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LOCAL EVOLUTION SYSTEMS IN GENERAL BANACH SPACES

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A local evolution system $\{U(t, s)\}$ is defined and constructed from a family of nonlinear, multi-valued operators $\{A(t)\}$ with common domain D, in a real Banach space X. In particular, it is shown that there is a family of operators $\{U(t, s)\}$ with domains $\{D(t, s)\}$ satisfying:

$$U(t, s)\colon D(t, s) \to \bar{D},$$

$$D \subset \bigcup_{s<t} D(t, s) \quad \text{for each } s,$$

$$D(t, r) \subset D(s, r) \quad \text{for} \quad r \leqq s \leqq t,$$

$$U(t, t)x = x \quad \text{for} \quad x \in D(t, t) \supset D, \quad \text{and}$$

$$U(s, r)D(t, r) \subset D(t, s) \quad \text{and} \quad U(t, s)U(s, r) \supset U(t, r).$$

The existence of $\{U(t, s)\}$ is established by showing that $\lim \Pi_s^t (I - \Delta t_i A(t_i))^{-1} x$ exists for $x \in D$, where "lim" denotes the refinement limit. When this limit exists it is called the product integral, and $U(t, s)x$ is defined to be this product integral.

The time dependent evolution equation

$$u'(t) \in A(t)u(t), \qquad u(s) = x,$$

is also studied, and it is shown that when X^* is uniformly convex, a strong solution exists on $[s, T]$. Finally, the notion of a solution of

$$u'(t) \in A(t)u(t), \qquad u(0) = x,$$

with respect to $\{D_n\}$ is defined, where $\{D_n\}$ is a non-decreasing sequence of sets whose union is D. Such solutions are shown to be unique, and an existence theorem is proved in the case when X^* is uniformly convex.

1. Local evolution systems.

If A assigns to each $x \in X$, a subset Ax of X then A will be called a *multi-valued* operator in X. The domain of A, $D(A)$, is the set $\{x \in X\colon Ax \neq \emptyset\}$. The range of A, $R(A)$, is the set $\cup\{Ax\colon x \in D(A)\}$.

The usual operations on operators are defined for multi-valued operators in a straightforward manner, see, e.g., **[4]**.

A multi-valued operator A in X is said to be *ω-dissipative* if ω is a non-negative real number and

$$\|(x_1 - \lambda y_1) - (x_2 - \lambda y_2)\| \geqq (1 - \lambda\omega)\|x_1 - x_2\|$$

whenever $x_1,\ x_2 \in D(A),\ y_1 \in Ax_1,\ y_2 \in Ax_2$ and $\lambda > 0$.

Multi-valued, ω-dissipative operators are studied by numerous authors. See, for example, **[5]**, **[7]**, **[19]**, **[16]**, **[14]**, **[18]**, and **[20]**.

PROPOSITION 1.1. *Suppose A is a multi-valued, ω-dissipative operator and $0 < \lambda\omega < 1$, then*

(1) $$(I - \lambda A)^{-1}$$

is a function, and

$$\|(I - \lambda A)^{-1}x - (I - \lambda A)^{-1}y\| \leqq (1 - \lambda\omega)^{-1}\|x - y\|$$

for $x,\ y \in R(I - \lambda A)$.

(2) $$\|(I - \lambda A)^{-1}x - x\| \leqq \lambda(1 - \lambda\omega)^{-1}|Ax|$$

for $x \in R(I - \lambda A) \cap D(A)$, where $|Ax| = \inf_{y \in Ax}\|y\|$.

(3) $$(I - \lambda A)^{-1}x - x \in \lambda A(I - \lambda A)^{-1}x$$

for $x \in R(I - \lambda A)$.

(4) *If $\lambda > 0$ and μ is a real number, then*

$$\frac{\mu}{\lambda}x + \frac{\lambda - \mu}{\lambda}(I - \lambda A)^{-1}x \in R(I - \mu A),$$

and

$$(I - \lambda A)^{-1}x = (I - \mu A)^{-1}\left(\frac{\mu}{\lambda}x + \frac{\lambda - \mu}{\lambda}(I - \lambda A)^{-1}x\right)$$

for $x \in R(I - \lambda A)$.

Proof. See Crandall and Liggett **[4]**.

Whenever working with a composition of functions the following conventions are used.

$$\prod_{i=j}^{j} T_i x = T_j x,$$

$$\prod_{i=1}^{k+1} T_i x = T_{k+1}\left(\prod_{i=1}^{k} T_i x\right),$$

$$\prod_{i=j}^{k} T_i x = x \quad \text{if} \quad k < j.$$

Let F be a mapping from $[a, b] \times [0, \lambda_0]$ into the set of mappings on X and let $x \in X$. For each partition $R = \{r_i\}_{i=0}^n$ of $[a, b]$ with $\max_{1 \leqq i \leqq n} \Delta r_i < \lambda_0$, let the product $\Pi_{i=1}^n F(r_i, \Delta r_i)x$ be denoted by $\Pi_a^b F(R)x$. If there is a partition P_0 of $[a, b]$ so that $\|\Pi_a^b F(P)x - w\| < \epsilon$ whenever P is a refinement of P_0, then one writes $\Pi_a^b F(I, dI)x = w$, and w is called the *product integral of F from a to b with respect to x.* Some theorems about product integrals are found in **[9]**, **[23]**, **[24]**, and **[22]**. Throughout the paper $\{A(t): 0 \leqq t \leqq T\}$ will denote a family of multi-valued operators with common domain D.

