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COMPONENTS OF ZERO SETS OF ANALYTIC FUNCTIONS IN C^2 IN THE UNIT BALL OR POLYDISC

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Let Ω denote either the unit ball or unit polydisc in C^2. Let f be a function analytic on a neighborhood of the closure of Ω, and let V be an irreducible component of $\{f=0\} \cap \Omega$. Then there is a bounded analytic function on Ω whose zero set is V.

1. Introduction. Let V_1 be the zero set of a polynomial in C^2. H. Alexander has posed the question of determining if an irreducible component of V_1 in $U^2 = \{(z, w) \in C^2 : |z| < 1, |w| < 1\}$, the unit polydisc, can be defined by a bounded holomorphic function in U^2 ([4] p. 233; see also [8], p. 90). We show here that this is the case for both U^2 and the unit ball $\{(z, w) \in C^2 : |z|^2 + |w|^2 < 1\}$. For the proof an explicit local construction is first made and then the patching theorems of Stout [9], [10], or Range and Siu [7] are used to prove the theorem.

2. Statement of theorem and outline of proof.

THEOREM. *Let f be holomorphic on a neighborhood of $\bar{\Omega}$, where either*

(i) $\Omega = U^2 = \{(z, w) \in C^2 \mid |z| < 1, |w| < 1\}$; *or*

(ii) *Ω is a strongly convex bounded open set in C^2 with* (real) *analytic boundary.*

Let V be an irreducible component of $\{z \in \Omega : f(z) = 0\}$, the variety of f in Ω. Then there exists a bounded holomorphic function F on Ω such that

$$V = \{z \in \Omega \mid F(z) = 0\} .$$

According to theorems of Stout [9], [10], or Range and Siu [7], it suffices to prove the following local version of the theorem.

PROPOSITION 1. *Let Ω, V be as in the theorem. For each $\zeta \in \bar{\Omega}$, there is an open set U_ζ in C^2 and a bounded holomorphic function f_ζ on $U_\zeta \cap \Omega$ such that for $\zeta, \eta \in \bar{\Omega}$,*

(i) $\zeta \in U_\zeta$;

(ii) $V \cap U_\zeta = \{z \in U_\zeta \cap \Omega : f_\zeta(z) = 0\}$;

(iii) *if $h = f_\zeta / f_\eta$, then h and $1/h$ are holomorphic and bounded on $U_\zeta \cap U_\eta \cap \Omega$.*

The only difficulty in finding the functions f_ζ of Proposition 1 occurs when $\zeta \in \partial\Omega$ and $f(\zeta) = 0$. The most obvious candidates to try are pieces of the Weierstrass polynomial occuring in the local factorization of f. This procedure works, and an outline of the steps is as follows.

Step 1. Choose local coordinates (z, w) at $\zeta \in \partial\Omega$ so that near ζ, f may be factored as a Weierstrass polynomial P in w times a unit U,

$$\begin{aligned} f(z, w) &= U(z, w) \cdot P(z, w) \\ &= U(z, w)[w^m + a_{m-1}(z)w^{m-1} + \cdots + a_0(z)] \\ &= U(z, w)\left[\prod_{j=1}^{m} (w - w_j(z))\right]. \end{aligned}$$

The $\{w_j(z)\}$ are not analytic functions, but may be thought of as multivalued analytic functions (see e.g. [2], p. 69, equation (2), or [1], Chapter 1, § 4, especially p. 20). For an appropriate choice of local coordinates (z, w), it is possible to choose branches of the multivalued functions $w_j(z)$ so that the functions

$$h_j(z, w) = w - w_j(z)$$

are single-valued analytic functions on the part on Ω near ζ.

Step 2. Show that the restriction of each of the functions h_j to Ω is irreducible (or a unit). Thus, any irreducible component of $V \cap \Omega$ must locally be the union of the zero sets of some of the h_j. A function f_ζ which works is then the product of these h_j.

We have encountered several problems in carrying through this program. These have required us to consider only $n = 2$. It may be that similar methods will work for $n > 2$, but the appropriate choice of local coordinates is not so evident. In case the boundary of Ω is only C^∞ instead of analytic (or piecewise analytic), there seems to be little hope that these methods will work, since the varieties $\{h_j = 0\} \cap \Omega$ can have infinitely many components.

Proposition 1 is a consequence of the following lemma.

