \mathcal{F}[H]$ that satisfies the following conditions.*

- (1) *If $F \in \mathcal{FS}(H)$ then $\nabla(F): H \rightarrow F$.*
- (2) *For each $V \in \mathcal{N}_\tau(0)$ and any non zero $x \in H$ there exists a positive $\lambda \in \mathbb{R}$ such that $\nabla(F)(\lambda x) \in V$ for all $F \in \mathcal{FS}(H)$.*

(3) If $x \in H$ such that $x \in {}^\circ F$ for $F \in {}^* \mathcal{F}\mathcal{S}(H)$ then ${}^* \nabla(F)(x) - x \in \mu_r(0)$.

(See [6], Theorem II.1.)

Now the stage is set for the proof of the main result for metrizable topological vector spaces. The plan of attack is straightforward. First a suitable chain of invariant internal linear subspaces for $T' = P^*TP$ is found. Using Propositions 1.7 and 2.10 we can easily obtain a chain of closed invariant subspaces for the compact operator T . Next we must insure that at least one member of the chain is non-trivial which therefore precipitates the result.

THEOREM 3.2. *Let (H, τ) be a complex metrizable topological vector space that satisfies Property 1. If $T \in \mathcal{L}(H)$ is a compact operator then T has at least one nontrivial τ -closed invariant linear subspace of H .*

Proof. We will assume that H is infinite dimensional and the compact operator T is not null on all of H . Also, $P \in {}^* \mathcal{L}(H)$, $H_\eta \in {}^* \mathcal{F}\mathcal{S}(H)$ are determined by Property 1 and $\nabla: \mathcal{F}\mathcal{S}(H) \rightarrow \mathcal{F}[H]$ is assumed to satisfy the conditions of Theorem 3.1.

Define $T' = P^*TP$ and let $W \in \mathcal{N}_r(0)$ such that

$$T'[*W] \subset \text{ns}_r(*H)$$

(see Proposition 2.12). If $T(y) \neq 0$ for $y \in H$ then there exists a positive $\lambda \in \mathbf{R}$ such that

$${}^* \nabla(F)(\lambda y) \in *W$$

for each $F \in {}^* \mathcal{F}\mathcal{S}(H)$ by Condition 2 of Theorem 3.1. Consequently, if we set $x_0 = \lambda y$ then

$$(3.3) \quad T'(x_0) \notin \mu_r(0)$$

by Proposition 2.9 and

$$(3.4) \quad T'({}^* \nabla(F)(x_0)) \in \text{ns}_r(*H)$$

for any $F \in {}^* \mathcal{F}\mathcal{S}(H)$.

Let T_η denote the restriction of T' to H_η . As in [2] and [4], we observe that in a *finite-dimensional* complex vector space E of dimension m , say, any linear operator possesses a *finite* chain of invariant subspaces

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_m = E$$

where $d(F_j) = j$ for $j = 0, \dots, m$ (recall from §1 that d is the dimension function defined on $\mathcal{FS}(H)$). Therefore there exists a **-finite* chain of **-finite dimensional* subspaces

$$(3.5) \quad \{0\} = F_0 \subset F_1 \subset \cdots \subset F_\omega = H_\eta$$

such that

$$(3.6) \quad T_\eta[F_\iota] \subset F_\iota \quad \text{for } \iota \in \{0, \dots, \omega\}$$

$$(3.7) \quad *d(F_\iota) = *d(F_{\iota-1}) + 1 \quad \text{for } \iota \in \{1, \dots, \omega\}$$

where $\omega = *d(H_\eta)$.

