PACIFIC JOURNAL OF MATHEMATICS
Vol 62, No 1, 1976

A CHARACTERIZATION OF TOPOLOGICAL LEFT THICK SUBSETS IN LOCALLY COMPACT LEFT AMENABLE SEMIGROUPS

JAMES C. S. WONG

In this paper, we define the concept of topological left thick subsets in a locally compact semigroup which is a generalisation and extension of the concept of left thick subsets in (discrete) semigroups introduced by T. Mitchell and prove that if T is a Borel measurable subset of a locally compact left amenable semigroup S, then T is topological left thick if and only if there is a topological left invariant mean M on S such that $M(\chi_T) = 1$ where χ_T is the characteristic functional of T in S, thus generalising, and extending a result of Mitchell for (discrete) semigroup.

1. Introduction. Let S be a semigroup and $T \subset S$. T is called left thick in S if for every finite subset $F \subset S$, there is some $s \in S$ such that $Fs \subset T$. In **[8]**, Mitchell proves that if S is left amenable (i.e. the bounded functions $m(S)$ on S has a left invariant mean) then a subset $T \subset S$ is left thick if and only if there is a left invariant mean μ on $m(S)$ such that $\mu(\xi_T) = 1$ where ξ_T is the characteristic function of T in S. In this paper, we extend this concept of left thick subsets to topological left thick subsets in locally compact semigroups and obtain a topological analogue (as well as an extension) of Mitchell's result. It is also an analogue of a result of Day in **[4]** for locally compact groups which as stated in **[4]** contains an error. A corrected version is given below in §5 (Remark 2).

2. Notations and terminologies. For definitions of topological left invariant means on locally compact semigroups, we follow Wong **[12]**. Let S be a locally compact semigroup, $M(S)$ its measure algebra with total variation norm and convolution as multiplication and $M_0(S)$ its probability measures. A Borel subset $T \subset S$ is called topological left thick if the following condition is satisfied: For each $0 < \epsilon \leqq 1$ and each compact subset $F \subset S$, there is some measure $\mu \in M_0(S)$ such that $\nu * \mu(T) > 1 - \epsilon$ for any $\nu \in M_0(S)$ with $\nu(F') = 0$. It will be proved below that this definition agrees with that in Mitchell **[8]** when S is discrete. (Our definition is necessarily more complicated because of the "continuity nature" involved.)

A net $\mu_\alpha \in M_0(S)$ is said to converge strongly to topological left invariance uniformly on compacta if for every compact set $F \subset S$,

$\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly for all $\mu \in M_0(S)$ with $\mu(F') = 0$. Since measures in $M(S)$ with compact supports are norm dense in $M(S)$, this implies that μ_α converges strongly to topological left invariance. That is $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ for any $\mu \in M_0(S)$. (This latter condition is equivalent to $M(S)^*$ having a topological left invariant mean by [**12**, Theorem 3.1]). The converse of the above implication need not be true except when S is discrete (for then compact sets are finite) or when S is a group (which is proved in §5 below).

3. A Lemma. Let M be a topological left invariant mean on $M(S)^*$, then there is a net $\mu_\alpha \in M_0(S)$ which converges w^* to M in $M(S)^{**}$. It is easy to show that the net μ_α converges weakly to topological left invariance. That is $\mu * \mu_\alpha - \mu_\alpha \to 0$ weakly in $M(S)$ for each $\mu \in M_0(S)$. In general, the convergence need not be uniform on compacta nor in the norm topology of $M(S)$. The following lemma states that if there is some net in $M_0(S)$ converging strongly to topological left invariance uniformly on compacta, then there is a net $\tau_\beta \in M_0(S)$ with the same property and such that $\tau_\beta \to M$ weak* in $M(S)^{**}$.