LEMMA 1. *Let Ω, V, f be as in Theorem* 1. *For each $\zeta \in \partial\Omega$ withf $(\zeta) = 0$, there exists open sets U_ζ, W_ζ in $\boldsymbol{C}^2$, and a holomorphic function f_ζ on $W_\zeta \cap \Omega$ such that* (i) *and* (ii) *of Proposition* 1 *hold and, further,*

(iv) $\zeta \in U_\zeta \subset \bar{U}_\zeta \subset W_\zeta$

(v) *f_ζ is continuous on $W_\zeta \cap \bar{\Omega}$, and $\bar{V} \cap W_\zeta \cap \bar{\Omega} = \{z \in W_\zeta \cap \bar{\Omega}: f_\zeta(z) = 0\}$.*

Also, condition (iii) *of Proposition* 1 *holds when* $\zeta, \eta \in \partial\Omega$ *and* $f(\zeta) = f(\eta) = 0$.

It is easy to deduce Proposition 1 from Lemma 1. In the rest of the paper we prove Lemma 1.

Proof of Proposition 1 *from Lemma* 1. Choose an analytic function g on Ω so that V is the variety of g and, further, that g defines the ideal of V locally (see e.g. [2, p. 251]). If $\zeta \in \Omega$ choose ε smaller than the distance from ζ to $\partial\Omega$ and set

$$V_\zeta = \text{ball of radius } \varepsilon \text{ about } \zeta = B(\zeta, \varepsilon)$$
$$f_\zeta = g\ .$$

Next, if $\zeta \in \partial\Omega$, but $f(\zeta) \neq 0$, choose $\varepsilon > 0$ so small that f is holomorphic on $B(\zeta, 2\varepsilon)$ and $B(\zeta, 2\varepsilon)$ does not meet the zero set of f. Then let $U_\zeta = B(\zeta, \varepsilon)$, $f_\zeta \equiv 1$. Finally, if $\zeta \in \partial\Omega$ and $f(\zeta) = 0$, let U_ζ, f_ζ be as given by Lemma 1. It is now easy to check that (i)-(iii) of Proposition 1 are satisfied.

3. **Structure of** V **at** $\partial\Omega$. In this section we prove Lemma 1 in case (ii) of Theorem 1. That is, we assume that locally Ω is defined as $\{\rho < 0\}$ where ρ is a real analytic, convex function with $\nabla\rho \neq 0$ on $\rho = 0$. We may also assume that f is irreducible on a neighborhood of $\bar{\Omega}$, since if not, a preliminary factorization can first be made. We may also assume that for any choice of affine coordinates on C^2, f is regular in w. That is, f vanishes on no open subset of a complex hyperplane in C^2. For, if this is the case, then $f(z, w) = [a + bz + cw] \cdot h(z, w)$, where h is holomorphic on a neighborhood of $\bar{\Omega}$. Since f is irreducible, we either have $a + bz + cw \neq 0$ on $\bar{\Omega}$, in which case we can replace f by h, or $h \neq 0$ on $\bar{\Omega}$, in which case we take $F = a + bz + cw$ and the Theorem is trivial.

Thus, let $\zeta \in \partial\Omega$. Choose orthonormal coordinates with origin at ζ so that for small values of z and w,

$$\rho(z, w) = 2\mathrm{Re}\, \gamma z + Q(z, w) + 2\mathrm{Re}\, P(z, w) + \varepsilon(z, w) \tag{3.1}$$

where $\gamma > 0$ ($\gamma = 1/2\,|\nabla\rho(\zeta)|$) and

$$Q(t_1, t_2) = \sum_{i,j=1}^{2} \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}(0, 0) t_i \bar{t}_j \geqq k(|t_1|^2 + |t_2|^2),\ k > 0$$

$$P(t_1, t_2) = \sum_{i,j=1}^{2} \frac{\partial^2\rho}{\partial z_i \partial z_j}(0, 0) t_i t_j \tag{3.2}$$

$$\varepsilon(z, w) = 0(|z|^3 + |w|^3)\ .$$

Thus, the direction of $\operatorname{Re} z$ is the direction of the outward normal to Ω at ζ.