Let $Q_\iota = *\nabla(F_\iota)$ for $\iota \in \{0, \dots, \omega\}$ and consider the expression

$$(3.8) \quad r_\iota = *\rho(T'(x_0) - T'Q_\iota(x_0), 0), \quad \iota = 0, 1, \dots, \omega$$

where ρ is a translation-invariant metric on H that generates τ . Note that $r_0 = *\rho(T'(x_0), 0)$ is not an infinitesimal by (3.3) since $F_0 = \{0\}$. Therefore, $\delta < r_0$ for some positive $\delta \in \mathbf{R}$. Since $x_0 \in H = {}^\circ H_\eta = {}^\circ F_\omega$ implies $x_0 - Q_\omega(x_0) \in \mu_\tau(0)$ (see (2.2), (3.5) and Condition 3 of Theorem 3.1) and $T'[\mu_\tau(0)] \subset \mu_\tau(0)$ implies

$$T'(x_0) - T'Q_\omega(x_0) = T'(x_0 - Q_\omega(x_0)) \in \mu_\tau(0),$$

it follows that r_ω is an infinitesimal. Consequently, $r_\omega < \frac{1}{2}\delta < r_0$ which implies

$$(3.9) \quad r_\nu < \frac{1}{2}\delta \leq r_{\nu-1}$$

for some $\nu \in \{1, \dots, \omega\}$ since $\{r_0, \dots, r_\omega\}$ is **-finite*.

Consider ${}^\circ F_{\nu-1}$ and ${}^\circ F_\nu$, the τ -standard parts of $F_{\nu-1}$ and F_ν (see (1.3)). Propositions 1.7, 2.10 and expressions (3.5), (3.6) imply ${}^\circ F_{\nu-1}$ and ${}^\circ F_\nu$ are τ -closed invariant linear subspaces of T since $T_\eta = T'$ on H_η . Now ${}^\circ F_{\nu-1}$ cannot coincide with H , in particular it cannot include x_0 . For if it did, then $r_{\nu-1}$ would be an infinitesimal (see (3.8) and Condition 3 of Theorem 3.1) contrary to (3.9).

On the other hand, ${}^\circ F_\nu$ cannot reduce to $\{0\}$. Indeed, if ${}^\circ F_\nu = \{0\}$ then ${}^\circ[T'Q_\nu(x_0)] = 0$ by (3.4) and (3.6) which would imply $T'Q_\nu(x_0) \in \mu_\tau(0)$. We would then have

$$\begin{aligned}
r_0 - r_\nu &= {}^*\rho(T'(x_0), 0) - {}^*\rho(T'(x_0) - T'Q_\nu(x_0), 0) \\
&\leq {}^*\rho(T'(x_0) - [T'(x_0) - T'Q_\nu(x_0)], 0) \\
&= {}^*\rho(T'Q_\nu(x_0), 0) < \frac{1}{2}\delta.
\end{aligned}$$

Therefore, $\delta < r_0$ and $r_0 - r_\nu < \frac{1}{2}\delta$ would imply $\frac{1}{2}\delta < r_\nu$ contrary to (3.9).

Finally from expressions (3.5), (3.7) and Proposition 1.8, we infer that either ${}^\circ F_{\nu-1}$ or ${}^\circ F_\nu$ is nontrivial.

4. Property 2. So far, we still have not shown that compact operators on *arbitrary* complex topological vector spaces satisfying Property 1 have nontrivial, closed invariant linear subspaces. We will fulfill this obligation in this section. We will do it by the usual mathematical ploy of defining other conditions, collectively called Property 2, that are more convenient. We then establish the result using these conditions and finally we show that spaces satisfying Property 1 also satisfy the new conditions, Property 2.

In this section, (H, τ) is an arbitrary complex topological vector space.

Define $\mathcal{M}[\tau]$ as follows: $\rho \in \mathcal{M}[\tau]$ if and only if ρ is a translation-invariant *pseudo-metric* on H such that for $g_\rho: H \rightarrow \mathbf{R}$ defined by $g_\rho(x) = \rho(x, 0)$, $x \in H$, we have $g_\rho(\lambda x) \leq g_\rho(x)$ for $|\lambda| \leq 1$ and g_ρ is continuous at $0 \in H$. Clearly, g_ρ is subadditive and therefore continuous on all of H . Furthermore, it can be shown that

$$\{S(\rho; \delta) \mid \rho \in \mathcal{M}[\tau], 0 < \delta \in \mathbf{R}\}$$

forms a filter basis for $\mathcal{N}_\tau(0)$ where

$$S(\rho; \delta) = \{x \in H \mid \rho(x, 0) \leq \delta\}$$

(see [9], Theorem 6.7).