LEMMA 3.1. *Let μ_α be a net in $M_0(S)$ which converges strongly to topological left invariance uniformly on compacta and let M be a topological left invariant mean on $M(S)^*$. Then there is a net τ_γ in $M_0(S)$ such that $\tau_\gamma \to M$ weak* in $M(S)^{**}$ and τ_γ converges strongly to topological left invariance uniformly on compacta.*

Proof. Let μ_α, $\alpha \in D$ be the net given and let ν_β, $\beta \in E$ be a net in $M_0(S)$ such that $\nu_\beta \to M$ weak* in $M(S)^{**}$. We shall construct the required net by a result on iterated limits (Kelley [**7**, Theorem 4, Chapter 2, p. 69]) consider the product directed sets $D \times \Pi\{E: \alpha \in D\}$ and $D \times E$. Define maps

$$R: D \times \Pi\{E: \alpha \in D\} \to D \times E \quad \text{and}$$
$$V: D \times E \to M(S)^{**} \quad \text{Where}$$

$M(S)^{**}$ is considered as a topological space with the weak* topology, by

$$R(\alpha, f) = (\alpha, f(\alpha))$$

and

$$V(\alpha, \beta) = \mu_\alpha * \nu_\beta.$$

Since M is topological left invariant, the iterated limit $w^* \lim_\alpha w^* \lim_\beta \mu_\alpha * \mu_\beta$ exists and is equal to M. In fact for $F \in M(S)^*$,

$$(\mu_\alpha * \nu_\beta)(F) = F(\mu_\alpha * \nu_\beta) = (\mu_\alpha \odot F)(\nu_\beta) = \nu_\beta(\mu_\alpha \odot F) \xrightarrow{\beta} M(\mu_\alpha \odot F) = M(F).$$

By [**6**, Theorem 4, Chapter 2, p. 69], the net $V \circ R$ converges to this iterated limit M in weak* topology of $M(S)^{**}$. It remains to show that $V \circ R(\alpha, f)$ converges strongly to topological left invariance uniformly on compacta. But for any $\mu \in M_0(S)$,

$$\begin{aligned} &\|\mu * V \circ R(\alpha, f) - V \circ R(\alpha, f)\| \\ &\quad = \|\mu * \mu_\alpha * \nu_{f(\alpha)} - \mu_\alpha * \nu_{f(\alpha)}\| \\ &\quad \leqq \|\mu * \mu_\alpha - \mu_\alpha\| \quad \text{and the result follows immediately.} \end{aligned}$$

NOTE. Uniform strong left amenability was first introduced by H. Reiter [**9**] for locally compact groups (in a slightly different form) and is equivalent to left amenability (see §5 below). We need this concept for semigroups in order to extend Mitchell's result. In fact, Lemma 3.1 is the crux of the idea. It is not known if it is equivalent to the existence of a topological left invariant mean on $M(S)^*$ for a general semigroup S.

4. Main results. Notations and definitions not explained here can be found in Wong [**12**].

THEOREM 4.1. *Let S be a locally compact semigroup for which there is a net in $M_0(S)$ converging strongly to topological left invariance uniformly on compacta and T a Borel subset of S. Then the following statements are equivalent:*

(1) *T is topological left thick in S.*

(2) *There is a topological left invariant mean M on $M(S)^*$ such that $M(\chi_T) = 1$.*

Here χ_T is the characteristic functional of T in S defined by $\chi_T(\mu) = \int_T 1\, d\mu = \mu(T)$, $\mu \in M(S)$.

Proof. (1) implies (2)

By assumption, $M(S)^*$ has a topological left invariant mean M (see §2). Suppose T is topological left thick. Consider the pairs $\alpha = (\epsilon, F)$ where $0 < \epsilon \leqq 1$ and $F \subset S$ compact. Define $\alpha \geqq \alpha_1$ to mean $\epsilon \leqq \epsilon_1$ and $F \supset F_1$. For each α, there is a measure $\mu_\alpha \in M_0(S)$ such that $\nu * \mu_\alpha(T) > 1 - \epsilon$ for all $\nu \in M_0(S)$ with $\nu(F') = 0$. We first prove that $\chi_T \odot \mu_\alpha(\nu) \to 1$ for any $\nu \in M_0(S)$. Let $\nu \in M_0(S)$ have compact support F_0. Given $0 < \epsilon_0 \leqq 1$, put $\alpha_0 = (\epsilon_0, F_0)$. If $\alpha \geqq \alpha_0$, then $\epsilon \leqq \epsilon_0$, $F \supset F_0$ and $\nu(F') = 0$. Hence

$$|1 - (\chi_T \odot \mu_\alpha)(\nu)| = 1 - \nu * \mu_\alpha(T) < \epsilon \leqq \epsilon_0.$$