In this coordinate system, factor f as a Weierstrass polynomial in w times a nonvanishing function,

$$f(z, w) = P(z, w)U(z, w) = \left[\prod_{j=1}^{m} (w - w_j(z))\right]F(z, w) , \qquad m = m(\zeta) . \tag{3.3}$$

on a neighborhood of ζ. Since f is irreducible on a neighborhood of $\bar{\Omega}$, the variety of common zeroes of f and ∇f is discrete, since it must have codimension of at least 2. Thus, for sufficiently small $z \neq 0$, the roots $w_j(z)$ of f given by (3.3) are all distinct. Further, on any simply connected open subset of a small punctured disc $0 < |z| < \delta$ such as

$$\{z \mid 0 < |z| < \delta, \operatorname{Re} z < 0\}$$

the $w_j(z)$ may be chosen as single valued analytic functions of z, and each of them $w(z) = w_j(z)$ has a Puiseux expansion

$$w(z) = \sum_{l=1}^{\infty} c_l (z^{1/q})^l \tag{3.4}$$

as a series of fractional powers of z (see e.g. [3], p. 346, or [1], pp. 7.22).

We want to study each of the "pieces" of the variety of f near ζ,

$$h_j(z, w) = 0 \tag{3.5}$$

where

$$h_j(z, w) = w - w_j(z) .$$

In particular, we want to prove that the h_j are irreducible analytic functions on the part of Ω near ζ; that is, the part of the zero set of h_j lying inside Ω is connected. Since Ω is defined by the inequality $\rho < 0$, this means we have to study when the function

$$u(z) = \rho(z, w(z)) , \tag{3.6}$$

where $w(z) = w_j(z)$ for some j, can be negative. The necessary facts are in the next two lemmas.

LEMMA 3.1. *There exists $\delta_0 > 0$ such that either*

(i) $u(re^{i\theta}) \geqq 0$, $\pi/2 \leqq \theta \leqq 3\pi/2$, $0 < r \leqq \delta_0$; *or*

(ii) *for every r, $0 < r < \delta_0$, there exists $\theta = \theta(r)$ with $\pi/2 \leqq \theta \leqq 3\pi/2$, such that*

$$u(re^{i\theta}) < 0 \ .$$

REMARK. If ρ is only C^∞ instead of real analytic, this lemma is false. A counterexample may be obtained as follows. Choose a C^∞ function χ of one real variable r with an infinite order zero at $r = 0$, and such that $\chi(r)$ changes sign infinitely often as $r \to 0$. For example, $\chi(r) = e^{-1/r^2} \sin(1/r)$. Then set

$$\varepsilon(z, w) = \left[\frac{-|w|^4}{4} + \frac{|w|^2}{2}\chi\left(\frac{|w|^2}{2}\right)\right]\phi(z, w)$$

where ϕ is a C^∞ function with $\phi = 1$ on a neighborhood of $(0, 0)$, $0 \leqq \phi \leqq 1$, and $\phi = 0$ outside a slightly larger neighborhood of $(0, 0)$.

We can do this keeping $\varepsilon(z, w)$ small in the C^2 norm. Then define

$$\begin{aligned}\rho(z, w) &= -2\mathrm{Re}\, z + |z|^2 + |w|^2 + \varepsilon(z, w)\\ &= \{|1 - z|^2 + |w|^2 - 1\} + \varepsilon(z, w) \ ,\end{aligned}$$

and set $\Omega = \{\rho < 0\}$. Since $\varepsilon(z, w)$ is small in the C^2 norm, Ω is strictly pseudoconvex and is, of course, a small perturbation of the ball with center at $(1, 0)$ and radius 1. Then set $f(z, w) = w^2 - 2z$ and $V = \{(z, w): f(z, w) = 0\}$. It is not hard to verify that $V \cap \Omega$ has infinitely many components, and also, all the sets $P_j(\delta)$ have infinitely many components. Thus, Lemma 3.1 fails. However, an extra argument will show that the Theorem is still correct for this example.

Proof. Set $z = t^n$, $v(t) = u(t^n)$ where $n \geqq 1$ is chosen so that $w(t^n)$, given by (3.4) is an analytic function of t near $t = 0$. Let

$$S_\delta = \left\{t = re^{i\varphi}: \frac{\pi}{2} \leqq n\varphi \leqq \frac{3\pi}{2},\ v(t) < 0,\ r < \delta\right\} .$$

If the lemma is false, then there exists sequences $\{r_j\}$, $\{r_j^*\}$, $\{\varphi_j\}$ such that $r_1 > r_1^* > r_2 > r_2^* \cdots$, $r_j \to 0$, $\pi/2 \leqq n\varphi_j \leqq 3\pi/2$, and $v(r_j e^{i\varphi}) \geqq 0$ for all $\pi/2 \leqq n\varphi \leqq 3\pi/2$, and $v(r_j^* e^{i\varphi_j}) < 0$.

