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ON TAME CANTOR SETS IN SPHERES HAVING THE SAME PROJECTION IN EACH DIRECTION

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The main result of this paper is that for each $n \geqq 2$ there exists in E^n a tame $(n-1)$-sphere S containing a twice tame Cantor subset C such that all projections of the two sets are the same. That is, if H_α is any $(n-1)$-dimensional linear hyperspace in E^n and $\pi_\alpha: E^n \to H_\alpha$ denotes the natural projection of E^n onto H_α, then for every α we have $\pi_\alpha(S) = \pi_\alpha(C)$. A number of interesting corollaries follow immediately from this result. One corollary is that there exists for each $n \geqq 2$ a countable collection of tame Cantor sets in E^n such that each straight line in E^n intersects a countable number of these Cantor sets.

1. Introduction and definitions. The main result (Corollaries 1 and 6) of this paper is that for each $n \geqq 2$ there exists in E^n a tame $(n-1)$-sphere S containing a twice tame Cantor subset C such that all projections of the two sets are the same (this is made precise in Theorem 1). This generalizes the results of **[5]**, where this result was obtained for $n = 3$ and 2. Our result is obtained by first showing in Theorem 1 that the above result holds for $n \geqq 2$ if we weaken the conclusion by only requiring that $S - C$ is locally flat, rather than S itself being tame. By **[3]**, an immediate corollary to Theorem 1 is the main result for $n \neq 3$ (Corollary 1). We obtain the result for $n = 3$ in Corollary 6 by applying **[6]** and showing that our construction, in fact, produces a 2-sphere having 1-ULC complementary domains. We note that our technique for constructing the desired spheres differs from what was done in **[5]** for $n = 3$. There, an ambient homeomorphism was also constructed along with the desired 2-sphere. The method for constructing the desired Cantor set C is simply a generalization to higher dimensions of the work done in **[5]**, which in turn is actually based on a clever idea of Borsuk in **[1]**. Interestingly enough, while the work needed to obtain the main result in **[5]** for $n = 3$ is almost as long and involved as our generalization here, an elementary eight line proof of the result when $n = 2$ is given in the last paragraph of this same paper. As in **[5]**, we also obtain a number of easy corollaries to our main result. One interesting corollary is that there exists for each $n \geqq 2$ a countable collection of tame Cantor sets in E^n such that each straight line in E^n intersects a countable number of these Cantor sets.

The organization of our papers is as follows. In the remainder of this section we give some standard definitions. In §2, we state our basic result, Theorem 1, and the main result for $n \neq 3$ as Corollary 1. We then note four additional corollaries and their short proofs, assuming the main result for $n \geqq 2$. In §3, we state two key theorems (Theorems 2 and 3) needed to prove Theorem 1, and then give a proof of Theorem 1 using these two results. In §4, we prove Theorem 2. In §5, we state and prove a result using a simple canonical geometrical construction which is then used in §6, along with Theorem 2, to prove Theorem 3. Finally, in Corollary 6, we show that if we make use of the particularly nice canonical construction suggested by the proof of Theorem 3 to obtain an example satisfying Theorem 1 when $n = 3$, then we in fact obtain a tame 2-sphere having the desired properties. We now give some definitions.

By a POLYHEDRON in E^n, we will mean a compact subset of E^n having the same point set as some finite rectilinear subcomplex K of E^n. Our REGULAR NEIGHBORHOODS will be standard as in [2]. We will call a k-sphere Σ (or a k-cell D) in E^n TAME if there exists an ambient homeomorphism carrying Σ (or D) onto a polyhedron in E^n. We say a Cantor set C in E^n (or S^n) is TAME if there exists a space homeomorphism carrying C into a subset of a line (or a circle). If C is a Cantor set in $S^k \subset E^n$, then we say that C is TWICE TAME in S^k if C is tame in both S^k and in E^n. If Σ is a tame k-sphere in E^n and H is a $(n-k-1)$-hyperplane in E^n, we say H LINKS Σ if $H \cap \Sigma = \phi$ and Σ is not homologous to zero in $E^n - H$. An open subset U of an $(n-1)$-sphere $S \subset E^n$ is LOCALLY FLAT if for any point $x \in U$ there exists an open subset W in E^n such that $(W, W \cap U)$ is homeomorphic, as pairs, to (E^n, E^{n-1}). If H_α is any $(n-1)$-dimensional linear hyperspace in E^n, let $\pi_\alpha : E^n \to H_\alpha$ denote the natural projection of E^n onto H_α. That is, if L is a straight line orthogonal to H_α, then $\pi_\alpha(L) = L \cap H_\alpha$. Finally, we give an important definition which will be used in one of our key theorems, Theorem 2. Given $X \subset E^n$ and $\epsilon > 0$, let $X^\epsilon = \{x \in X \mid d(x, \operatorname{Fr} X) \geqq \epsilon\}$. We note that X^ϵ can also be expressed as $\cup \{x \in X \mid N(x, \epsilon) \subset X\}$. Hence, X^ϵ is a closed subset of int X.

2. Main results.

THEOREM 1. *For each $n \geqq 2$, there exists in E^n an $(n-1)$-sphere S containing a twice tame Cantor subset C such that $S - C$ is locally flat and all projections of the two sets are the same. That is, for every $\pi_\alpha : E^n \to H_\alpha$, $\pi_\alpha(S) = \pi_\alpha(C)$.*

Applying [3], we immediately obtain the following result.

COROLLARY 1. *For each $n \geqq 4$ or $n = 2$, there exists in E^n a tame $(n-1)$-sphere S containing a twice tame Cantor subset C such that all projections of the sets are the same.*

For the remaining corollaries, we will assume our main result for all $n \geqq 2$. That is, we will assume Theorem 1 and Corollary 6, and hence have the analogous result to Corollary 1 for all $n \geqq 2$.

COROLLARY 2. *For each $n \geqq 3$, there exists in E^n a tame k-sphere Σ^k $(1 \leqq k \leqq n-2)$ which can not be linked by any $(n-k-1)$-hyperplane H.*

COROLLARY 3. *Let $n \geqq 2$, and let P be any subcomplex of E^{n-1} of dimension $\geqq 1$. Then there exists an embedding $h: P \to E^n$ and a Cantor subset C of P such that all projections of $h(P)$ and $h(C)$ are the same. Furthermore, we can pick h and C so that C is tame in P and in E^{n-1}, $h(C)$ is tame in E^n, and for each simplex σ of P, $h(\sigma)$ is tame in E^n.*

COROLLARY 4. *If we restrict the above two corollaries to the case where $n \geqq 3$, $k = n-2$, and P is an $(n-2)$-sphere, then for any tame codimension two knot $K^{n-2} \subset E^n$, we obtain the above two conclusions with $h(P) = \Sigma^{n-2}$ embedded equivalently to K.*

COROLLARY 5. *There exists a countable collection of tame Cantor sets in E^n $(n \geqq 2)$ such that each straight line interval in E^n intersects a countable number of these Cantor sets.*

Proof of 2. Let S and C be as in Corollary 1 (recall we are assuming Corollary 6 if $n = 3$). Since C is twice tame, there exists a homeomorphism $f: E^n \to E^n$ such that $f(S)$ is a polyhedron and $f(C)$ lies in a line segment in $f(S)$. Given k, $1 \leqq k \leqq n-2$, let $\hat{\Sigma}$ be a subpolyhedral k-sphere in $f(S)$ containing $f(C)$. Let $\Sigma = f^{-1}(\hat{\Sigma})$. Then $C \subset \Sigma \subset S$ and clearly Σ is tame. Suppose H is an $(n-k-1)$-hyperplane missing Σ. Since H then misses C and every projection of C and S are the same, H misses S. Since S is tame, $\overline{\operatorname{int} S}$ is an n-cell missing H. Therefore, Σ bounds a $(k+1)$-cell in $\overline{\operatorname{int} S}$ missing H, and H does not link Σ.

Proof of 3. Let S and f be as above. For notational purposes, denote the above C by $\hat{C}$. Since P has dimension $\geqq 1$, we can piecewise linearly embed P in $f(S)$ so that the image of some segment of P contains $f(\hat{C})$. Denote the embedding by $g: P \to f(S)$. Define $h: P \to E^n$ by $h = f^{-1} \circ g$. Let $C = h^{-1}(\hat{C})$. Since $\hat{C} \subset h(P) \subset S$ and $\pi_\alpha(S) = \pi_\alpha(\hat{C})$ for all α, we have $\pi_\alpha(h(P)) = \pi_\alpha(h(C))$. Clearly, the remaining conclusions also hold.