That is, $\lim_\alpha (\chi_T \odot \mu_\alpha)(\nu) = 1$ for such ν. Since the measures in $M_0(S)$ with compact supports are norm dense in $M_0(S)$, it follows that $\chi_T \odot \mu_\alpha \to 1$ weak* in $M(S)^*$ (Note that $M_0(S)$ spans $M(S)$). Here the functional 1 is defined by $1(\mu) = \mu(S)$, $\mu \in M(S)$ (i.e. $1 = \chi_S$). Let now N be a mean on $M(S)^*$ such that $\mu_\alpha \to N$ weak* in $M(S)^{**}$ (using a subnet if necessary) and consider the Arens' product $M \odot N$ (in the second conjugate Banach algebra $M(S)^{**}$, see Arens **[1]** and Day **[2**, §6**]** for definition). It is straightforward to verify that $M \odot N$ is a topological left invariant mean (since M is) on $M(S)^{**}$ such that $(M \odot N)(\chi_T) = 1$.

(2) implies (1)

Assume that there is a topological left invariant mean M on $M(S)^*$ such that $M(\chi_T) = 1$. By Lemma 3.1, there is a net $\mu_\alpha \in M_0(S)$ such that $\mu_\alpha \to M\ w^*$ in $M(S)^{**}$ and such that μ_α converges strongly to topological left invariance uniformly on compacta. Consider the net $\chi_T \odot \mu_\alpha$. Let $F \subset S$ be compact, $\mu \in M_0(S)$ with $\mu(F') = 0$. Then

$$\begin{aligned} &|1 - \chi_T(\mu * \mu_\alpha)| \\ &\quad \leqq |1 - \chi_T(\mu_\alpha)| + |\chi_T(\mu * \mu_\alpha) - \chi_T(\mu_\alpha)| \\ &\quad \leqq |1 - \chi_T(\mu_\alpha)| + \|\mu * \mu_\alpha - \mu_\alpha\|. \end{aligned}$$

Since $\lim_\alpha \chi_T(\mu_\alpha) = \lim_\alpha \mu_\alpha(\chi_T) = M(\chi_T) = 1$ and $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly for all $\mu \in M_0(S)$ with $\mu(F') = 0$, it follows that $(\chi_T \odot \mu_\alpha)(\mu) \to 1$ uniformly for all $\mu \in M_0(S)$ with $\mu(F') = 0$.

Now if T is not topological left thick, then there exists a pair (ϵ_0, F_0), $0 < \epsilon_0 \leqq 1$, $F_0 \subset S$ compact such that for any $\mu \in M_0(S)$, there is some $\nu \in M_0(S)$ with $\nu(F_0') = 0$ but $(\nu * \mu)(T) \leqq 1 - \epsilon_0$. Let $\nu_\alpha \in M_0(S)$, $\nu_\alpha(F_0') = 0$ and

$$(\nu_\alpha * \mu_\alpha)(T) \leqq 1 - \epsilon_0 \quad \text{for any} \quad \alpha.$$

Then

$$(\chi_T \odot \mu_\alpha)(\nu_\alpha) \leqq 1 - \epsilon_0 < 1 \quad \text{for any} \quad \alpha$$

which contradicts the fact that the net $\chi_T \odot \mu_\alpha(\nu) \to 1$, *uniformly* for all $\nu \in M_0(S)$ with $\nu(F_0') = 0$. This completes the proof.