For $\rho \in \mathcal{M}[\tau]$ let τ_ρ denote the topology on H generated by ρ . Obviously, $\rho \in \mathcal{M}[\tau]$ implies $\tau_\rho \subset \tau$ and (H, τ_ρ) is a complex topological vector space.

DEFINITION 4.1. A topological vector space (H, τ) is said to satisfy Property 2 if and only if there exists $\mathcal{A} \subset \mathcal{M}[\tau]$ such that the following conditions are fulfilled.

- (1) The collection $\{S(\rho; \delta) \mid \rho \in \mathcal{A}, 0 < \delta \in \mathbf{R}\}$ is a filter basis for $\mathcal{N}_\tau(0)$.
- (2) If $\rho \in \mathcal{A}$ then (H, τ_ρ) satisfies Property 1.

THEOREM 4.2. *Let (H, τ) be a complex topological vector space that satisfies Property 2. If τ is not the chaotic topology and if $T \in \mathcal{L}(H)$ is a compact operator then T has at least one nontrivial, τ -closed invariant linear subspace of H .*

Proof. Let $T \in \mathcal{L}(H)$ such that $\overline{T[V]}$ is τ -compact for some $V \in \mathcal{N}_\tau(0)$. It can be assumed that $V \neq H$ since $\tau \neq \{H, \phi\}$. Let $\mathcal{A} \subset \mathcal{M}[\tau]$ fulfill the conditions of Definition 4.1. Thus, $S(\rho; \delta_0) \subset V$ for some $\rho \in \mathcal{A}$ and some positive $\delta_0 \in \mathbf{R}$. Consequently, (H, τ_ρ) is a complex topological vector space such that $\tau_\rho \subset \tau$, $V \in \mathcal{N}_{\tau_\rho}(0)$ and (H, τ_ρ) satisfies Property 1.

Let $M = \{x \in H \mid \rho(x, 0) = 0\}$. If $x \in M$ and $\lambda \in \mathbf{C}$ then

$$\rho(\lambda x, 0) = \rho([\lambda n^{-1}]nx, 0) \leq \rho(nx, 0) = 0$$

for $n \in \mathbf{N}$ and $|\lambda| < n$. Consequently, M is a proper τ_ρ -closed linear subspace of H .

If $\delta \in \mathbf{R}$ is positive then by the τ_ρ -continuity of T , there exists a positive $\delta_1 \in \mathbf{R}$ such that

$$T[M] \subset T[S(\rho; \delta_1)] \subset S(\rho; \delta);$$

therefore, $T[M] \subset M$. So if $M \neq \{0\}$ then M is a nontrivial, τ -closed invariant linear subspace of T since τ_ρ is weaker than τ .

If $M = \{0\}$ then (H, τ_ρ) is a complex metrizable topological vector space that satisfies Property 1. By Theorem 3.2 there exists a nontrivial τ_ρ -closed, and thus τ -closed, linear subspace $F \subset H$ such that $T[F] \subset F$.

Next, we show that Property 1 implies Property 2. The basic idea is to start with an arbitrary balanced neighborhood of zero and use Condition 2 of Definition 2.1 to obtain a sequence of balanced neighborhoods of zero that generates a pseudo-metric linear topology satisfying Property 1.