Proof of 4. Take P in the above proof to be a polyhedral $(n-2)$-disk D and let $g: D \to f(S)$ be the above map, where $g(D)$ lies in some $(n-1)$-simplex σ of $f(S)$. Given the knot K, in a neighborhood of σ in $\operatorname{int} f(S)$ attach a polyhedral $(n-2)$-disk E to $\operatorname{Bd} g(D)$ so that $E \cap f(S) = \operatorname{Bd} g(D)$ and $E \cup g(D)$ is a polyhedral $(n-2)$-sphere belonging to the same knot class as K. Let F be a homeomorphism of E^n onto itself carrying K onto $E \cup g(D)$. Define $h: K \to E^n$ by $h = f^{-1} \circ F | K$. The result now easily follows. We note, any line intersecting $h(K)$ also intersects S, since $h(K) \subset \overline{\operatorname{int} S}$. Therefore, $\pi_\alpha(h(K)) = \pi_\alpha(S)$ for all α.

Proof of 5. Let S and C be as in Corollary 1, and let $x_0 \in \operatorname{int} S$. Let Z denote the subset of E^n consisting of all points having rational coordinates. Given $z \in Z$, let $T_z: E^n \to E^n$ be the translation carrying x_0 to z. Then $\{T_z(C) | z \in Z\}$ is the desired countable collection of Cantor sets. Let $\epsilon > 0$ be so small that $N(x_0, \epsilon) \subset \operatorname{int} S$. Given a straight line L, pick $z \in Z$ so that $\operatorname{dist}(z, L) < \epsilon$. Then $T_z(N(x, \epsilon)) \cap L \neq \phi$, and hence $L \cap T_z(S) \neq \phi$. But this implies that $L \cap T_z(C) \neq \phi$. Clearly, for any L, there are a countable number of such z's.

3. Proof of Theorem 1. The idea for the following theorem resulted from trying to understand the basic lemmas of [5] so as to isolate and state precisely the key ingredient which would allow us to generalize the results of [5] to higher dimensions.

THEOREM 2. *Suppose δ is a positive number, D is a polyhedral n-cell in E^n $(n \geqq 2)$, and $\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_m$ is a finite collection of polyhedral n-cells having disjoint interiors such that $\bigcup_{j=1}^m \tilde{B}_j = D$, and for each $j = 1, \cdots, m$,* $\operatorname{diam} \tilde{B}_j < \delta$. *If ϵ and η are positive numbers, T is a* (rectilinear) *triangulation of D containing each $\tilde{B}_j$ as a subcomplex, Z_1 is a compact subset of D missing the $(n-2)$-skeleton of T, and Z_2 is a compact subset of D missing* $\operatorname{Bd} D$, *then there exist a finite collection, $B_1, B_2, \cdots, B_m$ of polyhedral n-cells and a* PL *η-homeomorphism h of D onto itself such that*

(i) *h carries each $\tilde{B}_j$ onto B_j and is the identity on $Z_1 \cup Z_2 \cup$* $\operatorname{Bd} D$ *and) n all $\tilde{B}_j$ missing* $\operatorname{Bd} D$,

(ii) $\bigcup_{j=1}^m B_j = D$,

(iii) $\operatorname{diam} B_j < \delta$, *for $j = 1, 2, \cdots, m$, and*

(iv) *there exists a $\gamma > 0$ such that $\bigcup_{j=1}^m (B_j^\gamma)$ intersects any straight line intersecting D^ϵ. In fact, any straight line intersecting D^ϵ intersects some B_j^γ where $B_j \cap \operatorname{Bd} D \neq \phi$.*

To prove Theorem 1, we will start with the rectilinear $(n-1)$-sphere $\operatorname{Bd}([-1, 1]^n)$ in E^n. We then will define a sequence of PL homeomorphisms each modifying in elementary canonical fashion the previously given polyhedral $(n-1)$-sphere having certain "nice" properties into a new polyhedral $(n-1)$-sphere having similar "nice" properties. An application of Theorem 2 and then a sequence of applications of Theorem 3 (whose proof also will make use of Theorem 2) will then give us our desired Cantor set and $(n-1)$-sphere. The next result, Theorem 3, describes what "nice" is and how we do these modifications. Before stating this result, we first need two simple definitions.

Suppose F is a polyhedral n-cell in E^n and G is a subpolyhedron of F. We will call the pair (F, G) a codimension-one cell-cell pair if (F, G) is PL homeomorphic, as pairs, to $([-1,1]^{n-1}\times[-1,1],[-1,1]^{n-1}\times 0)$. We will call the pair (F, G) a codimension-one produce cell-sphere pair if (F, G) is PL homeomorphism, as pairs, to

$$([-2,2]^{n-1}\times[-1,1],\ \mathrm{Bd}([-1,1]^{n-1})\times[-1,1]).$$

Since we will only be considering codimension-one pairs here, for notational purposes we will drop the "codimension-one" and simply denote each of the above pairs as a c.c. pair or a p.c.s. pair. The dimension n will always be clear from the context.

THEOREM 3. *Let S be a polyhedral $(n-1)$-sphere in E^n $(n \geqq 2)$, X a compact subset of E^n containing S in its interior, ϵ' is a positive number, $B_1, B_2, \cdots, B_m$ a disjoint collection of polyhedral n-cells, and $G_1, G_2, \cdots, G_r$ a disjoint collection of polyhedral $(n-1)$-cells in S such that*

(1) *$S - \bigcup_{i=1}^r \operatorname{int} G_i$ misses $\bigcup_{j=1}^m B_j$,*

(2) *for each j, B_j intersects one and only one G_i, each G_i is interesected by some B_j, and if $B_j \cap G_i \neq \phi$, then $(B_j, B_j \cap G_i)$ is a* c.c. *or* p.c.s. *pair, and*

(3) *any straight line intersecting X also intersects some $B_j^{\epsilon'}$.*

Then, given δ_1 and $\delta_2 > 0$ and $0 < \eta < \operatorname{dist}(S, E^n - X)$, there exist

(4) *positive number γ and ϵ, with $0 < \epsilon \leqq \epsilon'$,*

(5) *for each $j = 1, 2, \cdots, m$, a finite collection of disjoint polyhedral n-cells $B_j, B_{j2}, \cdots, B_{jm_j}$ in $\operatorname{int} B_j$,*

(6) *for each $i = 1, 2, \cdots, r$, a finite collection of disjoint polyhedral $(n-1)$-cells $G_{i1}, G_{i2}, \cdots, G_{ir_i}$ in $\operatorname{int} G_i$, and*

(7) *a* PL *homeomorphism $f\colon E^n \to E^n$*

such that

(8) *$\operatorname{diam} B_{jk} < \delta_1$ and $\operatorname{diam} G_{il} < \delta_2$,*

(9) *$f = id$ on $(E^n - \bigcup_{j=1}^m B_j) \supset S - \bigcup_{i=1}^r \operatorname{int} G_i$ and moves points less than η on $\bigcup_{i=1}^r G_i - \bigcup_{i=1}^r \bigcup_{l=1}^{r_i} \operatorname{int} G_{il}$,*

(10) *$f(S - \bigcup_{i=1}^r \bigcup_{l=1}^{r_i} G_{il})$ misses $\bigcup_{j=1}^m \bigcup_{k=1}^{m_j} B_{jk}$,*

(11) *for each j, k, B_{jk} intersects one and only one $f(G_{il})$, each $f(G_{il})$ is intersected by some B_{jk}, and*
each such nonempty intersection is either a c.c. *or* p.c.s. *pair,*

(12) *for each $G_{ij} \subset \operatorname{int} G_i$, there exists an integer k such that G_{ij} and $f(G_{ij})$ each lie in* $\operatorname{int} B_k$,

(13) $f(S) \subset \operatorname{int}[X \cup (\bigcup_{j=1}^m B_j^\epsilon)]$ (recall $0 < \epsilon \leqq \epsilon'$),

(14) *any straight line intersecting $\bigcup_{j=1}^m B_j^\epsilon$ also intersects $\bigcup_{i=1}^m \bigcup_{k=1}^{m_j} B_{jk}^\gamma$, and*

(15) *any straight line intersecting $X \cup (\bigcup_{j=1}^m B_j^\epsilon)$ intersects some B_{jk}^γ.* (Hence any straight line interesting $f(S)$ also intersects some B_{jk}^γ.)

We now give a proof of Theorem 1 assuming Theorems 2 and 3. Theorem 2 will be proved in §4 and Theorem 3 in §6.