NOTES. (a) For implication (1) implies (2), we need only the existence of a net $\mu_\alpha \in M_0(S)$ such that $\mu * \mu_\alpha - \mu_\alpha \to 0$ weakly in $M(S)$ for each $\mu \in M_0(S)$, instead of a net converging to topological left invariance uniformly on compacta. Any w^*-cluster point of μ_α in $M(S)^{**}$ will give a topological left invariant mean M on $M(S)^*$.

(b) The preceeding theorem is a topological analogue of Mitchell's result in [**8**, Theorem 7, p. 257]. It is also an analogue of a result in Day [**4**, Theorem 7.8] for locally compact groups which involves the Haar measure. However, our result is valid for all locally compact semigroups and is formulated in terms of measures in $M(S)$. This result of Day does not make sense in our case because of the absence of a Haar measure.

The next theorem shows that the concept of topological left thickness agrees with that of left thickness in Mitchell [**8**], when S is a discrete semigroup.

THEOREM 4.2. *If S is a discrete semigroup and T any subset of S, then T is left thick iff T is topological left thick.*

Proof. Since S is discrete $M(S)=l_1(S)$, the absolutely summable functions on S and $M(S)^*=m(S)$ the bounded functions on S. Assume that T is topological left thick but not left thick. As in Mitchell [**8**, Theorem 7], there is some finite subset F in S such that for any $t \in S$

$$\sum_{s\in F} \xi_T(st) \leqq N-1 \quad \text{where} \quad \xi_T \quad \text{is the}$$

characteristic function of T in S and N is the number of elements in F. Define $\theta \in l_1(S)$ by $\theta(S)=0$ if $s \notin F$ and $\theta(S)=1/N$ if $s \in F$, then $\theta \geqq 0$, $\|\theta\|_1 = 1$ and

$$\begin{aligned} \chi_T(\theta * \varphi) &= \sum_s \sum_t \xi_T(st)\theta(s)\varphi(t) \\ &= \frac{1}{N} \sum_t \sum_{s\in F} \xi_T(st)\varphi(t) \\ &\leqq \frac{1}{N}(N-1)\sum_t \varphi(t) \\ &= \frac{N-1}{N} \end{aligned}$$

for any $\varphi \in l_1(S)$, $\varphi \geqq 0$ and $\|\varphi\|_1 = 1$.

But by topological left thickness, for the pair $(1/N, F)$, there is some $\varphi \in l_1(S)$ $\varphi \geqq 0$, $\|\varphi\|_1 = 1$ such that

$$\chi_T(\theta * \varphi) > 1-\epsilon = \frac{N-1}{N}$$

for any $\theta \in l_1(S)$, $\theta \geqq 0$, $\theta(s)=0$, $s \notin F$ and $\|\theta\|_1 = 1$. This is a contradiction.

Conversely, if S is left thick, then for any $0 < \epsilon \leqq 1$ and $F \subset S$ compact (i.e. finite), there is some $s_1 \in S$ such that $Fs_1 \subset T$. Put $\varphi(s) = 1$ if $s = s_1$ and $\varphi(s) = 0$ if $s \neq s_1$, then $\varphi \in l_1(S)$, $\varphi \geqq 0$, $\|\varphi\|_1 = 1$. Moreover, for any $\theta \in l_1(S)$ with $\theta \geqq 0$, $\|\theta\|_1 = 1$, $\theta(s) = 0$ if $s \notin F$, we have

$$\begin{aligned}\chi_T(\theta * \varphi) &= \sum_s \sum_t \xi_T(st)\theta(s)\varphi(t)\\ &= \sum_{s\in F} \sum_t \xi_T(st)\varphi(t)\theta(s)\\ &= \sum_{s\in F} \xi_T(ss_1)\theta(s)\\ &= \sum_{s\in F} \theta(s) = 1 > 1-\epsilon.\end{aligned}$$

Hence T is topological left thick.

REMARKS. Let S be discrete.