THEOREM 4.3. *If a topological vector space (H, τ) satisfies Property 1 then (H, τ) satisfies Property 2.*

Proof. Let $P \in {}^*\mathcal{L}(H)$, $H_\eta \in {}^*\mathcal{F}\mathcal{S}(H)$ be determined by Property 1 and let $V \in \mathcal{N}_\tau(0)$ be balanced. By Condition 2 of Definition 2.1 there exists $W \in \mathcal{N}_\tau(0)$ such that $P[{}^*W] \subset {}^*V$. Also, there exists a balanced $V_2 \in \mathcal{N}_\tau(0)$ such that

$$V_2 + V_2 + V_2 \subset V \cap W$$

which implies $P[*V_2] \subset *V$. Therefore define

$$(4.4) \quad \{V_n\}_{n=1}^\infty \subset \mathcal{N}_\tau(0)$$

inductively as follows: $V_1 = V$, for $n \in \mathbf{N}_+$ let $V_{n+1} \in \mathcal{N}_\tau(0)$ be a balanced neighborhood such that

$$(4.5) \quad V_{n+1} + V_{n+1} + V_{n+1} \subset V_n$$

and

$$(4.6) \quad P[*V_{n+1}] \subset *V_n.$$

Consequently, there exists a unique topology τ_V on H such that (H, τ_V) is a topological vector space and $\{V_n\}_{n=1}^\infty$ is a filter basis for $\mathcal{N}_{\tau_V}(0)$ (see [8], Theorem 2.3.1).

Conditions 1 and 2 of Definition 2.1 are satisfied by the definitions of P , H_η and $\{V_n\}_{n=1}^\infty$. Also τ_V is weaker than τ by (4.4) which implies $\mu_\tau(0) \subset \mu_{\tau_V}(0)$ ([6], Chapter I, §2). Thus, $P(x) - x \in \mu_\tau(0)$ implies $P(x) - x \in \mu_{\tau_V}(0)$ for $x \in H$. Therefore, (H, τ_V) satisfies Property 1.

Using (4.4) and (4.5) it can be shown that some $\rho \in \mathcal{M}[\tau]$ generates τ_V (see [9], Theorem 6.7 or [8], Theorem 2.6.1). Therefore, if we define

$$\mathcal{A} = \{\rho \in \mathcal{M}[\tau] \mid \rho \text{ generates } \tau_V \text{ for balanced } V \in \mathcal{N}_\tau(0)\}$$

then \mathcal{A} fulfills the conditions of Definition 4.1; that is, (H, τ) satisfies Property 2.

THEOREM 4.7. *Let (H, τ) be a complex topological vector space that satisfies Property 1. If τ is not the chaotic topology and if $T \in \mathcal{L}(H)$ is a compact operator then T has at least one nontrivial τ -closed invariant linear subspace of H .*

Proof. Theorems 4.3 and 4.2.

5. Metrizable topological vector spaces with Schauder basis. In this section we examine the conditions needed for a metrizable topological vector space with a Schauder basis to satisfy Property 1. In particular we will show that a Fréchet space (i.e., a complete, metrizable topological vector space) with a Schauder basis satisfies Property 1.

In this section, (H, τ) is assumed to be a metrizable complex topological vector space.

If $\{x_j\}$ is a sequence in H then $\mathcal{F}_r[x_j; H]$ will denote the filter, on H , generated by $\{\{x_j\}_{j=n}^\infty \mid n \in \mathbf{N}_+\}$.

DEFINITION 5.1. Let (H, τ) be a metrizable complex topological vector spaces. A sequence $\{e_i\} \subset H$ of distinct nonzero elements is a Schauder basis for (H, τ) if and only if for each $x \in H$ there exists a unique sequence $\{\xi_i\} \subset \mathbf{C}$ such that $\mathcal{F}_r[\sum_{i=1}^j \xi_i e_i; H]$ converges to x with respect to τ .