Proof of Theorem 1. Let $S_0 = \operatorname{Bd}([-1,1]^n)$ and let $B_0 = [-2,2]^n$. Let T_0 be any rectilinear triangulation of B_0 obtained by first triangulating $\operatorname{Bd} B_0$ and then letting simplexes of T_0 be those in $\operatorname{Bd} B_0$, the origin, and those of the form {the origin joined with a simplex in $\operatorname{Bd} B_0$}. We note, each n-simplex Δ of T_0 intersects S_0 and $(\Delta, \Delta \cap S_0)$ is a c.c. pair. Let $\Delta_1, \cdots, \Delta_m$ denote the n-simplexes of T_0. Let ϵ_0 and η each equal $\frac{1}{4}$. We now apply Theorem 2 using $\delta = 3$, $D = B_0$, $\tilde{B}_i = \Delta_i$, $\epsilon = \epsilon_0 = \frac{1}{4}$, $\eta = \frac{1}{4}$, $T = T_0$, $Z_1 = \phi$ and $Z_2 = [\frac{-3}{2}, \frac{3}{2}]^n$. Let $B_1, B_2, \cdots, B_m$ be the collection of polyhedral n-cells promised by Theorem 2. Since $S_0 \subset Z_2$, it follows that $B_j \cap S_0 = \Delta_j \cap S_0$ for each i. Hence, each B_j intersects S_0 and $(B_j, B_j \cap S_0)$ is a c.c. pair. Also, since each Δ_j has an $(n-1)$-face in $\operatorname{Bd} D$, each B_j intersects $\operatorname{Bd} D$. By Theorem 2, (iv) there exists an $\tilde{\epsilon}_1$ ($= \gamma$ there) such that $\bigcup_{j=1}^m B_{j1}^{\tilde{\epsilon}}$ intersects any straight line intersecting $B_0^{\epsilon_0}$. Let $X_1 = B_0^{\epsilon_0}$ and let $B_{11}, B_{12}, \cdots, B_{1m_1}$ be polyhedral n-cells in the interior of $B_1, B_2, \cdots, B_m$, respectively, such that $B_{j1}^{\tilde{\epsilon}} \subset \operatorname{int} B_{ij}$ and $(B_{ij}, B_{ij} \cap S_0)$ is a c.c. pair. (This just requires a small shrinking of each B_i using the collar structure of the c.c. pair.) The B_{ij}'s are now disjoint. Choose $\epsilon_1' > 0$ so that $B_j^{\epsilon_1'} \supset B_j^{\tilde{\epsilon}_1}$. Also, let $G_{11}, G_{12}, \cdots, G_{1m}$ be disjoint $(n-1)$-cells in S_0 so that $\operatorname{int} G_{1j} \supset B_{1j} \cap S_0$. Hence $S_0 - \bigcup_{j=1}^m G_{1j}$ misses $\bigcup_{j=1}^m B_{1j}$. For notational purposes, we now denote S_0 by S_1. We observe that the collection S_1, X_1, ϵ_1', $\{B_{1j}\}$ and $\{G_{1j}\}$ now satisfy the hypothesis of Theorem 3.

Inductively, suppose that for some fixed $i_0 \geqq 1$, we have for each $1 \leqq i \leqq i_0$ a collection S_i, X_i, ϵ_i', $\{B_{ij}\}$ and $\{G_{ij}\}$ satisfying the hypothesis of Theorem 3, and a PL homeomorphism $f_{i-1}: E^n \to E^n$ such that $f_{i-1}(S_{i-1}) = S_i$, $\operatorname{diam} B_{ij} < \dfrac{3}{2^{i-1}}$, and $\operatorname{diam}[(f_{i-1} \circ \cdots \circ f_1 \circ f_0)^{-1}(G_{ij})] < 2/2^{i-1}$. Also, for $1 < i \leqq i_0$, we suppose that

$$\cup f_{i-1}^{-1}(G_{ij}) \subset \cup \operatorname{int} G_{i-1,j}, \quad \cup B_{ij} \subseteq \cup \operatorname{int} B_{i-1,j}, \quad X_i = X_1 \cup \left[\bigcup_{k=1}^{i-1} \bigcup_{i=1}^{m_k} B_{kj}^{\epsilon_k} \right]$$

for some $0 < \epsilon_k \leqq \epsilon'_k$, $f_{i-1} = id$ on

$$\left(E^n - \bigcup_{j=1}^{m_{i-1}} B_{i-1,j}\right) \supset \left(S_{i-1} - \bigcap_{j=1}^{r_j} \operatorname{int} G_{i-1,j}\right),$$

and for each G_{ij}, there exist integers k and l so that $f_{i-1}^{-1}(G_{ij})$ lies in $(\operatorname{int} G_{i-1,k}) \subset (\operatorname{int} B_{i-1,l})$ and $G_{ij} \subset \operatorname{int} B_{i-1,l}$.

We observe, if we let $f_0 =$ identity, then we have our inductive step for $i_0 = 1$ (the "also" part is vacuous for $i_0 = 1$). Now suppose we have our inductive hypothesis for $i_0 = p$. We will show that the inductive hypothesis holds for $i_0 = p + 1$. Suppose we now have a collection S_p, X_p, ϵ'_p, $\{B_{pj}\}_{j=1}^{m_p}$ and $\{G_{pi}\}_{i=1}^{r_p}$ satisfying the hypothesis of Theorem 3. Let $\delta_1 = 3/2^p$ and choose δ_2 so small that for any subset $A \subset S_p$ of diameter $< \delta_2$, we have $\operatorname{diam}[(f_{p-1} \circ \cdots \circ f_1 \circ f_0)^{-1}(A)] < 2/2^p$. We now apply Theorem 3 using these δ's. The γ of Theorem 3, (4) becomes our new ϵ'_{p+1} and the ϵ our new ϵ_p. The n-cells of (5) become our collection $\{B_{p+1,j}\}_{j=1}^{m_{p+1}}$ and the concluded f of (7) is our desired f_p. We let $S_{p+1} = f_p(S_p)$ and the images of the $(n-1)$-cells of (6) under f_p become our collection $\{G_{p+1,i}\}_{i=1}^{r_{p+1}}$. Clearly the diameters of our new B's and G's are of the appropriate size. Clearly (5) and (6) of Theorem 3 give us the first two requirements of the "also" part of our induction hypothesis for $i = p + 1$. We let $X_{p+1} = X_p \cup (\bigcup_{j=1}^{m_p} B_{pj}^{\epsilon_p})$. This then satisfies the next statement of the "also" part and by (13) of Theorem 3, $S_{p+1} \subset \operatorname{int} X_{p+1}$. By (15) of that theorem, any straight line intersecting X_{p+1} also intersects some $B_{p+1,j}^{\epsilon'_{p+1}}$. Adding conclusions (10) and (11), we see that our new collection S_{p+1}, X_{p+1}, ϵ'_{p+1}, $\{B_{p+1,j}\}$ and $\{G_{p+1,j}\}$ also satisfies the hypothesis of Theorem 3. Conclusions (9) and (12) of Theorem 3 give us the final two statements of the "also" part of our inductive hypothesis for $i = p + 1$.

Hence, inductively, we obtain for each $i \geqq 1$ a PL homeomorphism $f_{i-1}: E^n \to E^n$ such that $f_{i-1}(S_{i-1}) = S_i$, with each f_{i-1} modifying the previous S_{i-1} as described above in the inductive hypothesis. For each $i \geqq 1$, let $h_i = f_i \circ \cdots \circ f_1 | S_1$; recall $S_1 = \operatorname{Bd}([-1, 1]^n)$. We ignore f_0, since it is the identity. Since the diameters of our B_{ij}'s are less than $3/2^{i-1}$, and $f_i = id$ outside the B_{ij}'s, we have that $d(h_i(x), h_{i-1}(x)) < 3/2^{i-1}$ for all $x \in S_1$. Hence $h = \lim_{i\to\infty} h_i$ is a continuous function carrying S_1 into E^n.

We recall that f_i only modifies S_i on the union of the G_{ij}'s. Hence, if we let $H_{ij} = h_{i-1}^{-1}(G_{ij})$ for all i and j, then for $k \geqq i$ we have that $h_k = h_{i-1}$ on $S_1 - \bigcup_{j=1}^{r_i} H_{ij}$. We recall that in defining f_i we picked the $G_{i+1,j}$'s so that $\operatorname{diam} H_{i+1,j} < 2/2^i$ for all i and j. Since for each i, the collection $\{H_{ij}\}$ is a disjoint collection of polyhedral $(n-1)$-cells in S_1 and $\bigcup H_{i+1,j} \subset \bigcup \operatorname{int} H_{ij}$ (by the first part of the "also" inductive hypothesis) it follows that $C^* = \bigcap_{i=1}^{\infty} \bigcup_j H_{ij}$ is a tame Cantor set in S_1

[4]. By similar reasoning, $C = \bigcap_{i=1}^{\infty} \bigcup_j B_{ij}$ is also a tame Cantor set in E^n.