We can assume that for $\varphi = 3\pi/2n$ or $\varphi = \pi/2n$ we have $v(re^{i\varphi}) \geqq 0$ for all sufficiently small $r > 0$. Otherwise, since $r \to v(re^{i\varphi})$ is real analytic, we have $v(re^{i\varphi}) < 0$ on an interval $(0, \delta)$ and the lemma is true. Consequently the assumption that the lemma is false leads to the conclusion that for any neighborhood U of $t = 0$, the points $r_j^* e^{i\varphi_j}$ all belong to different components of $\mathrm{O} = U \backslash \{t: v(t) = 0\}$. However, since $v(t)$ is real analytic, the set O can have only finitely many components ([5], p. 96, Lemma 1), which is a contradiction.

LEMMA 3.2. *There exists* $\delta_0 = \delta_0(\zeta) > 0$ *so small that if* $S(\delta) = \{z: \mathrm{Re}\, z < 0, |z| < \delta\}$ *then*

(i) *for each* j *the set*

$$P_j(\delta) = \{z\colon z \in S(\delta),\ \rho(z, w_j(z)) < 0\}$$

is either empty for all $0 < \delta < \delta_0$, *or else* $P_j(\delta)$ *is nonempty and connected for every* $0 < \delta \leqq \delta_0$;

(ii) *if* $P_j(\delta)$ *is not empty for some* (and hence all) $0 < \delta < \delta_0$, *then its closure in* $S(\delta)$ *is equal to* $\{z\colon z \in S(\delta),\ \rho(z, w_j(z)) \leqq 0\}$;

(iii) *if* $\zeta' \in \partial\Omega$ *and the length of the* z *coordinate of* ζ' *is less then* δ_0, *and if the function* $z \to \rho(z, w_j(z))$ *vanishes at* ζ' *for some* j *for which* $P_j(\delta)$ *is not empty, then* $m(\zeta') = 1$, *where* $m(\zeta')$ *is as defined in* (3.3).

Proof. We, of course, choose δ so small that all the points with coordinates $(z, w_j(z))$ are in the domain of ρ. From the expansion (3.4), we see that if $w(z)$ is one of the roots of $f(z, w) = 0$, then

$$w(z) = az^{\beta} + h(z) \tag{3.7}$$

where $a \neq 0$, $\beta > 0$, and $h(z)$ is a power series in fractional powers of z higher than β which converges on some neighborhood of $z = 0$. Writing $z = re^{i\theta}$, we have, in particular, that

$$|h(z)| = 0(r^{\alpha}) \text{ for some } \alpha > \beta\ .$$

Let $u(z) = \rho(z, w(z))$, so we wish to study the set $u < 0$.

We will distinguish the three cases $\beta < 1/2$, $\beta = 1/2$, $\beta > 1/2$.

Case 1. $\beta < 1/2$.

With $z = re^{i\theta}$, $\pi/2 \leqq \theta \leqq 3\pi/2$, we claim that

$$\begin{aligned} u(z) &= r^{2\beta}[|a|^2 + 2|c|\cos(2\beta\theta - \theta_0)] + o(r^{2\beta}) \\ \frac{\partial u}{\partial\theta}(z) &= -4\beta|c|r^{2\beta}\sin(2\beta\theta - \theta_0) + o(r^{2\beta}) \\ \frac{\partial^2 u}{\partial\theta^2}(z) &= -8\beta^2|c|r^{2\beta}\cos(2\beta\theta - \theta_0) + o(r^{2\beta}) \end{aligned} \tag{3.8}$$

for some real number θ_0 and complex numbers $a \neq 0$ and c. The number $a \neq 0$ is a constant multiple of the number a of (3.7) while c is a multiple of the coefficient of w^2 in the Taylor series expansion of $\rho(z, w)$ about $(z, w) = (0, 0)$. The equations (3.8) follow from direct substitution of the formula (3.7) for $w(z)$ into the Taylor series expansion of ρ given by (3.1), and standard estimates for the remainder in Taylor's formula. We omit the calculations.