$\{A(t)\}$ is said to satisfy *Condition* $\mathscr{K}$ if there is a non-decreasing function $\mathscr{L}: [0, \infty) \to [0, \infty)$ such that

$$|A(t)x| \leqq |A(s)x| + |t - s| \mathscr{L}(\|x\|)(1 + |A(s)x|)$$

for $x \in D$ and $0 \leqq s,\ t \leqq T$. Families of multi-valued operators which satisfy Condition $\mathscr{K}$ are studied in **[4]**, **[5]**, **[13]**, and **[10]**.

$\{A(t)\}$ is said to satisfy *Condition* $\mathscr{C}$ if there is a non-decreasing function $\mathscr{L}: [0, \infty) \to [0, \infty)$ and $\lambda_0 > 0$ such that

$$\|(I - \lambda A(t))^{-1}x - (I - \lambda A(s))^{-1}x\| \leqq \lambda |t - s| \mathscr{L}(\|x\|)(1 + |A(s)x|)$$

whenever $0 \leqq s,\ t \leqq T,\ 0 \leqq \lambda < \lambda_0$, and $x \in D$. Families of operators which satisfy Condition $\mathscr{C}$ are studied in **[4]**, **[5]**, and **[10]**.

Let $\Lambda_x(A(t)) = \Lambda_x$ be defined by

$$\Lambda_x = \{(\tau, r): r > 0 \quad \text{and} \quad B(x, r) \cap D \subset R(I - \lambda A(t)) \quad \text{for each} \quad t \in [0, T] \quad \text{and} \quad 0 < \lambda < \tau\}.$$

$\{A(t)\}$ is said to satisfy Condition $\mathscr{D}$ if $\Lambda_x \neq \emptyset$ for each $x \in D$. *Condition* $\mathscr{D}$ on the family $A(t) = A$ implies Condition I of **[1]**. See also **[16]**.

Let $S \subset X$, $0 < T$, and let $D(t, s) \subset \bar{S}$ for $0 \leqq s \leqq t \leqq T$. A family of operators $\{U(t, s)\}$ is called a *local evolution system on S if* $U(t, s): D(t, s) \to \bar{S}$, and

(i) $S \subset \bigcup_{s<t} D(t, s)$ for each $s \in [0, T)$

(ii) $D(t, r) \subset D(s, r)$ for $0 \leqq r \leqq s \leqq t \leqq T$

(iii) $U(t, t)x = x$ for $x \in D(t, t) \supset S$

(iv) $U(s, r)D(t, r) \subset D(t, s)$ and $U(t, s)U(s, r) \supset U(t, r)$ for $0 \leqq r \leqq s \leqq t \leqq T$.

In [7] Dorroh gives the definition of a local transformation semigroup. The above definition may be viewed as a generalization of that definition. Also, in a manner analogous to that in [7], one can show that a natural way for local evolution systems to arise is from solutions of time dependent nonlinear evolution equations.

THEOREM 1.2. *Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of ω-dissipative, multi-valued operators with common domain D, which satisfies Conditions $\mathscr{D}$, $\mathscr{K}$, and $\mathscr{C}$. Let $x \in D$, $(\lambda_0, r) \in \Lambda_x$, and $0 \leqq s \leqq t \leqq T$. Then there exists a positive real number $b = b(s, x, r)$ such that if $0 < t - s < b$, then $\Pi_s^t J(I, dI)x$ exists, where $J(t, \lambda) = (I - \lambda A(t))^{-1}$.*

The proof of this theorem will be given after the proof of Theorem 1.3.

THEOREM 1.3. *Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of ω-dissipative, multi-valued operators with common domain D, which satisfies Conditions $\mathscr{D}$, $\mathscr{K}$, and $\mathscr{C}$. Let $E(t, s)$ denote the set of all vectors x belonging to D for which $\Pi_s^{t'} J(I, dI)x$ exists for $s \leqq t' \leqq t$. Then*

$$\|\Pi_s^t J(I, dI)x - \Pi_s^t J(I, dI)y\| \leqq \exp(\omega(t - s))\|x - y\|$$

for x, $y \in E(t, s)$, and if we define $U(t, s)$: $\overline{E(t, s)} \to \bar{D}$ by

$$U(t, s)x = \begin{cases} \Pi_s^t J(I, dI)x, \ x \in E(t, s) \\ \lim_{n \to \infty} \Pi_s^t J(I, dI)x_n, \ x_n \in E(t, s), \ x_n \to x \in \overline{E(t, s)}, \end{cases}$$

then $\{U(t, s)\}$ is a local evolution system on D with $D(t, s) = \overline{E(t, s)}$.

Proof. First note that for $x, y \in E(t, s)$ there exists a partition P of $[s, t]$ such that if $0 < \|P\|\omega < 1$, and $P' = \{r_1, r_2, \cdots, r_n\}$ is a refinement of P, then

$$\|\Pi_s^t J(P')x - \Pi_s^t J(P')y\| \leqq \prod_{i=1}^{n} \frac{1}{1 - \mu_i \omega} \|x - y\|,$$

where $\mu_i = r_i - r_{i-1}$. Therefore, for x, $y \in E(t, s)$

$$\|\Pi_s^t J(I, dI)x - \Pi_s^t J(I, dI)y\| \leqq \exp(\omega(t - s))\|x - y\|,$$

and the first statement of the theorem is proved.