We now claim that h is 1–1 on S_1 and that $h(C^*) = C$. We already have observed that h is 1–1 on $S_1 - C^*$, since each h_i is a homeomorphism and for any $x \in S_1 - C^*$, there exists an i such that $h = h_i$ in a neighborhood of x. Also, since $S_i - \bigcup G_{ij}$ misses $\bigcup B_{ij}$, for every i, no $x \in S_1 - C^*$ is carried to a point of C. It remains to show that $h \mid C^*$ is a homeomorphism of C^* onto C. Given any G_{ij}, there exist integers k and l so that $f_{i-1}^{-1}(G_{ij}) \subset (\operatorname{int} G_{i-1,k}) \cap (\operatorname{int} B_{i-1,l})$ and $G_{ij} \subset \operatorname{int} B_{i-1,l}$. Therefore $h_{i-1}(H_{ij}) \subset \operatorname{int} B_{i-1,l}$. Since $h_k = id$ outside $\bigcup B_{i-1,j}$ for $k \geqq i-1$, it follows that $h(H_{ij}) \subset B_{i-1,l}$. Similarly, given $H_{i+1,k} \subset H_{ij}$, there exists a $B_{i,s}$ such that $h(H_{i+1,j}) \subset B_{i,s}$. Since h is continuous, $B_{i,s} \subset B_{i-1,l}$. Since each point of C^* is the intersection of H_{ij}'s $i = 1, 2, \cdots$, and each point of C is the intersection of B_{ij_i}'s, it follows that for every $x \in C^*$, $h(x) \in C$. We see that h is 1–1 on C^* because each B_{ij} intersects one and only one G_{ik}. $h \mid C^*$ is onto because each G_{ik} is intersected by some B_{ij}. Another way to observe that $h \mid C^*$ is onto is that clearly $h(S_1) \supset C$ and no point of $S_1 - C^*$ is carried to C.

We now claim that $h(S_1) = S \supset C$ satisfies Theorem 1. Since $h(C^*) = C$, C is twice tame in S. To prove that all projections of C and S are the same, it suffices to show that any straight line L intersecting S also intersects C. Hence, suppose we are given L, such that $L \cap S \neq \phi$. If L does not intersect C then for some large i, L misses $\bigcup_{j=1}^{m_i} B_{ij}$. Since all further modifications of S_i occur in $\bigcup_{j=1}^{m_i} B_{ij}$ and $L \cap S \neq \phi$, it follows that $L \cap S_i \neq \phi$. Now $S_i \subset X_i$ and any line intersecting X_i also intersects some $B_{ij}^{\epsilon_j} \subset B_{ij}$. This contradiction shows that $L \cap C \neq \phi$, and this completes the proof of Theorem 1.

4. *Proof of Theorem* 2. Given D and $\bigcup_{j=1}^{m} \tilde{B}_j = D$ such that $\operatorname{diam} \tilde{B}_j < \delta$, suppose T is a triangulation of D containing each $\tilde{B}_j$ as a subcomplex and Z_1 is a compact subset of D missing the $(n-2)$-skeleton of T. Given ϵ and $\eta > 0$, we may suppose by taking smaller numbers if necessary (certainly, if (iv) holds for a smaller ϵ', it will hold for the given one—since if $0 < \epsilon' \leqq \epsilon$, then $D^{\epsilon} \subset D^{\epsilon'}$) that $D^{\epsilon} \supset Z_2$ $D^{\epsilon} \supset Z_2 \cup \{\tilde{B}_j \mid \tilde{B}_j \cap \operatorname{Bd} D = \phi\} \cup \{\text{simplexes } \Delta \in T \mid \Delta \cap \operatorname{Bd} D = \phi\}$ and that $\eta < 1/4 \ \min(\{\delta - \operatorname{diam} \tilde{B}_j \mid j = 1, 2, \cdots, m\},\ \operatorname{dist}(Z_1, T^{(n-2)}))$, where $T^{(n-2)}$ is the $(n-2)$-skeleton of T. Since T contains each $\tilde{B}_j$ as a subcomplex and $\bigcup_{j=1}^{m} \tilde{B}_j = D$, it follows that the mesh of $T < \delta$.

Let $\Delta_1, \cdots, \Delta_r$ be the n-simplexes of T, ordered so that the n-simplexes of T having faces in $\operatorname{Bd} D$ appear first in this ordering. That is, suppose each of $\Delta_1, \cdots, \Delta_t$ intersects $\operatorname{Bd} D$, while each of $\Delta_{t+1}, \cdots, \Delta_r$ lie in $\operatorname{int} D$. By our assumption on ϵ, $\Delta_{t+1} \cup \cdots \cup \Delta_r \subset D^{\epsilon}$ and given any Δ_i $(1 \leqq i \leqq t)$, then each face of Δ_i missing $\operatorname{Bd} D$ also lies in D^{ϵ}. We will now consider each Δ_1 through Δ_t in turn, and construct small "ridges" on them so that if R_i denotes the ridge constructed on Δ_i, then $\Delta_i \cup R_i$ is a polyhedral n-cell, each R_i misses $D^\epsilon \cup \mathrm{Bd}\, D \cup Z_1$ (hence misses Z_2, since $Z_2 \subset D^\epsilon$), the R_i's are disjoint, and $R_i \cap \mathrm{Bd}\,\Delta_i$ separates $\mathrm{Bd}\,\Delta_i \cap D^\epsilon$ from $\mathrm{Bd}\,\Delta_i \cap \mathrm{Bd}\, D$ (in $\mathrm{Bd}\,\Delta_i$). Also, each ridge R_i will be so small that if σ is any simplex of T missing Δ_i, then $R_i \cap \sigma = \phi$. Actually, each R_i will be PL equivalent to $N_i \times [-\frac{1}{2}, \frac{1}{2}] \times [0, \frac{1}{2}]$, where N_i is a polyhedral $(n-2)$-manifold in $\mathrm{Bd}\,\Delta_i$ separating $\mathrm{Bd}\,\Delta_i \cap D^\epsilon$ from $\mathrm{Bd}\,\Delta_i \cap \mathrm{Bd}\, D$ and $R_i \cap \Delta_i \subset \mathrm{Bd}\,\Delta_i$ will correspond to $N_i \times [0,1] \times 0$. Since all the ridges will be of this form and will be disjoint and will all miss $Z_1 \cup \mathrm{Bd}\, D \cup D^\epsilon$, we can construct for each $i = 1, 2, , \cdots, t$ PL η-homeomorphism h_i taking D onto itself such that h_i only moves points in a small neighborhood W_i of R_i and takes Δ_i onto $\Delta_i \cup R_i$. The W_i's will be small enough neighborhoods so as to be disjoint and also miss $Z_1 \cup \mathrm{Bd}\, D \cup D^\epsilon$. The desired h will then be $h_t \circ h_{t-1} \circ \cdots \circ h_1$ and B_i will be defined by $B_i = h(\tilde{B}_i)$. Clearly, h is a PL η-homeomorphism and conclusions (i), (ii) and (iii) follow immediately.

We first consider Δ_1. We know $\Delta_1 \cap \mathrm{Bd}\, D \neq \phi$. We can assume for each i, $1 \leqq i \leqq t$, that $\mathrm{Bd}\,\Delta_i \neq \mathrm{Bd}\, D$. Otherwise $\Delta_1 = D = \tilde{B}_1$ and we can take $h = id$ and $\gamma = \epsilon$. Let M_1 be a "small" regular neighborhood of $\Delta_1 \cap \mathrm{Bd}\, D$ in $\mathrm{Bd}\,\Delta_1$ so that M_1 misses $\Delta_1 \cap D^\epsilon$ and so that the various components of M_1 correspond to regular neighborhoods of the various components of $\Delta_1 \cap \mathrm{Bd}\, D$. Also, since Z_1 misses the $(n-2)$-skeleton of T, if we take a small enough regular neighborhood so that $\mathrm{Bd}\, M_1$ lies in a small neighborhood of $T^{(n-2)}$, we can suppose $\mathrm{Bd}\, M_1 = N_1$ misses Z_1. We now simply repeat the procedure for each Δ_i $(i = 2, 3, \cdots, t)$ in turn, taking smaller and smaller regular neighborhoods each time to insure that the N_i's ($= \mathrm{Bd}\, M_i$'s) are disjoint. (We can think of the N_i's as lying in smaller and smaller neighborhoods of $\mathrm{Bd}\, D$ as i increases. That is, we can think of M_1 as being the simplicial neighborhood of $\Delta_1 \cap \mathrm{Bd}\, D$ in $\mathrm{Bd}\,\Delta_1$ under some sufficiently high barycentric subdivision of T, and the consecutive M_i's will be similar except each lying in higher and higher barycentric subdivisions of T.)