Next note that if $c = 0$, then $u(z) > 0$ for small $r > 0$, so $P_j(\delta)$ is empty for all small δ. We can therefore assume $c \neq 0$. We claim that

$$I_r = \{\theta\colon \pi/2 \leqq \theta \leqq 3\pi/2,\ u(re^{i\theta}) < 0\}$$

is an (possibly empty) open interval. This follows from a (slightly) tedious analysis of (3.8). For, from the last two equations of (3.8) we see that for sufficiently small r, on any interval of length $\pi/2\beta > \pi$, the function $\theta \longrightarrow u(re^{i\theta})$ either

(A) decreases to a minimum and increases thereafter; or

(B) increases to a maximum and decreases thereafter.

Thus, the only way I_r can fail to be an interval is to have $u(ir) < 0$, $u(-ir) < 0$ and $u(re^{i\theta}) \geqq 0$ for some θ, $\pi/2 < \theta < 3\pi/2$. This implies

$$\cos(\beta\pi - \theta_0) < -\frac{|a|^2}{2|c|} + o(1) < 0$$

$$\cos(3\beta\pi - \theta_0) < -\frac{|a|^2}{2c} + o(1) < 0$$

when r is small. Then, since the interval $\beta\pi - \theta_0 \leqq x \leqq 3\beta\pi - \theta_0$ has length $2\beta\pi < \pi$ and the cosine function is negative at both end points, it follows that it is negative on the entire interval, and in fact, smaller than the largest of the endpoint values. In this case, we then have $u(re^{i\theta}) < 0$ on the entire interval, for sufficiently small r. Thus, I_r, must be an interval.

It then follows from Lemma 3.1 that the set $P(\delta) = P_j(\delta)$ of (i) of the Lemma is either empty or has the property that it is an open set which meets every circle $|z| = r < \delta$ in an arc. In this latter case, it is clear that $P(\delta)$ is connected.

Part (ii) of the lemma follows from (3.8) in much the same way. The equations show that for sufficiently small $r > 0$, near any point $\theta \in (\pi/2, 3\pi/2)$ the function u is either strictly increasing, strictly decreasing, strictly concave, or strictly convex. Thus, near any point where $u(re^{i\theta}) = 0$, there are either points with $u < 0$ or else u has a strict relative maximum or minimum. The case of a relative maximum cannot occur, since then I_r would not be an interval. In the case of a relative minimum, we see from (3.8) that it must be an absolute minimum, so I_r is empty. Thus, (ii) follows. Note also that when I_r is not empty, we also have that

$$\frac{\partial}{\partial\theta} u(re^{i\theta}) \neq 0 \text{ at points } \theta \text{ with } u(re^{i\theta}) = 0.$$

We now prove part (iii). Let $\zeta \neq \zeta' \in \partial\Omega$ be a point near ζ with coordinates (a, b), $a \neq 0$. Assume that $P(\delta)$ is not empty and that $u(a) = 0$. As noted at the end of the last paragraph, $(\partial u)/(\partial\theta)(a) \neq 0$. Consider the factorization of f in the (z', w') coordinate system near ζ',

$$f(z', w') = \left[\prod_{k=1}^{m(\zeta')} (w' - w_k'(z'))\right] F'(z', w') .$$

Suppose by way of contradiction that $m(\zeta') \geqq 2$. Then $\partial f/\partial w' = 0$ at the point ζ'. However, since the zeroes $\{w_j(z)\}$ are all distinct, $\partial f/\partial w \neq 0$ at ζ', so $\nabla f \neq 0$ and $\partial f/\partial z' \neq 0$ at ζ. By the implicit function theorem, we can write the zeroes of f near ζ' as $z' = h(w') =$ const. $(w')^p +$ higher order terms. If $p \geqq 2$, the manifold $(h(w'), w')$ is tangent to $\partial\Omega$ at ζ', and so the restriction of ρ to this manifold must have a vanishing gradient. However, this restriction is just what we are calling $u(z)$, and we have already seen that $\partial u/\partial\theta \neq 0$, hence $\nabla u \neq 0$. Thus, $p \geqq 2$ does not occur, so, locally, w' is an analytic function of z' on $f = 0$. Thus, $m(\zeta') = 1$, as asserted. This completes the discussion of Case 1.