Now let $D(t, s) = \overline{E(t, s)}$ and check the four properties of a local evolution system.

Proof of (i). Let $s \in [0, T)$ and $x \in D$. Choose $t - s < b(s, x, r)$ (see Theorem 1.2) for some $r > 0$. By Theorem 1.2 it follows that $x \in E(t, s)$, and hence $D \subset \bigcup_{s<t} D(t, s)$.

Proof of (iv). One proves that $U(s, r)E(t, r) \subset \overline{E(t, s)}$. Then by definition of $U(s, r)$ the result will follow for $\overline{E(t, r)}$. Let $x \in E(t, r)$ and $t' \in [s, t]$. Choose a sequence $\{R_n\}$ of partitions of $[r, t']$ such that $s \in R_n$, and so that if $P_n = R_{n|[r,s]}$, then any refinement R'_n of R_n and any refinement P'_n of P_n have the property that

$$\|\Pi_r^{t'} J(R'_n)x - U(t, r)x\| < \frac{1}{n},$$

and

$$\|\Pi_r^{s} J(P'_n)x - U(s, r)x\| < \frac{1}{n}.$$

Let P'_n be a refinement of P_n and $y_n = \Pi_r^s J(P'_n)x$. One shows that $y_n \in \overline{E(t, s)}$ for large n, and noting that $y_n \to U(s, r)x$, it will follow that $U(s, r)x \in \overline{E(t, s)}$. Let $\epsilon > 0$ be given, and choose n large enough so that $2/n < \epsilon$. Let $Q_n = R_{n|[s,t']}$. Suppose that Q'_n is a refinement of Q_n, then $\Pi_r^{t'} J(Q'_n \cup P'_n)x$ is defined since $Q'_n \cup P'_n$ is a refinement of R_n. However,

$$\begin{aligned}\Pi_r^{t'} J(Q'_n \cup P'_n)x &= \Pi_s^{t'} J(Q'_n)\Pi_r^s J(P'_n)x \\ &= \Pi_s^{t'} J(Q'_n)y_n.\end{aligned}$$

Thus $\Pi_s^{t'} J(Q'_n)y_n$ is defined for each refinement Q'_n of Q_n. Next, let Q'_n and Q''_n be refinements of Q_n, then

$$\begin{aligned}&\|\Pi_s^{t'} J(Q'_n)y_n - \Pi_s^{t'} J(Q''_n)y_n\| \\ &\qquad \leqq \|\Pi_s^{t'} J(Q'_n \cup P'_n)x - U(t, r)x\| + \|U(t, r)x - \Pi_r^{t'} J(Q''_n \cup P'_n)x\| \\ &\qquad \leqq \frac{1}{n} + \frac{1}{n} \\ &\qquad < \epsilon,\end{aligned}$$

because $Q'_n \cup P'_n$ and $Q''_n \cup P'_n$ are refinements of R_n. Hence, by definition of $E(t, s)$, $y_n \in E(t, s)$, and thus it follows that $U(s, r)x \in \overline{E(t, s)}$.

Now one proves that $U(t, s)U(s, r)x = U(t, r)x$ for $x \in E(t, r)$. Let y_n, R_n, P_n, and Q_n be as above. Then by definition of $U(t, s)$,

$U(t,s)y_n \to U(t,s)U(s,r)x$. One proves that $U(t,s)y_n \to U(t,r)x$. Let $\epsilon > 0$ be given. Choose n large enough so that $1/n < \epsilon/2$. Choose Q'_n, a partition of $[s,t]$, so that $\|\Pi_s^t J(Q'_n)y_n - U(t,s)y_n\| < \epsilon/2$. Now for each such n,

$$\begin{aligned}\|U(t,s)y_n - U(t,r)x\| &< \|U(t,s)y_n - \Pi_s^t J(Q''_n)y_n\| \\ &\quad + \|\Pi_r^t J(Q''_n \cup P'_n)x - U(t,r)x\| \\ &< \epsilon\end{aligned}$$

where $Q''_n = Q'_n \cup Q_n$. This is the desired result.

In order to prove Theorem 1.2, the following four lemmas are needed. In these lemmas the notation given below is used.

(a) $M(x) = \sup_{0 \leqq t \leqq T} |A(t)x|$ for $x \in D$,
(b) $J(t,\lambda) = (I - \lambda A(t))^{-1}$,
(c) $\sigma(\mu, n) = \sum_{i=1}^{n} \mu_i$ for any sequence $\{\mu_i\}$ of real numbers,
(d) $u(a,\lambda,k) = \Pi_{i=1}^{k} J(a + \sigma(\lambda, i), \lambda_i)$ if $\sigma(\lambda,k) < T$.