(a) Since compact sets are finite, it is easy to see that for any net $\varphi_\alpha \in l_1(S)$, $\varphi_\alpha \geqq 0$ and $\|\varphi_\alpha\|_1 = 1$, the following statements are equivalent (i) $\|l_s\varphi_\alpha - \varphi_\alpha\|_1 \to 0$ for each $s \in S$, (ii) for any compact $F \subset S$. $\|l_s\varphi_\alpha - \varphi_\alpha\|_1 \to 0$ uniformly for $s \in F$ and (iii) for each compact $F \subset S$, $\|\theta * \varphi_\alpha - \varphi_\alpha\|_1 \to 0$ uniformly for any $\theta \in l_1(S)$ with $\theta \geqq 0$, $\|\theta\|_1 = 1$ and $\theta(s) = 0$ if $s \notin F$. Each of these statements is equivalent to left amenability of S.

(b) Under the canonical isometric isomorphism φ of $m(S)$ onto $M(S)^*$, ξ_T becomes χ_T (i.e. $\varphi(\xi_T) = \chi_T$) and each mean M on $M(S)^*$ induces a mean $m = \varphi^*M$ on $M(S)$ such that $m(\xi_T) = M(\chi_T)$ (and vice versa).

(c) In view of remarks (a) and (b) and Theorem 4.2, it follows that Mitchell's result [**8**, Theorem 7] is a special case of Theorem 4.1.

5. Locally compact groups. If S is a locally compact group, then left amenability is equivalent to "uniform strong left amenability". This is implicitly contained in Day [**3**] who attributes it to Reiter [**10**]. We give the proof here for completeness. For notations in abstract harmonic analysis on groups, we follow Hewitt and Ross [**6**].

LEMMA 5.1 (Reiter, Day). *Let S be a locally compact group. Then the following statements are equivalent:*

(1) *There is a net $\mu_\alpha \in M_0(S)$ such that $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ for each $\mu \in M_0(S)$, i.e. S is left amenable.*

(2) *There is a net $\mu_\alpha \in M_0(S)$ such that for each compact net $F \subset S$, $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly for all $\mu \in M_0(S)$ with $\mu(F') = 0$.*

Proof. (2) implies (1)

This is because the measures in $M_0(S)$ with compact supports are norm dense in $M_0(S)$.

(1) implies (2)

Let $\mu_\alpha \in M_0(S)$ be a net such that $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ for each $\mu \in M_0(S)$. We can assume that $\mu_\alpha \in M_a(S) \cong L_1(S)$ otherwise consider the net $\nu * \mu_\alpha$ for some fixed $\nu \in M_a(S)$ (which is an ideal of $M(S)$) with $\nu \geqq 0$ and $\|\nu\| = 1$. By the canonical isometric isomorphism of $M_a(S) \cong L_1(S)$, this implies that there is a net $\varphi_\alpha \in P(S) = \{\varphi \in L_1(S)$: $\varphi \geqq 0, \|\varphi\|_1 = 1\}$ such that $\|\mu * \varphi_\alpha - \varphi_\alpha\| \to 0$ for each $\mu \in M_0(S)$. We can assume that φ_α is left equicontinuous (that is, given $\epsilon > 0$, there is some neighborhood U of the identity in S such that $\|l_s\varphi_\alpha - \varphi_\alpha\|_1 < \epsilon$ for any α and any $s \in U$) otherwise replace φ_α by $\varphi * \varphi_\alpha$ where φ is a fixed element in $P(S)$ (see Day **[3]** for details). It follows from [**3**, Theorem 1] that for each compact set F in S, $\|l_{t^{-1}}\varphi_\alpha - \varphi_\alpha\|_1 \to 0$ uniformly for $t \in F$. Now let $\mu \in M_0(S)$ with $\mu(F') = 0$ and $\epsilon > 0$, choose α_0 such that $\|l_{t^{-1}}\varphi_\alpha - \varphi_\alpha\|_1 < \epsilon$ for any $\alpha \geqq \alpha_0$ and $t \in F$. Then

$$\begin{aligned}\|\mu * \varphi_\alpha - \varphi_\alpha\|_1 &\leqq \int\int |l_{t^{-1}}\varphi_\alpha(s) - \varphi_\alpha(s)|\, ds\, d\mu(t)\\ &= \int_F \|l_{t^{-1}}\varphi_\alpha - \varphi_\alpha\|_1\, d\mu(t) < \epsilon, \quad \text{for any}\end{aligned}$$