Assume that a sequence $\{e_i\} \subset H$ is a Schauder basis for (H, τ) . Let

$$(5.2) \quad E = \text{sp}(\{e_i\}) \quad \text{and} \quad E_j = \text{sp}(e_1, \dots, e_j)$$

for $j \in \mathbf{N}_+$, i.e., E and E_j are the smallest linear subspaces of H that contain $\{e_i\}$ and $\{e_1, \dots, e_j\}$ respectively. Clearly, $E = \bigcup_{j=1}^\infty E_j$ and $\{E_j \mid j \in \mathbf{N}_+\} \subset \mathcal{F}\mathcal{S}(H)$. Furthermore it is easily seen that $H = \bar{E} = \bigcup_{j=1}^\infty E_j$ from which we conclude (H, τ) is separable. We will call $\{E_j \mid j \in \mathbf{N}_+\}$ the sequence of coordinate spaces generated by $\{e_i\}$.

If, for each $i \in \mathbf{N}_+$, we define $\pi_i: H \rightarrow \mathbf{C}$ by $\pi_i(x) = \xi_i$ for $x \in H$ and $\{\xi_i\}$, the unique complex sequence determined for x by $\{e_i\}$, then it can be shown that π_i is a linear functional on H that maps e_i to 1 and e_j to 0 for $j \in \mathbf{N}_+$ and $j \neq i$.

For each $j \in \mathbf{N}_+$ define $P_j: H \rightarrow E_j$ by

$$(5.3) \quad P_j(x) = \sum_{i=1}^j \pi_i(x) e_i$$

for $x \in H$. Consequently, $P_j \in \mathcal{L}(H)$, i.e., P_j is a linear transformation and $P_j(y) = y$ for $y \in E_j$. We will call $\{P_j \mid j \in \mathbf{N}_+\}$ the sequence of projections generated by $\{e_i\}$.

PROPOSITION 5.4. Let $\{e_i\}$ be a Schauder basis for (H, τ) and let $\{P_j \mid j \in \mathbf{N}_+\}$ be the sequence of projections generated by $\{e_i\}$. If there exists a filter basis $\mathcal{B} \subset \mathcal{N}_\tau(0)$ such that $P_j[V] \subset V$ for $V \in \mathcal{B}$ and $P_j \in \{P_j \mid j \in \mathbf{N}_+\}$ then (H, τ) satisfies Property 1.

(See [6], Proposition III.1.)

Let ρ be a translation-invariant metric on H that generates τ . We will call $\{e_i\}$ a *monotone* Schauder basis (with respect to ρ) if and only if the sequence $\{\rho(P_j(x), 0) \mid j \in \mathbf{N}_+\}$ is monotonely increasing for each $x \in H$, that is, if $i, j \in \mathbf{N}_+$ and $i \leq j$ then $\rho(P_i(x), 0) \leq \rho(P_j(x), 0)$ for each $x \in H$.

PROPOSITION 5.5. *If (H, τ) has a monotone Schauder basis then (H, τ) satisfies Property 1.*

Proof. Let ρ be a translation-invariant metric on H that generates τ and let $\{e_j\}$ be a monotone Schauder basis for (H, τ) with respect to ρ . If $x \in H$ then $\rho(P_j(x), 0) \leq \rho(x, 0)$ for each $j \in \mathbf{N}_+$ by the continuity of ρ . Therefore, if $\delta \in \mathbf{R}$ is positive then $x \in S(\rho; \delta)$ implies $\rho(P_j(x), 0) \leq \rho(x, 0) \leq \delta$ for $j \in \mathbf{N}_+$. Consequently $P_j[S(\rho; \delta)] \subset S(\rho; \delta)$ for each $P_j \in \{P_j \mid j \in \mathbf{N}_+\}$. We conclude that (H, τ) satisfies Property 1 from Proposition 5.4.

In the following theorem, a Fréchet space is a complete metrizable topological vector space.