Now let each W_i be an appropriately "small" regular neighborhood of N_i in $\mathrm{int}\, D$ under some sufficiently high further barycentric subdivision of T. By small we mean small enough so that the W_i's are disjoint (components of each W_i correspond to components of each N_i), $\cup W_i$ misses $D^\epsilon \cup \mathrm{Bd}\, D \cup Z_1$, and $\Delta_i \cup W_i$ lies in an η-neighborhood of Δ_i. It follows easily from standard PL theory ([2]) that each $W_i \approx N_i \times [-1,1]^2$ with $W_i \cap \Delta_i \approx N_i \times [-1,1] \times [-1,0]$, $W_i \cap \mathrm{ext}\,\Delta_i \approx N_i \times [-1,1] \times [0,1]$, and $W_i \cap \mathrm{Bd}\,\Delta_i \approx N_i \times [-1,1] \times 0$. Since the PL equivalence can be obtained carrying $x \in N_i$ to $(x, 0, 0) \in N_i \times [-1,1]^2$, by taking a smaller W_i, if necessary, we can suppose that the preimage of $x \times [-1,1]^2$ lies in a η-neighborhood of x in $\mathrm{int}\, D$ for each $x \in N_i$. The required $R_i \subset \mathrm{int}\, W_i$ will then be the regular neighborhood of

N_i in $\operatorname{ext}\Delta_i \cap \operatorname{int} W_i$ corresponding to $N_i \times [-\frac{1}{2}, \frac{1}{2}] \times [0, \frac{1}{2}] \subset N_i \times [-1, 1]^2$. To define the desired η-homeomorphism $h_i: D \to D$ keeping points of $D - \operatorname{int} W_i$ fixed and carrying Δ_i onto $\Delta_i \cup R_i$ is now elementary.

As we have already noted, our PL homeomorphism h is $h_t \circ h_{t-1} \circ \cdots \circ h_1$. We observe that h only moves points in simplexes intersecting $\operatorname{Bd} D$ and given any Δ_i, $1 \leqq i \leqq t$, $h(\Delta_i) \supset R_i \cup (\Delta_i - [\text{small nbd. of } \cup_{j=1\, i \neq j}^{t} R_j])$, and for $i > t$, $h(\Delta_i) = \Delta_i$.

It now remains to prove cconclusion (iv). We claim that any straight line L intersecting D^ϵ intersects the interior of some $B_i = h(\tilde{B}_i)$. Since each $\tilde{B}_i$ is the union of certain Δ_j's, it suffices to show that L intersects the interior of some $h(\Delta_j)$. In fact, we will show that L intersects the interior of some $h(\Delta_j)$ where $h(\Delta_j) \cap \operatorname{Bd} D = \Delta_j \cap \operatorname{Bd} D \neq \phi$. Given L, there exists a $p \in \operatorname{Bd} D^\epsilon$ and a $p' \in \operatorname{Bd} D$ such that the open segment U of L between p and p' misses $D^\epsilon \cup \operatorname{Bd} D$. If U lies in no proper face of any n-simplex Δ_i of T, then for some point p'' of U near p', we could have the open segment V of L between p'' and p' lying in the interior of some Δ_i. If we pick p'' close enough to p' so as to miss all the ridges and their neighborhoods, the W_i's, then $V \subset h(\operatorname{int}\Delta_i)$. If U lies in some proper face of some Δ_i, then since N_i separates p from p', there exists a point $\hat{p}$ of L in N_i. But then, $\hat{p} \in h(\operatorname{int}\Delta_i)$, since $N_i \subset h(\operatorname{int}\Delta_i)$.

To complete the proof now, we make use of the following argument given in [1], p. 274. Let $\tilde{\Lambda}$ denote the space of all lines L in E^n. That is, points of $\tilde{\Lambda}$ are straight lines in E^n. The topology of $\tilde{\Lambda}$ is generated by the following basis. Given $L \in \tilde{\Lambda}$, a finite collection $p_1, p_2, \cdots, p_m$ of points of L, and $\epsilon_1, \epsilon_2, \cdots, \epsilon_m > 0$, let $N(L, (p_1, p_2, \cdots, p_m), (\epsilon_1, \cdots, \epsilon_m)\}$ be the set of all lines $\tilde{L}$ in E^n such that for each i there exists $\tilde{p}_i \in \tilde{L}$ with $\operatorname{dist}(p_i, \tilde{p}_i) < \epsilon_i$. This has a subbasis of the form $\{N(L, p, \epsilon) \mid \in \tilde{\Lambda}, p \in L, \text{ and } \epsilon > 0\}$. For given any $U = N(L, (p_1, \cdots, p_m), (\epsilon_1, \cdots, \epsilon_m))$ and $\tilde{L} \in U$, then $\bigcap_{i=1}^{m} N(\tilde{L}, \tilde{p}_i, \tilde{\epsilon}_i) \subset U$, where $\operatorname{dist}(p_i, \tilde{p}_i) = \delta_i < \epsilon_i$ and $\tilde{\epsilon}_i \leqq \epsilon_i - \delta_i$. We note $\tilde{\Lambda}$ also has a basis of the form $\{N(L(r, s), (\epsilon, \epsilon))\}$ where for each $L \in \tilde{\Lambda}$ we used fixed points $r \neq s \in L$. For given any U as above and $\tilde{L} \in U$, we consider $N(\tilde{L}, \tilde{p}_i, \tilde{\epsilon}_i)$ $i = 1, 2, \cdots, m$ where $\bigcap_{i=1}^{m} N(\tilde{L}, \tilde{p}_i, \tilde{\epsilon}_i) \subset U$. Let r and s be any two distinct points of $\tilde{L}$. For each i, there exists a $\delta_i > 0$ so that any $\hat{L} \in N(\tilde{L}, (r, s), (\delta_i, \delta_i))$ intersects the $\tilde{\epsilon}_i$-neighborhood of $\tilde{p}_i$ in E^n. Hence, if $\epsilon = \min\{\delta_i \mid i = 1, 2, \cdots, m\}$, then

$$N(\tilde{L}, (r, s), (\epsilon, \epsilon)) \subset \bigcap_{i=1}^{m} N(\tilde{L}, \tilde{p}_i, \tilde{\epsilon}_i) \subset U.$$

Let Λ denote the subspace of $\tilde{\Lambda}$ consisting of all lines L in E^n intersecting D^ϵ. For each point $p \in \operatorname{Bd} D^\epsilon$ and $p' \in \operatorname{Bd} D$, let $L(p, p')$ denote the straight line in E^n containing p and p'. Let $F: \operatorname{Bd} D^\epsilon \times$

$\operatorname{Bd} D \to \Lambda$ be defined by $F(p,p') = L(p,p')$. Since $\operatorname{dist}(p,p') \geqq \epsilon$, $L(p,p')$ depends continuously on (p,p'). Since $\operatorname{Bd} D^{\epsilon} \times \operatorname{Bd} D$ is compact and F carries this space continuously onto Λ, Λ is compact. Now for each $L \in \Lambda$, let $r(L) = \sup\{s \mid N(x,s) \subset \text{some } B_i, \text{ with } B_i \cap \operatorname{Bd} D \neq \phi, \text{ and } x \in L\}$. By the above paragraph, $r(L) > 0$ for all $L \in \Lambda$. Since $r(L)$ depends continuously on L, there exists a γ_0 such that $r(L) \geqq \gamma_0$ for every $L \in \Lambda$. But then, given any γ, $0 < \gamma < \gamma_0$, it follows that if $L \in \Lambda$, then L meets $\bigcup_{i=1}^{m} B_i^{\gamma}$. This completes the proof of Theorem 2.