LEMMA 1.4. *Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of ω-dissipative, multi-valued operators with common domain D, satisfying Conditions $\mathcal{D}$ and $\mathcal{K}$. Let $x \in D$, $s \in [0,T)$, $(\lambda_0, r) \in \Lambda_x$, $b = b(s,x,r) = \min\{T - s, r[\exp(2\omega(T-s))M(x)]^{-1}\}$, $\{r_i\}_{i=0}^{n}$ be a partition of $[s, s+b]$, and let $\mu_i = r_i - r_{i-1}$ for $i = 1,2,\cdots,n$. Suppose that $0 < \mu_i < \lambda_0$ and that $0 < \mu_i\omega < \frac{1}{2}$ for $i = 1,2,\cdots,n$. If $\sigma(\mu,k) < b$ for $k = 1,2,\cdots,n$ then $u(s,\mu,k)x$ is defined, $u(s,\mu,k)x \in B(x,r) \cap D$, and $\|u(s,\mu,k)x - x\| < \sigma(\mu,k)M(x)$.*

LEMMA 1.5. *Let the hypotheses of Lemma 1.4 be satisfied. If $\sigma(\mu,k) < b$ for $k = 1,2,\cdots,n$, then*

$$|A(r_k)u(s,\mu,k)x| \leqq \exp(2\omega\sigma(\mu,k))M_k,$$

where

$$M_k = M(x)\prod_{i=1}^{k}(1 + \mu_i L) + \sum_{i=1}^{k} \mu_i L \prod_{l=i+1}^{k}(1 + \mu_l L),$$

and $L = \mathcal{L}(r + \|x\|)$. Furthermore, there is a constant $R = R(s,x,r)$ so that if $\sigma(\mu,k) < b$, then

$$|A(r_k)u(s,\mu,k)x| \leqq R \quad \text{for} \quad k = 1,2,\cdots,n.$$

The proofs of Lemma 1.4 and Lemma 1.5 follow by induction on k, using the Proposition 1.1. See also **[1]**.

As an aid in stating the following lemmas, the following notation is introduced.

Let $\{a_k\}$, $\{b_k\}$ be sequences of nonnegative integers. For $r \leqq s$ and $0 \leqq t \leqq s-r+1$ let

$$A(r,s,t) = \{(x_r, x_{r+1}, \cdots, x_s) \in R^{s-r+1}\text{: exactly } t \text{ of the components are 1 and the remaining components are 0}\}.$$

Let $f\colon A(r,s,t) \to R$ be given by

$$f(x_r, x_{r+1}, \cdots, x_s) = \prod_{i=1}^{s} \eta_i \quad \text{where} \quad \begin{cases} \eta_i = a_i & \text{if } x_i = 1 \\ \eta_i = b_i & \text{if } x_i = 0. \end{cases}$$

Finally, define

$$\overset{s}{\underset{r}{[}} a, b]_t = \sum_{y \in A(r,s,t)} f(y).$$

For notational convenience if $s < r$ and $t \geqq 0$ define

$$\overset{s}{\underset{r}{[}} a, b]_t = 1.$$

LEMMA 1.6. *Let the hypotheses of Lemma* 1.4 *be satisfied. Let* $\mu_1, \mu_2, \cdots, \mu_n \leqq \lambda$ *and* $0 < \lambda < \lambda_0$. *In addition, let* $a_k = \mu_k/\lambda$, $b_k = 1 - a_k$, $s_k = k\lambda$, $m\lambda = b$, *for* $k = 1, 2, \cdots, n$, *and*

$$d_{m,n} = \left\| \prod_{k=1}^{n} J(r_k, \mu_k)x - \prod_{k=1}^{m} J(s_k, \lambda)x \right\|.$$

Then

$$\text{(i)} \qquad d_{k,j} \leqq \exp(2\omega\mu_j)\{a_j d_{k-1,j-1} + b_j d_{k,j-1} + e_{k,j}\},$$

and

$$\text{(ii)} \qquad d_{m,n} \leqq \exp(2\omega\sigma(\mu, n)) \left\{ \sum_{j=0}^{m} \overset{n}{\underset{1}{[}} a, b]_j d_{m-j,0} + \sum_{j=m}^{n} a_{n-j+1} \overset{n}{\underset{n-j+2}{[}} a, b]_{m-1} d_{0,n-j} + \sum_{k=0}^{m-1} \sum_{i=1}^{n-k} \overset{n}{\underset{i+1}{[}} a, b]_k e_{m-k,i} \right\}$$

for $1 \leqq m \leqq n$, *where*

$$e_{k,j} = \left\| J(s_k, \mu_j) \prod_{i=1}^{j-1} J(r_i, \mu_j) \prod_{i=1}^{j-1} J(r_i, \mu_i)x - J(r_j, \mu_j) \prod_{i=1}^{j-1} J(r_i, \mu_i)x \right\|.$$

Proof of (i). See Crandall and Pazy **[1]**.

The proof of (ii) involves a rather lengthy induction argument and can be found in the appendix to the author's dissertation.

LEMMA 1.7. *Let the hypotheses of Lemma* 1.6 *and Condition* $\mathscr{C}$ *be satisfied. If* $\sigma(a, n) = m$, *then*

(i) $$\sum_{j=0}^{m} \overset{n}{\underset{1}{[}} a, b]_j (m - j) = \sum_{j=m}^{n} \sum_{k=1}^{n-j} a_k a_{n-j+1} \overset{n}{\underset{n-j+2}{[}} a, b]_{m-1},$$

(ii) $$\sum_{j=0}^{m} (m - j) \overset{n}{\underset{1}{[}} a, b]_j \leqq \sqrt{m},$$

(iii) $$\sum_{k=0}^{m-1} \sum_{i=0}^{n-k} a_i \overset{n}{\underset{i+1}{[}} a, b]_k \,|(m - k) - \sigma(a, i)| \leqq m\sqrt{m}, \text{ and}$$

(iv) $d_{m,n} \leqq \exp(2\omega m\lambda)\{2\lambda\sqrt{m} + \lambda^2 m\sqrt{m}\}C$ *for some constant* C.