Case 2. $\beta = 1/2$.
Exactly as in Case 1, we find, with $a \neq 0$,

$$\begin{aligned} u(z) &= r[|a|^2 + 2|b|\cos(\theta - \theta_1)] + o(r) \\ \frac{\partial u}{\partial\theta}(z) &= -2|b| r \sin(\theta - \theta_1) + o(r) \qquad (3.9) \\ \frac{\partial^2 u}{\partial\theta^2}(z) &= -2|b| r\cos(\theta - \theta_1) + o(r) . \end{aligned}$$

The proofs of (i)-(iii) are then the same as in Case 1.

Case 3. $\beta > 1/2$.
Exactly as in Case 1, we find

$$\begin{aligned} u(z) &= 2\gamma r\cos\theta + o(r) \\ \frac{\partial u}{\partial\theta}(z) &= -2\gamma r\sin\theta + o(r) \qquad (3.10) \\ \frac{\partial^2 u}{\partial\theta^2}(z) &= -2\gamma r\cos\theta + o(r) \end{aligned}$$

and we can again proceed as in the earlier cases.

REMARK. In this last case, which is always the one if $m(\zeta) = 1$, we definitely have that $P(\delta')$ is not empty. The first two cases only occur when $\{f = 0\}$ is tangent to $\partial\Omega$ at ζ.

Proof of Lemma 1. Let $\zeta \in \partial\Omega$, $f(\zeta) = 0$. Choose $\delta = \delta(\zeta) > 0$ so small that the conclusions of Lemma 3.2 all hold. Then let U_ζ be the collection of all points with (z, w) coordinate satisfying $|z| < 1/2\,\delta(\zeta)$. By an abuse of notation, we will write

$$U_\zeta = \left\{(z, w): |z| < \frac{1}{2}\delta(\zeta)\right\}, \tag{3.11}$$

Similarly, let

$$W_\zeta = \{(z, w): |z| < \delta(\zeta)\} . \tag{3.12}$$

For $\delta < \delta(\zeta)$, let

$$X_j(\delta) = \{(z, w_j(z)) \in \Omega: \operatorname{Re} z < 0, |z| < \delta\} .$$

From (i) of Lemma (3.1), each $X_j(\delta)$ is empty or a connected variety in $\Omega \cap \{|z| < \delta, \operatorname{Re} z < 0\}$. Thus, if we let $S(\delta) = \{(z, w): \operatorname{Re} z < 0, |z| < \delta\}$, and $V_1 = \{f = 0)$, then

$$V_1 \cap \Omega \cap S(\delta) = \bigcup_{j=1}^{m} X_j(\delta)$$

and the latter union is the decomposition of V_1 into irreducible components in $\Omega \cap S(\delta)$. Since V is an irreducible component of $V_1 \cap \Omega$, if we put $J = J(\zeta) = \{j: 1 \leqq j \leqq m, X_j(\delta) \subset V, X_j(\delta) \neq \phi\}$. Then

$$V \cap S(\delta) = \bigcup_{j \in J} X_j(\delta) .$$

Thus, with $h_j(z, w) = w - w_j(z)$, we define

$$f_\zeta = \prod_{j \in J(\zeta)} h_j . \tag{3.13}$$

Now, conditions (i), (ii), (iv) and (v) of Lemma 1 hold by construction. We only have to check condition (iii). Thus, assume $\zeta, \zeta' \in \partial\Omega$, $f(\zeta) = f(\zeta') = 0$, and there is a point $p \in U_\zeta \cap U_{\zeta'} \cap \Omega$. We have to prove that $u = f_\zeta/f_{\zeta'}$, is holomorphic, nonvanishing, and $|u|$ is bounded above and away from zero on $U_\zeta \cap U_{\zeta'} \cap \Omega$. Actually, we will see that u is analytic and nonvanishing on the closure of $U_\zeta \cap U_{\zeta'} \cap \Omega$.