5. A geometrical construction. In order to state our next result, we first need some notation. Let rD^n be the polyhedral n-cell in E^n defined by $rD^n = [-r,r]^n$ and let Σ^{n-1} denote the polyhedral $(n-1)$-sphere in E^n bounding $D^n = [-1,1]^n$. For a given $n \geqq 2$, we are going to want to again consider c.c. and p.c.s. polyhedral pairs. Recall these are polyhedral pairs PL homeomorphic to (D^n, D^{n-1}) or to $(2D^{n-1} \times [-1,1], \Sigma^{n-1} \times [-1,1])$, respectively. Those of the first type divide naturally into two polyhedral n-cells and those of the second type divide naturally into three polyhedral n-cells as follws. We express D^n as $D_+^n \cup D_-^n$, where $D_+^n = D^{n-1} \times [0,1]$ and $D_-^n = D^{n-1} \times [-1,0]$. We express $2D^{n-1} \times [-1,1]$ as $F_0 \cup F_+ \cup F_-$, where $F_0 = D^{n-1} \times [-1,1]$ and $F_\alpha = (2D_\alpha^{n-1} - \operatorname{int} D^{n-1}) \times [-1,1]$, $\alpha = +$ or $-$. We note $D_+^n \cap D_-^n = D^{n-1} \times 0$ (which we have identified with D^{n-1}),

$$F_+ \cup F_- = (2D^{n-1} - \operatorname{int} D^{n-1}) \times [-1,1],\ F_+ \cap F_- = (2D^{n-2} - \operatorname{int} D^{n-2}) \times [-1,1],\ F_0 \cap (F_+ \cup F_2) = \Sigma^{n-2} \times [-1,1],$$

and $F_0 \cap F_\alpha = \Sigma_\alpha^{n-2} \times [-1,1]$, $\alpha = +$ or $-$, where $\Sigma_\alpha^{n-2} = \Sigma^{n-2} \cap B_\alpha^{n-1}$, $\alpha = +$ or $-$.

Given any $m \geqq 0$, let $D_{(m)}^n$ and $(2D^{n-1} \times [-1,1])_{(m)}$ denote the "cellular" rectilinear subdivision of D^n and $(2D^{n-1} \times [-1,1])$ obtained by considering the union of the cells of the form $\times_{j=1}^{n} [i_j, k_j]$ and $\times_{j=1}^{n} [s_j, t_j]$, respectively, plus their faces, where i_j ($j = 1, \cdots, n$) and s_n is of the form $p/2^m$ and p is an integer such that $-2^m \leqq p < 2^m$, s_j ($j = 1, \cdots, n-1$) is of the form $q/2^m$ and q is an integer such that $-2^{m+1} \leqq q < 2^{m+1}$, and in each case $k_j = i_j + 1/2^m$ and $t_j = s_j + 1/2^m$. We note each of $D_{(m)}^n$ and $(2D^{n-1} \times [-1,1])_{(m)}$ contains D^{n-1} and $\Sigma^{n-1} \times [-1,1]$ as a "cellular" subcomplex. Also, for each $m \geqq 1$, $D_{(m+1)}^n$ and $(2D^{n-1} \times [-1,1])_{(m+1)}$ subdivides $D_{(m)}^n$ and $(2D^{n-1} \times [-1,1])_{(m)}$, respectively. For each $n \geqq 2$ and $m \geqq 1$, let $\tilde{A}_{+(m)}^{n-1}$ and $\tilde{A}_{-(m)}^{n-1}$ denote the unique disjoint $(n-1)$-cells in D^{n-1} containing the point $(1, \cdots, 1)$ and $(-1, -1, \cdots, -1)$, respectively. For each $n \geqq 2$ and $m \geqq 1$, let $\tilde{C}_{0(m)}^{n-1}$,

$\tilde{C}^{n-1}_{+(m)}$, $\tilde{C}^{n-1}_{-(m)}$ denote the disjoint $(n-1)$-cells in $\Sigma^{n-2}\times[-1,1]$ defined by $(\tilde{A}^{n-2}_{+(m)}\times 1)\times[-1,-1+1/2^m]$, $(\tilde{A}^{n-2}_{+(m)}\times 1)\times[1-1/2^m,1]$, and $(\tilde{A}^{n-2}_{-(m)}\times\{-1\})\times[-1,-1+1/2^m]$, respectively.

Observation 1. Given any $n\geqq 2$ and $m\geqq 1$, $D^n_{(m)}$ and $(2D^{n-1}\times[-1,1])_{(m)}$ induce "cellular" rectilinear subdivisions of D^n_α ($\alpha = +$ or $-$) and F^n_β ($\beta=0$, $+$ or $-$). We will denote these by $D^n_{\alpha(m)}$ and $F^n_{\beta(m)}$, respectively. Moreover, by merely moving back and forth, layer by layer, it is not too difficult to see that the n-cells of $D^n_{\alpha(m)}$ can be ordered, starting with the one containing $\tilde{A}^{n-1}_{\alpha(m)}$ or $\tilde{C}^{n-1}_{\beta(m)}$, respectively, so that any two consecutive cells of our ordering have an $(n-1)$-face in common. Given such an ordering, let $\tilde{a}_{\alpha(m)}$ (or $\tilde{c}_{\beta(m)}$) denote the rectilinear path in D^n_α (or F^n_α) obtained by taking the union of the following segments: the segment joining the barycenter of $\tilde{A}^{n-1}_{\alpha(m)}$ (or $\tilde{C}^{n-1}_{\beta(m)}$) to the barycenter of our first n-cell (which contains this given $(n-1)$-cell) and then adding the segments joining the barycenters of our consecutive cells in our ordering. We note, $\tilde{a}_{\alpha(m)}$ (or $\tilde{c}_{\alpha(m)}$) minus the barycenter of $\tilde{A}^{n-1}_{\alpha(m)}$ (or $\tilde{C}^{n-1}_{\alpha(m)}$) lies in the interior of D^n_α (or F^n_α) and $\tilde{a}_{\alpha(m)}$ (or $\tilde{c}_{\alpha(m)}$) intersects each n-cell of $D^n_{\alpha(m)}$ (or $F^n_{\alpha(m)}$) and this intersection is, in fact, a simple spanning path, except in the case of the last n-cell involved.

We now can state our desired result.

THEOREM 4. *Suppose S is a polyhedral $(n-1)$-sphere in E^n ($n\geqq 2$), $B_1, B_2,\cdots, B_m$ are disjoint polyhedral n-cells, and $G_1, G_2,\cdots, G_r$ are disjoint polyhedral $(n-1)$-cells in S such that*

(1) *$S-\bigcup_{i=1}^r \operatorname{int} G_i$ misses $\bigcup_{j=1}^m B_j$,*

(2) *for each j, B_j intersects one and only one G_i* (different B_j's may intersect the same G_i)

(3) *each G_i is intersected by some B_j, and*

(4) *if $B_j\cap G_i\neq\phi$, then there exists a* PL *homeomorphism h_j carrying the polyhedral pair $(B_j, B_j\cap G_i)$ onto one of the two types described above.*

If $h_j: (B_j, B_j\cap G_i)\to(D^n, D^{n-1})$ and we let $B_{j\alpha}=h_j^{-1}(D^n_\alpha)$ and $A_{j\alpha}=h_j^{-1}(\tilde{A}_{\alpha(1)})$ ($\alpha = +$ or $-$), *or if $h_k: (B_k, B_k\cap G_i)\to(2D^{n-1}\times[-1,1], \Sigma^{n-2}\times[-1,1])$ and we let $B_{k\beta}=h_k^{-1}(F_\beta)$ and $C_{k\beta}=h_k^{-1}(\tilde{C}_{\beta(1)})$* ($\beta=0$, $+$ or $-$), *then given positive numbers ϵ_1 and ϵ_2, and a rectilinear triangulation T of E^n containing all the above polyhedra as subcomplexes* (such a T exists by [2]), *there exist*

(5) *a subdivision $\tilde{T}$ of T such that for each $j\alpha$ ($k\beta$) there exists a finite collection $B_{j\alpha 1}, B_{j\alpha 2},\cdots, B_{j\alpha m(j\alpha)}$ of polyhedral n-cells having disjoint interiors such that $\cup B_{j\alpha i}=B_{j\alpha}$ and each $B_{j\alpha i}$ is a subcomplex of $\tilde{T}$ of diameter $<\epsilon_1$* (the corresponding conclusion holds for each $k\beta$), *and*

(6) *a* PL *homeomorphism* $h: E^n \to E^n$ *and for each* $j\alpha$ $(k\beta)$ *there exists a polyhedral* $(n-1)$*-cell* $G_{j\alpha}$ *lying in the interior of* $A_{j\alpha}$ *such that* $\operatorname{diam} G_{j\alpha} < \epsilon_2$, $h(\operatorname{int} G_{j\alpha}) \subset \operatorname{int} B_{j\alpha}$, $h(G_{j\alpha})$ *misses the* $(n-2)$*-skeleton of* $\tilde{T}$ *and intersects each* $B_{j\alpha i}$ $(i = 1, 2, \cdots, m(j\alpha))$ *so as to give a polyhedral pair of one of the two types described above* (the analogous statements holds for each $k\beta$ where $G_{k\beta} \subset \operatorname{int} C_{k\beta}$), *each of hypothesis* (2) *and* (3) *holds for the new "smaller"* B's *and* $h(G)$'s, *and* $h = id$ *on*

$$\left(E^n - \bigcup_{j=1}^{m} B_j\right) \cup [S - (\cup \operatorname{int} G_{j\alpha}) \cup (\cup \operatorname{int} G_{k\beta})].$$