Proof. The proof of the (i) involves another lengthy induction argument and is given in the author's dissertation. The proof of (ii) is similar to the proof of (iii) and is easier, so only the proof of (iii) is given

Proof of (iii).

$$\sum_{k=0}^{m-1} \sum_{i=0}^{n-k} a_i \overset{n}{\underset{i+1}{[}} a, b]_k \,|(m - k) - \sigma(a, i)|$$

$$= \sum_{i=1}^{n-m} \sum_{k=0}^{m-1} a_i \overset{n}{\underset{i+1}{[}} a, b]_k \,|(m - k) - \sigma(a, i)|$$

$$+ \sum_{i=n-m+1}^{n-1} \sum_{k=0}^{n-i} a_i \overset{n}{\underset{i+1}{[}} a, b]_k \,|(m - k) - \sigma(a, i)|$$

$$\leqq \sum_{i=1}^{n} a_i \sum_{k=0}^{n-i} \overset{n}{\underset{i+1}{[}} a, b]_k \,|k - (a_{i+1} + \cdots + a_n)|.$$

However,

$$\sum_{k=0}^{n-i} \overset{n}{\underset{i+1}{[}} a, b]_k \, |k - (a_{i+1} + \cdots + a_n)|$$

$$\leqq \left\{ \sum_{k=0}^{n-i} \overset{n}{\underset{i+1}{[}} a, b]_k \right\}^{1/2} \left\{ \sum_{k=0}^{n-i} \overset{n}{\underset{i+1}{[}} a, b]_k (k - (a_{i+1} + \cdots + a_n))^2 \right\}^{1/2}$$

$$= \sum_{k=0}^{n-i} k^2 \overset{n}{\underset{i+1}{[}} a, b]_k - 2(a_{i+1} + \cdots + a_n) \sum_{k=0}^{n-i} k \overset{n}{\underset{i+1}{[}} a, b]_k$$

$$+ (a_{i+1} + \cdots + a_n)^2 \sum_{k=0}^{n-i} \overset{n}{\underset{i+1}{[}} a, b]_k \Bigg\}^{1/2}$$

$$= \left\{ \left(\sum_{k=i+1}^{n} a_k \right)^2 - \sum_{k=i+1}^{n} a_k^2 + \sum_{k=i+1}^{n} a_k \right.$$

$$\left. - 2(a_{i+1} + \cdots + a_n) \sum_{k=i+1}^{n} a_k + \left(\sum_{k=i+1}^{n} a_k \right)^2 \right\}^{1/2}$$

$$= \left\{ \sum_{k=i+1}^{n} a_k - \sum_{k=i+1}^{n} a_k^2 \right\}^{1/2}$$

$$\leqq \left\{ \sum_{k=i+1}^{n} a_k \right\}^{1/2}$$

$$\leqq \sqrt{m}.$$

Therefore,

$$\sum_{k=0}^{m-1} \sum_{i=0}^{n-k} a_i \overset{n}{\underset{i+1}{[}} a, b]_k \, |(m-k) - \sigma(a, i)|$$

$$\leqq \sum_{i=1}^{n} a_i \sqrt{m}$$

$$= m\sqrt{m}.$$

This completes the proof of (iii).

Proof of (iv). Using Lemma 1.4, Lemma 1.5, Lemma 1.6 and Condition $\mathscr{C}$ one can show that there are constants $c_1 = c_1(x)$ and $c_2 = c_2(s, x, r)$ so that

$$d_{m,n} \leqq \exp(2\omega\sigma(\mu, n)) \left\{ \lambda \sum_{j=0}^{m} \overset{n}{\underset{1}{[}} a, b]_j (m-j) c_1 \right.$$

$$+ \lambda \sum_{j=m}^{n} \sum_{k=1}^{n-j} a_k a_{n-j+1} \overset{n}{\underset{n-j+2}{[}} a, b]_{m-1} c_1$$

$$\left. + \lambda^2 \sum_{k=0}^{n-1} \sum_{i=1}^{n-k} a_i \overset{n}{\underset{i+1}{[}} a, b]_k \, |(m-k) - \sigma(a, i)| c_2 \right\}$$

for $1 \leqq m \leqq n$.

Letting $C = \max(c_1, c_2)$ it follows from (i), (ii) and (iii) that

$$d_{m,n} \leqq \exp(2\omega m\lambda)\Big\{2\lambda \sum_{j=0}^{m} \prod_{1}^{n} [a, b]_j (m-j)$$

$$+ \lambda^2 \sum_{k=0}^{m-1} \sum_{i=1}^{n-k} a_i \prod_{i+1}^{n} [a, b]_k \,|(m-k) - \sigma(a, i)|\Big\}\, C$$

$$\leqq \exp(2\omega m\lambda)\{2\lambda \sqrt{m} + \lambda^2 m \sqrt{m}\} C.$$

This completes the proof of Lemma 1.7.