$\alpha \geqq \alpha_0$ and $\mu \in M_0(S)$ with $\mu(F') = 0$. This completes the proof.[1]

THEOREM 5.2. *Let S be a left amenable locally compact group and T a Borel subset of S. Then the following statements are equivalent*:

(1) *T is topological left thick.*

(2) *There is a topological left invariant mean M on $M(S)^*$ such that $M(\chi_T) = 1$.*

(3) *There is a topological left invariant mean m on $L_\infty(S)$ such that $m(\xi_T) = 1$.*

Proof. (1) and (2) are equivalent by Theorems 4.1 and 5.1. Now if (2) holds with M a topological left invariant mean on $M(S)^*$ and $M(\chi_T) = 1$, then using the "iterated limit" construction as in Lemma 3.1, a net μ_α in $M_0(S)$ can be found such that (i) for each $\mu \in M_0(S)$, $\lim_\alpha \|\mu * \mu_\alpha - \mu_\alpha\| = 0$ and (ii) $w*$ — limit of μ_α in $M(S)^{**}$ is M. Let $\mu_0 \in M_a(S) \cap M_0(S)$ be fixed and put $\nu_\alpha = \mu_0 * \mu_\alpha \in M_a(S)$ (which is an ideal in $M(S)$), then ν_α also satisfies (i) and (ii) (M is topological left

[1] $l_s\varphi$ is the left translate of the function φ by the element s in S. In Hewitt and Ross **[6]**, the notation ${}_s\varphi$ is used.

invariant). Let $\varphi_\alpha \in P(S) = \{\varphi \in L_1(S)\colon \varphi \geqq 0, \|\varphi\|_1 = 1\}$ correspond to ν_α under the isometric isomorphism $M_a(S) \cong L_1(S)$. By w^*-compactness of the means on $L_\infty(S)$, some subnet φ_α, also denoted by φ_α, can be found satisfying (iii) for each $\varphi \in P(S)$, $\lim_\alpha \|\varphi * \varphi_\alpha - \varphi_\alpha\| = 0$ and (iv) w^*-limit of φ_α in $L_\infty(S)^*$ is m. Then m is a topological left invariant mean on $L_\infty(S)$ and

$$m(\xi_T) = \lim_\alpha \varphi_\alpha(\xi_T) = \lim_\alpha \int \xi_T \varphi_\alpha d\lambda = \lim_\alpha \int \xi_T d\nu_\alpha = \lim_\alpha \nu_\alpha(\chi_T) = M(\chi_T) = 1$$

(λ a fixed left Haar measure). Hence (2) implies (3). Conversely, if (3) holds with m a topological left invariant mean on $L_\infty(S)$ and $m(\xi_T) = 1$, then again, a net $\varphi_\alpha \in P(S)$ can be found satisfying (iii) and (iv). Let $\mu_\alpha \in M_a(S)$ correspond to φ_α. By w^*-compactness of the means in $M(S)^{**}$, some subnet of μ_α, again denoted by μ_α, can be found satisfying (i) and (ii). Thus M is a topological left invariant mean on $M(S)^*$ and $M(\chi_T) = m(\xi_T) = 1$. So (3) implies (2). This completes the proof.

REMARKS. 1. Each φ in $P(S)$ is a mean on $L_\infty(S)$ if we define $\varphi(f) = \int f \cdot \varphi d\lambda$. This cannot be done for μ in $M_0(S)$ since $\int f d\mu$ is not well defined on $L_\infty(S)$ whose elements are equivalence classes of functions. This is why in the proof above, we have to consider measures μ_α in $M_a(S) \cong L_1(S)$, instead of $M_0(S)$.