Observation 2. We note, given G_i, if G_i intersects $B_{j1}, \cdots, B_{jm_j}$, where for some k_0, $0 \leqq k_0 \leqq m_j$, $(B_{jk}, B_{jk} \cap G_i)$ is of the first type for $1 \leqq k \leqq k_0$ $(= \phi$ if $k_0 = 0)$ and $(B_{jk}, B_{jk} \cap G_i)$ is of the second type for $k_0 + 1 \leqq k \leqq m_j$ $(= \phi$ if $k_0 = m_j)$, then the above result gives us a new collection of disjoint polyhedral $(n-1)$-cells $G_{jk\alpha}$ $(1 \leqq k \leqq k_0,\ \alpha = +$ or $-)$ and $G_{jk\beta}$ $(k_0 + 1 \leqq k \leqq m_j,\ \beta = 0, +$ or $-)$ in $\operatorname{int} G_i$ each lying in the interior of the appropriate B_{jk}. Moreover, if we shrink each $B_{jk\alpha i}$ or $B_{jk\beta i}$ (where jk corresponds to some j or k in (5) above) using the appropriate collar structure, we can obtain a disjoint collection of polyhedral n-cells $\{\tilde{B}_{jk\alpha i}, \tilde{B}_{jk\beta i}\}$ so that

$$S - (\cup \operatorname{int} G_{jk\alpha}) \cup (\cup \operatorname{int} G_{jk\beta}) = h(S - (\cup \operatorname{int} G_{jk\alpha}) \cup (\cup \operatorname{int} G_{jk\beta})$$

misses the union of the $\tilde{B}_{jk\alpha i}$'s and $\tilde{B}_{jk\beta i}$'s, and each $h(G_{jk\alpha}) \cap \tilde{B}_{jk\alpha i}$ and $h(G_{jk\beta}) \cap \tilde{B}_{jk\beta i}$ is a polyhedral pair of one of the two types above. That is, the new collection of $(n-1)$-cells and n-cells (the $\tilde{B}_{jk\alpha i}$'s) are disjoint and satisfy the hypothesis (1)–(4) of Theorem 4.

Proof of Theorem 4. Let T be the triangulation of E^n containing all the given polyhedra. Given, the h_j and h_k, choose an integer M large enough so that the preimage of each n-cell of $D^n_{(M)}$ under h_j and each n-cell of $(2D^{n-1} \times [-1, 1])_{(M)}$ under h_k has diameter less than ϵ_i. For each j (or k) let T_j be a triangulation of B_j (or T_k be a triangulation of B_k) and $\sigma_j(D^n_{(M)})$ (or $\sigma_k(2D^{n-1} \times [-1, 1])_{(M)}$) be a triangulation of D^n (or $2D^{n-1} \times [-1, 1]$) such that h_j (or h_k) is a simplicial homeomorphism. Let $\tilde{T}$ be the subdivision of T containing the T_j's and T_k's as subcomplexes. Let $\tilde{a}_{\alpha(M)}$ and $\tilde{c}_{\beta(M)}$ be the polyhedral arcs in $D^n_{\alpha(M)}$ and in $F^n_{\beta(M)}$ as described in Observation 1. By very small adjustments of the given arcs we can obtain new polyhedral arcs having similar properties, but now missing the $(n-2)$-skeletons of $\sigma_j D^n_{(M)}$ or $\sigma_k(2D^{n-1} \times [-1, 1])_{(M)}$. Let $a_{j\alpha}$ and $c_{k\beta}$ be the polyhedral arcs in $B_{j\alpha}$ and $F_{j\alpha}$ obtained as images of our adjusted $\tilde{a}_{\alpha(M)}$'s and $\tilde{c}_{\beta(M)}$'s under h_j^{-1} and h_k^{-1}. For each $B_{j\alpha}$, let $\{B_{j\alpha 1}, \cdots, B_{j\alpha m(j\alpha)}\}$ be the collection of polyhedral

n-cells obtained as images of the n-cells of $D^n_{\alpha(M)}$ under h_j^{-1}. Similarly, the $\{B_{k\beta 1}, \cdots, B_{k\beta m(k\beta)}\}$ are obtained as images of the n-cells of $F^n_{\alpha(M)}$ under h_k^{-1}. Conclusion (5) now follows.

Now $a_{j\alpha}$ (or $c_{k\beta}$) intersects each $B_{j\alpha i}$ (or $B_{k\beta i}$) in a spanning arc, except for the last cell. Also, each $a_{j\alpha}$ or $c_{k\beta}$ misses the $(n-2)$-skeleton of $\tilde{T}$. Let $P_{j\alpha}$ or $Q_{k\beta}$ be small regular neighborhoods of $a_{j\alpha}$ or $c_{k\beta}$ in $B_{j\alpha}$ or $B_{k\beta}$, respectively, and in $\operatorname{int} B_j$ so that each of the $P_{j\alpha}$ and the $Q_{k\beta}$ miss the $(n-2)$-skeleton of $\tilde{T}$ and so that $G_{j\alpha} = P_{j\alpha} \cap A_{j\alpha}$ and $G_{k\beta} = Q_{k\beta} \cap C_{k\beta}$ are all $(n-1)$-cells of diameter $< \epsilon_2$. Let $\hat{G}_{j\alpha} = \operatorname{Bd} P_{j\alpha} - \operatorname{int} G_{j\alpha}$ and let $\hat{G}_{k\beta} = \operatorname{Bd} Q_{k\beta} - \operatorname{int} G_{k\beta}$. If our regular neighborhoods are small enough and taken in a nice enough fashion, then each of $(B_{j\alpha i}, \hat{G}_{j\alpha} \cap B_{j\alpha i})$ and $(B_{k\beta i}, G_{k\beta} \cap B_{k\beta i})$ will be a polyhedral pair of type one or two above [2]. In a small neighborhood of each $P_{j\alpha}$ or $Q_{k\beta}$, say $\tilde{P}_{j\alpha}$, or $\tilde{Q}_{k\beta}$, we can define a PL homeomorphism $h_{j\alpha}$ or $h_{k\beta}$ carrying $\tilde{P}_{j\alpha}$ onto itself or $\tilde{Q}_{k\beta}$ onto itself so that $h_{j\alpha} = id$ on $\operatorname{Bd} \tilde{P}_{j\alpha} \cup [(S - \operatorname{int} G_{j\alpha}) \cap \tilde{P}_{j\alpha}]$ and $h_{j\alpha}(G_{j\alpha}) = \hat{G}_{j\alpha}$ or $h_{k\beta} = id$ on $\operatorname{Bd} \tilde{Q}_{j\alpha} \cup [(S - \operatorname{int} G_{k\beta}) \cap \tilde{Q}_{k\beta}]$ and $h_{k\beta}(G_{k\beta}) = \hat{G}_{k\beta}$. The PL homeomorphism $h\colon E^n \to E^n$ is obtained by defining h to be the identity on $E^n - (\cup \tilde{P}_{j\alpha}) \cup (\cup \tilde{Q}_{k\beta})$, equal to $h_{j\alpha}$ on $\tilde{P}_{j\alpha}$, and equal to $h_{k\beta}$ on $\tilde{Q}_{k\beta}$. If the $\tilde{P}_{j\alpha}$'s and $\tilde{Q}_{k\beta}$'s are small enough so as to lie in the interiors of the appropriate B_j's and B_k's, then Conclusion (6) easily follows.

6. *Proof of Theorem* 3. The proof of Theorem 3 will now follow quite easily from Theorems 2 and 4 as follows. Let S, X, ϵ', $\{B_j\}$, and $\{G_j\}$ be given as in the hypothesis of Theorem 3. We now apply Theorem 4, where T is some triangulation of E^n as required there and $\epsilon_1 = \delta_1$ and $\epsilon_2 = \delta_2$. Let $\tilde{T}$ be the subdivision of T and $h\colon E^n \to E^n$ be the PL homeomorphism promised by Theorem 4. Our desired PL homeomorphism will be a slight modification of this given h. We will denote the new n-cells in B_j by $\tilde{B}_{j1}, \cdots, \tilde{B}_{jm_j}$ and the new $(n-1)$-cells in $\operatorname{int} G_i$ by $G_{i1}, \cdots, G_{ir_i}$. (Recall, each B_j was divided into two or three parts and then further divided into the appropriate n-cells $\{\tilde{B}_{ji}\}$ and r_i is actually the sum of 2's and 3's.) The given G_{ij}'s will be the desired $(n-1)$-cells. We now claim that if we modify the $\tilde{B}_{ji}$'s slightly and then "shrink" them appropriately, as suggested by Observation 2, then our result will immediately follow. We will use Theorem 2 to tell us how little to shrink and modify the various $\tilde{B}_{ji}$'s, and how to modify h.