Now for the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\epsilon < 0$ be given. Let

$$P_m = \left\{ s + i\left(\frac{t-s}{m}\right)\right\}_{i=0}^{m}, \qquad m = 1, 2, \cdots.$$

Choose m_0 large enough so that

$$\exp(2\omega(t-s))\{2(t-s) + (t-s)^2\} \frac{C}{\sqrt{m_0}} < \frac{\epsilon}{2},$$

where C is chosen as in Lemma 1.7. Let $P' = \{r'_i\}_{i=0}^{n}$ and $P'' = \{r''_i\}_{i=0}^{k}$ be refinements of P_{m_0} satisfying Lemma 1.4. Then,

$$\|\Pi_s^t(P')x - \Pi_s^t(P'')x\|$$

$$\leqq \|\Pi_s^t(P')x - \Pi_s^t(P_{m_0})x\| + \|\Pi_s^t(P_{m_0})x - \Pi_s^t(P'')x\|$$

$$= d_{m_0,n} + d_{m_0,k}$$

$$\leqq 2\exp\left(2\omega m_0\left(\frac{t-s}{m_0}\right)\right)\left\{2\,\frac{t-s}{m_0}\sqrt{m_0} + \frac{(t-s)^2}{m_0^2}\, m_0\sqrt{m_0}\right\} C$$

$$< \epsilon.$$

Thus $\Pi_s^t J(I, dI)x$ exists, and Theorem 1.2 is proved.

REMARK. Notice that if $\bar{D} \subset R(I - \lambda A(t))$, for $t \in [0, T]$ and $0 < \lambda < \lambda_0$, then choosing $r = \infty$ and $b(s, x, r) = T - s$, the condition $0 \leqq t - s < b(s, x, r)$ becomes $s \leqq t \leqq T$. Hence Theorem 1.2 implies Theorem 2.1 of Crandall and Pazy **[6]**.

After obtaining these results, it was discovered that A. T. Plant **[22]** had proven the existence of a slightly more general type of product integral. He uses a stronger substitute for Condition $\mathscr{D}$ and his results only apply when there is a global evolution system. This proof is very similar to Plant's except that he uses probabilistic methods instead of mathematical induction.

2. Time dependent evolution equations. The results found in this section are similar to those of Crandall and Pazy **[6]** and Brezis and Pazy **[1]**.

Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of multi-valued operators with common domain D. Let $x \in D$ and $s \in [0, T)$.

A function $u: [s, T] \to X$ is said to be a *strong solution* of

$$u'(t) \in A(t)u(t), \quad u(s) = x$$

if u satisfies the following conditions:

(i) u is Lipschitz continuous;

(ii) $u'(t)$ exists a.e. on (s, T); and

(iii) $u(t) \in D$ a.e. on (s, T), $u(s) = x$, and $u'(t) \in A(t)u(t)$ a.e. on (s, T).

THEOREM 2.1. *Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of ω-dissipative, multi-valued operators with common domain D, satisfying Conditions $\mathscr{D}$, $\mathscr{K}$, and $\mathscr{C}$. Let $x \in D$, $s \in [0, T)$, and $(\lambda_0, r) \in \Lambda_x$. If the problem*

$$u'(t) \in A(t)u(t), \; u(s) = x$$

has a strong solution u on $[s, T]$, then

$$U(t, s)x = u(t) \quad on \quad [s, s + b),$$

where b is chosen as in Theorem 1.2.

Proof. The proof of this theorem follows from Theorem 1.2 using arguments like those found in **[6]**.

Let X^* denote the dual space of X. Let $\langle x, f\rangle$ denote the value of f at x, for $x \in X$ and $f \in X^*$. The *duality mapping F* is the mapping of X into X^* defined by

$$F(x) = \{f \in X^*: \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

The following three facts are well-known, see **[13]**.

(1) If X^* is uniformly convex then F is single-valued and uniformly continuous on bounded sets.

(2) Let $x, y \in X$. Then $\|x\| \leqq \|x + \alpha y\|$ for every $\alpha > 0$, if and only if there is an $f \in F(x)$ so that $\langle y, f\rangle \geqq 0$.

(3) Let u be an X-valued function on an interval of real numbers. Suppose u has a weak derivative $u'(s) \in X$ at s. If $\|u(\cdot)\|$ is also differentiable at s, then

$$\|u(s)\| \frac{d}{ds} \|u(s)\| = \langle u'(s), f \rangle$$

for every $f \in F(u(s))$.

Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of multi-valued operators with common domain D. $\{A(t)\}$ is said to satisfy *Condition* $\mathcal{M}$ if whenever $t_n \in (0, T)$ and $(x_n, y_n) \in A(t_n)$, $n = 1, 2, 3, \cdots$, and $t_n \to t$, $x_n \to x$, and $y_n \rightharpoonup y$, then $(x, y) \in A(t)$.

Multi-valued operators satisfying this type of condition are studied in **[17]**.

The existence theorem which follows and its proof are generalizations of the work of Brezis and Pazy **[1]** to the time dependent case.

THEOREM 2.2. *Let X^* be uniformly convex. Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of ω-dissipative, multi-valued operators with common domain D, satisfying Conditions $\mathcal{D}$, $\mathcal{K}$, $\mathcal{C}$, and $\mathcal{M}$. Let $x \in D$ and $s \in [0, T)$, then there exists a unique strong solution to the initial value problem*

$$u'(t) \in A(t)u(t),\ u(s) = x$$

on the interval $[s, T]$.