2. It follows from Theorem 5.2 that for a locally compact group S, topological left thickness of a measurable subset T of S is equivalent to Day's condition (in [**4**, Theorem 7.8]) that for each compact set F in S and each $\epsilon > 0$, there is some $s \in S$ such that $\lambda(F \cap Ts) \geqq (1 - \epsilon) \cdot \lambda(F)$ where λ is the left Haar measure. Note that Day's original condition in [**4**, Theorem 7.8] is similar but with the set theoretical difference $F \sim Ts$ in place of the intersection $F \cap Ts$ above. His result as stated in Day [**4**, Theorem 7.8] is false as is easily seen by taking both S and T to be the real numbers. The correct version using set theoretical difference should be $\lambda(F \sim Ts) \leqq \epsilon \cdot \lambda(F)$.

6. An example.

Let S be the real numbers with addition. It is easy to see that $M(S)$ has a topological left invariant mean (since it is commutative). In fact, there is a net in $M_0(S)$ converging strongly to topological left invariance uniformly on compacta (Theorem 5.1). Let T be a measurable subset of S, then T is topological left thick if T contains intervals of arbitrarily large Lebesgue measure (i.e. length) no matter how thinly these intervals are scattered (cf. Day [**4**, Remarks after Corollary 7.5, p. 24]). For in this case, given any $0 < \epsilon \leqq 1$ and any finite interval I, we can choose some $a \in S$ such that $T - a \supset I$. Now if ϵ_a is the Dirac measure at a, then $\nu * \epsilon_a(T) = \nu(T - a) \geqq \nu(I) = 1 > 1 - \epsilon$ for any ν in $M_0(S)$ with $\nu(I) = 1$. Since each compact set in S is bounded, the result follows immediately. This result yields many mutually singular topological left invariant means on the real numbers, analogous to the situation of the integers. (See also Mitchell [**8**, Corollary 5, p. 258]). Of course, the same can also be done for the positive real numbers under addition, which is a locally compact semigroup.

Addendum. After the submission of this paper, the author learned that M. Day has obtained yet another similar characterisation of those Borel subsets on which some topological left invariant mean concentrates, namely, the topological left lumpy subsets which also extend Mitchell's concept of left thick subsets. His result are valid assuming only (topological) left amenability but not necessarily uniform strong left amenability.

REFERENCES

1. R. Arens, *The adjoint of a bilinear operator*, Proc. Amer. Math. Soc., **2** (1951), 839–848.
2. M. Day, *Amenable semigroups*, Illinois J. Math., **1** (1957), 509–544.
3. ———, *Amenability and equicontinuity*, Studia Math., **31** (1968), 481–494.
4. ———, *Semigroups and amenability*, Proceedings, Symposium on Semigroups, Wayne State University, Edited by K. Folley, Academic Press, 1969.
5. F. P. Greenleaf, *Invariant means on topological groups*, Van Nostrand Math. Stud. No. 16, Van Nostrand, New York, 1969.
6. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer–Verlag, Berlin, 1963.
7. J. L. Kelley, *General Topology*, Van Nostrand, New York, 1968.
8. T. Mitchell, *Constant functions and left invariant means on semigroups*, Trans. Amer. Math. Soc., **119** (1965), 244–261.
9. H. Reiter, *The convex hull of translates of a function in* L^1, J. London Math. Soc., **35** (1960), 5–16.
10. ———, *On some properties of locally compact groups*, Indag. Math., **27** (1965), 687–701.
11. James C. S. Wong, *Topologically stationary locally compact groups and amenability*, Trans. Amer. Math. Soc., **144** (1969), 351–363.
12. ———, *An ergodic property of locally compact amenable semigroups*, Pacific J. Math., **48** (1973), 615–619.
13. ———, *A characterisation of locally compact amenable subsemigroups*, to appear.

Received July 8, 1975. Research supported by the National Research Council of Canada Grant No. A8227.

UNIVERSITY OF CALGARY, CANADA