THEOREM 5.6. *Let (H, τ) be a Fréchet space. If (H, τ) has a Schauder basis then (H, τ) satisfies Property 1.*

Proof. Let $\{e_j\}$ be a Schauder basis for (H, τ) and let $\{P_j \mid j \in \mathbf{N}_+\}$ be the sequence of projections generated by $\{e_j\}$. Also, let ρ be a translation-invariant metric on H that generates τ . Define $H \rightarrow \mathbf{R}$ as follows:

$$\rho_0(x, y) = \sup \{\rho(P_j(x - y), 0) \mid j \in \mathbf{N}_+\}$$

for $x, y \in H$. Since (H, τ) is complete, it can be shown that ρ_0 is a translation-invariant metric on H that generates τ (see [13], Theorem 11.4.1). Furthermore, $\{e_j\}$ is a monotone Schauder basis with respect to ρ_0 . Therefore, (H, τ) satisfies Property 1 by Proposition 5.5.

COROLLARY 5.7. *If (H, τ) is a complex Fréchet space with a Schauder basis and if $T \in \mathcal{L}(H)$ is a compact operator then T has at least one nontrivial closed invariant linear subspace of H .*

Proof. Theorems 5.6 and 3.2.

EXAMPLE 5.8. Let $\mathcal{S}q(\mathbf{C})$ be the collection of all complex sequences and let (ξ_j) symbolize an element of $\mathcal{S}q(\mathbf{C})$, i.e., $\xi_j \in \mathbf{C}$ for $j \in \mathbf{N}_+$. The set $\mathcal{S}q(\mathbf{C})$ is a complex vector space under the usual pointwise definition of addition and scalar multiplication.

Let $\{p_j\}$ be a sequence of real numbers satisfying the inequalities $0 < p_j \leq 1$ for each $j \in \mathbf{N}_+$ and define

$$\ell(p_j) = \{(\xi_j) \in \mathcal{S}q(\mathbf{C}) \mid \sum_{j=1}^{\infty} |\xi_j|^{p_j} < \infty\}.$$

Since the identities $|\lambda + \beta|^p \leq |\lambda|^p + |\beta|^p$ and $|\lambda\beta|^p \leq \max(1, |\lambda|)|\beta|^p$ are valid for all $\lambda, \beta \in \mathbf{C}$ and $0 < p \leq 1$, it follows that $\ell(p_j)$ is a vector subspace of $\mathcal{S}q(\mathbf{C})$. Also, we can define a translation-invariant metric $\rho(p_j)$ on $\ell(p_j)$ as follows:

$$\rho(p_j)(x, y) = \sum_{j=1}^{\infty} |\xi_j - \zeta_j|^p$$

for $x = (\xi_j), y = (\zeta_j) \in \ell(p_j)$. We will let $\tau(p_j)$ denote the topology on $\ell(p_j)$ generated by $\rho(p_j)$.

S. Simons made an extensive study of $(\ell(p_j), \tau(p_j))$ in [12]. Among the facts he established are the following.

(5.9) $(\ell(p_j), \tau(p_j))$ is a Fréchet space.

(5.10) $(\ell(p_j), \tau(p_j))$ is locally convex if and only if $\ell(p_j) = \ell^1$.

(5.11) $(\ell(p_j), \tau(p_j))$ is locally bounded if and only if $\liminf p_j > 0$.

(See [12], Lemma 1, Theorems 5 and 6.)

If, for $j \in \mathbf{N}_+$, we define $e_j = (\xi_k)$ such that $\xi_j = 1$ and $\xi_k = 0$ for $k \in \mathbf{N}_+$ and $k \neq j$ then $\{e_j\}$ is a Schauder basis for $(\ell(p_j), \tau(p_j))$ by definition of $\rho(p_j)$. Therefore, $(\ell(p_j), \tau(p_j))$ satisfies Property 1 by (5.9) and Theorem 5.6. Furthermore, if $p_j \rightarrow 0$ as $j \rightarrow \infty$ then $\ell(p_j) \neq \ell^1$ since $x_0 = (j^{-q_j}) \notin \ell(p_j)$, where $q_j = p_j^{-1}$, and $x_0 \in \ell^1$. Consequently, from (5.10) and (5.11) we can infer that if $\{p_j\}$ converges to 0 then $(\ell(p_j), \tau(p_j))$ is neither locally convex nor locally bounded. In particular, the space $(\ell(j^{-1}), \tau(j^{-1}))$ is not locally convex and not locally bounded.