The three lemmas below and the following definition are needed in the proof of this theorem.

Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of multi-valued operators. $\{A(t)\}$ is said to satisfy *Condition* $\mathcal{R}$ if whenever $t_n \in [0, T]$ and $(x_n, y_n) \in A(t_n)$ $n = 1, 2, \cdots$ and $x_n \to x$, $y_n \to y$, $t_n \to t$, then $(x, y) \in A(t)$.

LEMMA 2.3. *Let X^* be uniformly convex. Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of ω-dissipative, multi-valued operators with common domain D, satsfying Conditions $\mathcal{D}$, $\mathcal{K}$, and $\mathcal{M}$. Let $x \in D$, $(\lambda_0, r) \in \Lambda_x$, $s \in [0, T)$ and $b = b(s, x, r)$. If the sequence of functions $\{u_n\}$ defined below converges pointwise to a function u on $[s, s + b)$, then $u(t) \in D$ for $t \in [s, s + b)$.*

Proof. Define the sequences $u_n : [s, s + b) \to X$,

$$v_n : [s, s + b) \to X$$

as follows:

Let $\{\lambda_n\}=\{b/n\}$ and $\{r_{i,n}\}=\{s+i\lambda_n\}$ for $i=0,1,2,\cdots,n$. Then

$$u_n(t)=u(s,\lambda,k-1)x \quad \text{for} \quad r_{k-1,n}\leqq t<r_{k,n},\ k=1,2,\cdots,n$$

and

$$v_n(t)=u(s,\lambda,k-1)x+\frac{1}{\lambda_n}[u(s,\lambda,k)x-u(s,\lambda,k-1)x](t-r_{k,n})$$

for $r_{k-1,n}\leqq t<r_{k,n}$, $k=1,2,\cdots,n$.

Let $t\in[s,s+b)$ and choose a sequence of nonnegative integers $\{k_n\}$ so that $t_n=s+k_n\lambda_n$, $t_n<t$, and $t-t_n<\lambda_n$. Letting $x_n=J(t_n,\lambda_n)v_n(t_n)$, and $y_n=\lambda_n^{-1}[J(t_n,\lambda_n)v_n(t_n)-v_n(t_n)]$, we obtain that $y_n\in A(t_n)x_n$ and that $\|y_n\|\leqq(1-\lambda_n\omega)^{-1}|A(t_n)v_n(t_n)|$. Thus, it follows from Lemma 1.5 that $\{y_n\}$ is bounded. Because X^* is uniformly convex, $y_{n_k}\rightharpoonup y$ for some subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and some $y\in X$. Using Lemma 1.5 and the fact that $v_n(t)\to u(t)$ if and only if $u_n(t)\to u(t)$, it is not hard to show that $x_n\to u(t)$. Thus by Condition $\mathcal{M}$ $(u(t),y)\in A(t)$, i.e., $u(t)\in D$.

LEMMA 2.4. *Let X^* be uniformly convex. Let $\{A(t)\colon 0\leqq t\leqq T\}$ be a family of ω-dissipative, multi-valued operators with common domain D satisfying Condition $\mathcal{R}$. Let $A(t)x\subset B(t)x$ for $x\in D$, where $\{B(t)\colon 0\leqq t\leqq T\}$ is also ω-dissipative. Suppose that $D(B(t))=D_B$ for each $t\in[0,T]$. Suppose that for each $x\in D_B$, there is a ball $B(x,r)$ and a number $\lambda_0>0$ such that*

$$B(x,r)\cap D_B\subset R(I-\lambda A(t))$$

for $0<\lambda<\lambda_0$, and $0\leqq t\leqq T$. Suppose that $0\leqq s<b'<T$, and that there is a function $u\colon[s,b']\to X$ satisfying: $u(t)\in D_B$ for $t\in[s,b']$, u is differentiable a.e. on (s,b'), and $u'(t)\in B(t)u(t)$ a.e. on (s,b'). Then $u(t)\in D$ a.e. on (s,b'), and $u'(t)\in A(t)u(t)$ a.e. on (s,b').

Proof. Choose $t\in(s,b')$ so that $x\equiv u(t)\in D_B$, u is differentiable at t, and $u'(t)\in B(t)u(t)$. Let λ_0 and r be chosen such that

$$B(x,r)\cap D_B\subset R(I-\lambda A(t))$$

for $0<\lambda<\lambda_0$ and $0\leqq t\leqq T$. Note that for t' close to t, $u(t')\in B(x,r)$, since u is continuous at t. Choose an increasing sequence $\{t_n\}$ so that $t_n\to t$ and let $\lambda_n=t-t_n$. Then

$$u(t_n)\in B(x,r)\cap D_B\subset R(I-\lambda_nA(t_n))$$

for large n. So $(I-\lambda_nA(t_n))^{-1}u(t_n)\equiv x_n\in D$, and $u(t_n)=x_n-\lambda_ny_n$ for some $y_n \in A(t_n)x_n \subset B(t_n)x_n$. Since $u'(t) \in B(t)u(t)$ and $B(t)$ is ω-dissipative,

$$\langle u'(t) - y_n, F(x - x_n)\rangle \leqq \omega \|x - x_n\|^2.$$

Now let

$$\phi(s) = u'(t) - \frac{u(s) - u(t)}{s - t}.$$

Then

$$\begin{aligned}\phi(t_n) &= u'(t) - \frac{u(t_n) - u(t)}{t_n - t}\\ &= u'(t) - \frac{x_n - \lambda_n y_n - x}{-\lambda_n}\\ &= u'(t) - \frac{x - x_n}{\lambda_n} - y_n.\end{aligned}$$