That is, since $S \subset \operatorname{int} X$, $h = id$ on $S - \cup G_{ij}$, and $h(\cup G_{ij}) \subset \cup \operatorname{int} B_j$, we have $h(S - \cup G_{ij}) \subset \operatorname{int} X$ and we can pick $0 < \epsilon \leqq \epsilon'$ so that $h(\cup G_{ij}) \subset \cup \operatorname{int} B_j^{\epsilon}$. For each B_j with triangulation $\tilde{T}$ and our given ϵ, we apply Theorem 2, where $\eta < \operatorname{dist}(S, E^n - X)$ and $Z_1 = Z_2 = \cup h(G_{rs})$, the two or three new $(n-1)$-cells in $\operatorname{int} B_j$ intersecting the various $\tilde{B}_{ji}$'s so nicely. Theorem 2 gives us an PL η homeomorphism $h_j\colon B_j \to B_j$ such that $h_j = id$ on $\operatorname{Bd} B_j \cup (\cup h(G_{rs}))$

and the collection $h_j(\tilde{B}_{ji})$ satisfy the conclusion of that theorem (the δ there is $\epsilon_1 = \delta_1$ above). Since h_j is fixed on $\cup h(G_{rs})$, each $(h_j(B_{ji}), h_j(B_{ji}) \cap hh_j(G_{rs}))$ is still a c.c. or p.c.s. pair (so we can still shrink-in and preserve this property).

We now define $f\colon E^n \to E^n$ by $f = h$ on $E^n - \cup B_j$ and $f = h_j \circ h$ on each B_j. We observe, that since $f = h = id$ on $E^n - \cup B_j$, $f = h_j$ on $G_i \cap B_j - \cup G_{rs}$ (where G_i is the unique $(n-1)$-cell intersecting B_j and $\cup G_{rs}$ is the two or three given sub $(n-1)$-cells; recall $h = id$ on this set), and h_j moves points less than η, it follows that $f(S - \cup_{i,j} G_{ij}) \subset \operatorname{int} X$. Also, since $h_j = id$ on $\cup h(G_{rs})$, it follows by a choice of above that $f(\cup_{ij} G_{ij}) \subset \cup \operatorname{int} B_j^{\epsilon}$. Thus conclusion (13) holds. We now want to shrink each $h_j(\tilde{B}_{ji})$ in slightly (using the collar structure as a c.c. or p.c.s. pair) to obtain our desired disjoint B_{ji}'s so that we have the required intersections with $f(S)$. We note that since $h(S - \cup G_{ij}) \subset B_j$ lies in $\cup \operatorname{Bd} \tilde{B}_{ji}$ and $B_{ji} \subset \operatorname{int} h_j(\tilde{B}_{ji})$, $f(S - \cup G_{ij})$ will miss the union of the B_{ji}'s. By Theorem 2, there exists a $\gamma_j > 0$ such that any straight line intersecting B_j^{ϵ} also intersects some $h_j(\tilde{B}_{ji})^{\gamma_j}$. We now shrink each $h_j(\tilde{B}_{ji})$ in slightly to obtain a B_{ji} with the correct intersection property so that int $B_{ji} \supset h_j(\tilde{B}_{ji})^{\gamma_j}$. Having done this for each j, we finally pick one fixed $\gamma > 0$ so that for each j, $B_{ji}^{\gamma} \supset h_j(\tilde{B}_{ji})^{\gamma_j}$. It is not too difficult now to see that conclusions (4)–(14) easily follow.

Conclusion (15) is simply a consequence of hypothesis (3) and conclusion (14). That is, any straight line intersecting $X \cup (\cup B_j^{\epsilon})$ must intersect some $B_j^{\epsilon} \supset B_j^{\epsilon'}$ by (4) and hence some B_{ji}^{γ} by (14). This completes the proof of Theorem 3.

COROLLARY 6. *If S^2 is the 2-sphere in E^3 and C is the Cantor subset of S constructed as in the proof of Theorem 1 using the canonical modifications given by the proof of Theorem 3, then S^2 is tame in E^2.*

Proof. By [6], it will suffice to show that each complementary domain of S^2 is locally simply connected. That is, we must show that if U is a complementary domain of S^2 and p is any point of S^2, then given any neighborhood W of p in E^3 there exists a neighborhood V of p in E^3 such that every map of the boundary of a disk D^2 into $V \cap U$ extends to a map of D^2 into $W \cap U$. Since $S^2 - C$ is locally flat, we only have to show that each complementary domain of S^2 is locally simply connected at points of C.

Let $p \in C$ and let W be a neighborhood of p in E^3. Recall $C = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{m_i} B_{ij}^3$. Choose i_0 large enough so that the $B_{i_0j}^3$ containing p lies in W_i. Recall $(B_{i_0j}^3, B_{i_0j}^3 \cap S_{i_0}^2)$ is a c.c. or p.c.s. pair, which we can think of as the union of 3-cells $D_1 \cup D_2$ or $F_1 \cup F_2 \cup F_3$ corresponding to $D_+^3 \cup D_-^3$ or to $F_0^3 \cup F_-^3 \cup F_+^3$ as defined in §5. Let $\tilde{B} = B_{i_0+1,k}^3$ be the 3-cell in int $B_{i_0j}^3$ containing our given point p. Now $\tilde{B}$ lies in the interior of the image of one of the 3-cells above under f_{i_0}. Let us denote this open image by R. Now $R \subset \operatorname{int} B_{i_0j}$ and R misses $S_{i_0+1} - \{\text{some } G_{i_0+1,j}\}$. Also, it follows easily that for any $k \leqq i_0$, $(B^3_{i_0j}, B^3_{i_0j} \cap S^2_{i_0})$ is PL homeomorphic to

$$(B_{i_0j}, B_{i_0j} \cap f_k \circ \cdots \circ f_{i_0}(S_{i_0})) = (B^3_{i_0j}, B^3_{i_0j} \cap S^2_{k+1}).$$

Moreover, if we use the nice canonical modifications as suggested in the proof of Theorem 3 (i.e., using the constructions of §5), it is not too difficult to see that each component of

$$\operatorname{int} B_{i_0j} - [(S^2_{k+1} \cap B_{i_0j}) \cup (\cup \{B_{k+1,j} \mid B_{k+1,j} \subset \operatorname{int} B_{i_0j}\})]$$

is an open 3-cell, except perhaps for one which is homeomorphic to $S^1 \times E^2$. This latter case occurs only if $\operatorname{int} B_{i_0j} - S_{i_0}$ has a component homeomorphic to $S^1 \times E^2$. We now claim that any map of $\operatorname{Bd} D^2$ into $\operatorname{int} \tilde{B} - S^2$ extends to a map of D into $(\operatorname{int} B_{i_0j}) - S^2$. Suppose $f\colon \operatorname{Bd} D \to (\operatorname{int} \tilde{B}) - S^2$ is any map. Choose $k > i_0 + 1$, large enough, so that $\bigcup_{i=1}^{m_k} B_{kj}$ misses $f(\operatorname{Bd} D)$. Then $f(\operatorname{Bd} D)$ misses $S \cup (\bigcup_{j=1}^{m_k} B_{kj})$ and hence misses $S_k \cup (\bigcup_{j=1}^{m_k} B_{kj})$ and this latter set contains S (since all further modifications of S_k leading to S occur in $\cup \operatorname{int} B_{kj}$). Let $X = (S_k \cap B_{i_0j}) \cup (\cup \{B_{kj} \mid B_{kj} \subset \operatorname{int} B_{i_0j}\})$. Consider $\operatorname{int} B - X$. As above, each component of this is an open 3-cell except one, which is homeomorphic to $S^1 \times E^2$. However, it is not too difficult to see that any loop in the component homeomorphic to $S^1 \times E^2$ shrinks to a point in $R - X$. Therefore, either f extends in $\operatorname{int} \tilde{B} - X$ (if $f(\operatorname{Bd} D)$ lies in some component homeomorphic to an open 3-cell) or f extends in $R - X$. Since each of these lie in $\operatorname{int} B_{i_0j} - S$, the result follows.

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