EXAMPLE 5.12. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two infinite matrices of real numbers such that $0 < a_{ij}$ and $0 < b_{ij} \leq 1$ for $i, j \in \mathbf{N}_+$. Define $\ell(A, B)$ in the following manner. If $x = (\xi_j)$ is a sequence of complex numbers then $x \in \ell(A, B)$ if and only if

$$\sum_{j=1}^{\infty} a_{ij} |\xi_j|^{b_{ij}} < \infty$$

for each $i \in \mathbf{N}_+$. Using the same arguments of Example 5.8, it can be shown that $\ell(A, B)$ is a complex vector space.

If we define

$$W_n = \{(\xi_j) \in \ell(A, B) \mid \sum_{j=1}^{\infty} a_{ij} |\xi_j|^{b_{ij}} \leq n^{-1} \text{ for } i = 1, \dots, n\}$$

for each positive integer n , then there exists a unique topology $\tau(A, B)$ on $\ell(A, B)$ such that $(\ell(A, B), \tau(A, B))$ is a topological vector space and $\{W_n\}_{n=1}^{\infty}$ is a filter basis for the $\tau(A, B)$ -neighborhoods of zero. Indeed, it can be shown that $(\ell(A, B), \tau(A, B))$ is a *Fréchet* space having $\{e_j\}$ (see Example 5.8) as a Schauder basis ([6], Propositions IV.2 and IV.3). Consequently, $(\ell(A, B), \tau(A, B))$ satisfies Property 1 by Theorem 5.6.

Note that the spaces $(\ell(p_j), \tau(p_j))$ of Example 5.8 form a subcollection of the spaces $(\ell(A, B), \tau(A, B))$, i.e., let $a_{ij} = 1$ for all $i, j \in \mathbf{N}_+$ and let $b_{ij} = p_j$ for each $i \in \mathbf{N}_+$. However, it is possible to define a subclass of the spaces $(\ell(A, B), \tau(A, B))$ that is neither locally convex nor locally bounded and different from the collection $(\ell(p_j), \tau(p_j))$.

We will say that $[A : B : \{q_i\}]$ satisfies p.3, where $A = [a_{ij}]$ and $B = [b_{ij}]$ are infinite matrices of real numbers and $\{q_i\}$ is a sequence of real numbers, if and only if

$$(5.13) \quad a_{ij} = 1 \text{ and } 0 < q_{i+1} < q_i < 1 \text{ for all } i, j \in \mathbf{N}_+,$$

$$(5.14) \quad q_{2i} < b_{i,j+1} \leq b_{ij} < q_{2i-1} \text{ for all } i, j \in \mathbf{N}_+.$$

Now if $[A : B : \{q_i\}]$ satisfies p.3, then $(\ell(A, B), \tau(A, B))$ is not locally convex, not locally bounded and is distinct from $(\ell(p_j), \tau(p_j))$ for any sequence $\{p_j\} \subset (0, 1)$ ([6], Propositions IV.4, IV.5 and IV.6).

In particular, if we let $(\ell[i^{-1}], \tau[i^{-1}])$ symbolize the $(\ell(A, B), \tau(A, B))$ space where $a_{ij} = 1$ and $b_{ij} = i^{-1}$ for $i, j \in \mathbf{N}_+$ then $\ell(i^{-1})$, of Example 5.8, is a proper subset of $\ell[i^{-1}]$ and $\tau[i^{-1}]$ induces a weaker topology on $\ell(i^{-1})$ than $\tau(i^{-1})$.

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