Now, on the set $W_\zeta \cap \Omega$, the function f_ζ satisfies $\nabla f_\zeta \neq 0$ on $f_\zeta = 0$. Thus, since f_ζ and $f_{\zeta'}$ have the same zero set on $W_\zeta \cap W_{\zeta'} \cap \Omega$, it follows that u is analytic and nonzero on $W_\zeta \cap W_{\zeta'} \cap \Omega$. To prove u and $1/u$ are bounded on $U_\zeta \cap U_{\zeta'} \cap \Omega$, we only have to prove u is bounded near each point q in the boundary of $U_\zeta \cap U_{\zeta'} \cap \Omega$. If $q \notin \partial\Omega$, this is clear since then $q \in W_\zeta \cap W_{\zeta'} \cap \Omega$, and u is analytic and nonvanishing at q. Thus, assume $q \in \partial\Omega$. We consider three cases.

Case 1. $q \in \partial\Omega$, $q \neq \zeta, \zeta'$.

In this case u is again analytic and nonzero on a neighborhood of q. For if, for example, $f_\zeta(q) = 0$, then $h_j(q) = 0$ for a unique

$j \in J(\zeta)$, because all the $\{w_j(z)\}$ are distinct. Then by (ii) of Lemma 3.2, we have that $q \in \bar{V}$. Thus, if $f_{\zeta'} = \prod_{k \in J(\zeta')} h'_k$ we have $h'_k(q) = 0$ for a unique $k \in J(\zeta')$. Because all the $\{w_j(z)\}$, $\{w'_k(z')\}$ are distinct, and because $\nabla h_j(q) \neq 0$, $\nabla h'_k(q) \neq 0$, we must have that h_j/h'_k is analytic and nonvanishing at q, and therefore also u since none of the other h_j, h'_k can vanish at q.

Case 2. $q = \zeta'$, $q \in \overline{U_\zeta \cap U_{\zeta'} \cap \Omega}$.

Since $f(\zeta') = 0$, there is a unique j such that $h_j(\zeta') = 0$. If $j \notin J(\zeta)$, then $\zeta' \notin \bar{V}$, so $f_{\zeta'} \equiv 1$ and $f_\zeta(\zeta') \neq 0$. Also, f_ζ is analytic on a neighborhood of ζ', so we are done. Thus, we may assume $j \in J(\zeta)$. Then by (iii) of Lemma 3.2 we have $m(\zeta') = 1$, so near ζ',

$$f = [w' - w'(z')] \cdot F'$$

where $w'(z')$, F' are analytic and $F' \neq 0$. Since only one of the h_j vanishes at ζ', we have for that j

$$h_j = [w' - w'(z')] \cdot G$$

where G is a nonvanishing analytic function near ζ'. Therefore also,

$$f_\zeta = [w' - w'(z')] \cdot H = f_{\zeta'} \cdot H$$

where H is analytic and nonvanishing at ζ', which proves this case.

Case 3. $q = \zeta$, $q \in \overline{U_\zeta \cap U_{\zeta'} \cap \Omega}$.

Same as Case 2.

This completes the proof of Theorem 1 for case (ii).

4. The case $\Omega = U^2$. This case is much the same as the earlier case, so we will not give many details. There is one new difficulty, however, which we will show how to avoid.

We assume that f is analytic and irreducible on a neighborhood of the closed unit polydisc and that f does not vanish on any line $z = a$, or $w = b$. Near a point (z_0, w_0) with $f(z_0, w_0) = 0$, we have, as in (3.4), the Puiseux expansions for the (multiple valued) solutions of $f(z, w) = 0$ near (z_0, w_0),

$$\begin{aligned} z - z_0 &= \sum_{j=1}^{\infty} a_j (w - w_0)^{j/p} = F((w - w_0)^{1/p}) \\ &= a_\alpha (w - w_0)^\alpha + \text{higher order terms}, \ a_\alpha \neq 0\,. \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} w - w_0 &= \sum_{i=1}^{\infty} b_i(z - z_0)^{i/q} = G((z - z_0)^{1/q}) \\ &= b_\beta(z - z_0)^\beta + \text{higher order terms}, \ b_\beta \neq 0 . \end{aligned} \tag{4.2}$$

For later reference, note that

$$\alpha\beta = 1 \tag{4.3}$$

since

$$\frac{|z - z_0|}{|w - w_0|^\alpha} \longrightarrow |a_\alpha| \neq 0 , \qquad \frac{|w - w_0|}{|z - z_0|^\beta} \longrightarrow |b_\beta| \neq 0 .$$

If $\zeta = (z_0, w_0) \in \partial U^2$, $f(\zeta) = 0$, we want to show how to define the functions f_ζ of Lemma 1. If ζ is not in the distinguished boundary, then there is no problem. To be specific, suppose $|z_0| < 1$. We have near (z_0, w_0) the factorization of f as in (3.3),

$$f(z, w) = \left[\prod_{j=1}^{m} h_j(z, w)\right] U(z, w)$$

where $U(z, w) \neq 0$ and h_j has the form

$$h_j(z, w) = (z - z_0) - F_j((w - w_0)^{1/q}) .$$

The zero set of $(h_j \cap U^2)$ is connected, so near (z_0, w_0), we can define

$$f_\zeta = \prod_{j \in J(\zeta)} h_j$$

where $J(\zeta)$ is the set of all j for which the zero set of h_j in U^2 is a subset of V.

When $\zeta = (z_0, w_0)$ has $|z_0| = |w_0| = 1$, there is again some difficulty in determining if $\{h_j = 0\} \cap U^2$ is connected. If

$$\begin{aligned} h_j(z, w) &= (z - z_0) - F_j((w - w_0)^{1/p}) \\ &= (z - z_0) - a_\alpha(w - w_0)^\alpha - \cdots \end{aligned}$$

then it is not hard to check, as in Lemma 3.2, that for small $\delta > 0$, the set

$$P_j(\delta) = \{w \colon |w - w_0| < \delta, \ |w| < 1, \ h_j(z, w) = 0\}$$

is connected provided that $\alpha < 2$. From equation (4.3), we see that $\alpha \cdot \beta = 1$. Thus, at least one of α, β is less than 2 (even 1). Therefore, to find f_ζ in this case we proceed as follows. First, factor f into irreducible factors near ζ,

$$f = \prod_{i=1}^{p} f_i$$

We then have the Puiseux expansions (4.1), (4.2) for the zeroes of each f_i near ζ. Further, the α, β are the same for all zeroes of f_i ([1], p. 22) although they could possibly be different for different factors f_i. The functions f_i can then be factored in the form

$$f_i = \left[\prod_{j=1}^{m} h_{i,j}\right] U_i$$

where $U_i \neq 0$ and

$$h_{i,j} = (z - z_0) - F_{i,j}((w - w_0)^{1/p}) \text{ if } \alpha < 2$$

or, if $\alpha \geqq 2$, take the factorization in the w-variable so that

$$h_{i,j} = (w - w_0) - G_{i,j}((z - z_0)^{1/q}) .$$

The functions $h_{i,j} \mid_{U^2}$ are then irreducible so f_ζ can be defined as

$$f_\zeta = \prod_i \Big(\prod_{j \in J_i(\zeta)} h_{i,j} \Big)$$

where $J_i(\zeta) = \{j\colon \text{zero set of } h_{i,j} \cap U^2 \subset V\}$. It can then be verified that these functions f_ζ will work for Lemma 1.

5. **Remarks.** It is possible to obtain a better conclusion in the Theorem than the result that V is defined by bounded functions. In fact, since the functions f_ζ which define V locally are Lipshitz continuous of some small order ε (i.e. $|f_\zeta(p) - f_\zeta(q)| \leqq C|p - q|^\varepsilon$, some $\varepsilon > 0$), we should be able to conclude that the function which defines V is also Lipshitz continuous of the same order. It is possible to show this is the case. In fact, in §§ 3 and 4, we actually showed that the quotients f_ζ/f_η are nonvanishing and analytic on the closure of $U_\zeta \cap U_\eta \cap \Omega$ for appropriate choices of the U_ζ. Thus, instead of using the Theorem of Stout or Range and Siu to carry out the patching arguments, one can explicitely carry out the patching arguments by taking logarithms and using the result that there are solutions of $\bar\partial u = f$ smooth up to the boundary if f itself is smooth up to the boundary and $\bar\partial f = 0$. (See [6] for this theorem in the case of the polydisc.)

We also note that the Theorem remains valid for strictly pseudoconvex sets Ω in $\boldsymbol{C}^2$ with real analytic boundaries and $H^2(\Omega, \boldsymbol{Z}) = 0$; since the only difficulties in the proof arise locally and, locally, a holomorphic change of coordinates can be made so that Ω is convex in the new coordinate.

Finally, it is a consequence of the patching arguments that the function F of the Theorem has the property that it locally generates the ideal of V, since it has the form $F = f_\zeta e^{a_\zeta}$ where a_ζ is an analytic function.

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