Thus,

$$\phi(t_n) + \frac{x - x_n}{\lambda_n} = u'(t) - y_n,$$

and

$$\left\langle \phi(t_n) + \frac{x - x_n}{\lambda_n}, F(x - x_n)\right\rangle \leqq \omega \|x - x_n\|^2.$$

So

$$\langle \phi(t_n), F(x - x_n)\rangle \leqq \frac{\lambda_n \omega - 1}{\lambda_n} \|x - x_n\|^2,$$

or

$$\frac{1 - \lambda_n \omega}{\lambda_n} \|x - x_n\|^2 \leqq \langle \phi(t_n), -F(x - x_n)\rangle \leqq \|\phi(t_n)\| \cdot \|x - x_n\|,$$

or

$$\|x - x_n\| \leqq \frac{\lambda_n}{1 - \lambda_n \omega} \|\phi(t_n)\|.$$

Therefore, $x_n \to x$. Now

$$\|u'(t) - y_n\| \leqq 3\|\phi(t_n)\|$$

for large n, so that $y_n \to u'(t)$. Hence Condition $\mathcal{R}$ shows that $u'(t) \in A(t)u(t)$ and the proof is complete.

LEMMA 2.5. *Let X be a reflexive Banach space and let $\{v_n(t)\}$ be a sequence in $L_p(a,b;X)$, $p>1$, such that $\{v_n(t)\}$ is bounded for almost all $t\in(a,b)$. Let $V(t)$ denote the set of weak cluster points of $\{v_n(t)\}$. If v_n converges weakly to u in $L_p(a,b;X)$ then $u(t)$ belongs to the closed convex hull of $V(t)$ a.e. on (a,b).*

Proof. See Kato **[14]**.

Proof of Theorem 2.2. The proof of the uniqueness of solution is standard and will not be given here.

Define $B(t)$, *an extension of* $A(t)$ for $t\in[0,T]$, as follows:

$$D(B(t))=D \text{ for } 0\leqq t\leqq T, \text{ and}$$

$$B(t)x=\text{closed convex hull of } A(t)x.$$

$B(t)$ is ω-dissipative for each $t\in[0,T]$. Let $0<b'<b$, and $x\in D$. Then by Theorem 1.2, the sequence $\{u_n\}$ of functions defined in Lemma 2.1 converges uniformly to $U(t,s)x$ on $[s,b']$.

Let $u(t)=U(t,s)x$. Then by Lemma 2.3, $u(t)\in D$ for $t\in[s,b']$. Also, note that since u is Lipschitz continuous and X^* is uniformly convex, $u'(t)$ exists almost everywhere on $(s,s+b')$.

Let $\{v_n\}$ be the sequence of functions defined in Lemma 2.3. We show that $v'_n\rightharpoonup u'$ in $L_p(s,s+b';X)$ for $1<p<\infty$. Since u_n converges uniformly to u, it follows that v_n also converges uniformly to u on $[s,s+b']$. Thus

$$\int_s^{s+b'} v'_n(t)f(t)dt\to-\int_s^{s+b'} u(t)f(t)dt,$$

for $f\in C'_0(s,s+b';R)$ where $C'_0(a,b;R)$ denotes the continuously differentiable real-valued functions which vanish outside of (a,b). Note also that

$$\int_s^{s+b'} v'_n(t)f(t)dt\to\int_s^{s+b'} u'(t)f(t)dt$$

for $f\in C'_0(s,s+b';R)$, so that

$$g\left(\int_s^{s+b'} v'_n(t)f(t)dt\right)\to g\left(\int_s^{s+b'} u'(t)f(t)dt\right)$$

for $f\in C'_0(s,s+b';R)$ and $g\in X^*$. Since the Lipschitz norm of v'_n is bounded, it follows that some subsequence $\{v'_{n_k}\}$ of $\{v'_n\}$ converges to some element w belonging to L_p, for $1<p<\infty$. Thus,

$$g\left(\int_s^{s+b'} v'_{n^k}(t) f(t) dt\right) \to g\left(\int_s^{s+b'} w f(t) dt\right)$$

for $f \in C_0'(s, s+b'; R)$ and $g \in X^*$, so it follows that $w = u'$ and $v'_n \rightharpoonup u'$ in L_p.

Next observe that from Condition $\mathcal{M}$ it follows that the set of weak cluster points of $\{v'_n(t)\}$, denoted by $V(t)$, is contained in $A(t)u(t)$. Thus, from Lemma 2.5,

$$\begin{aligned} u'(t) \in V(t) &\subset \text{closed convex hull of } A(t)u(t) \\ &= B(t)u(t). \end{aligned}$$

Now since $\{A(t)\colon 0 \leqq t \leqq T\}$ and $\{B(t)\colon 0 \leqq t \leqq T\}$ satisfy the hypothesis of Lemma 2.4,

$$u'(t) \in A(t)u(t) \quad \text{a.e.} \quad \text{on} \quad (s, s+b').$$

Next, show that u is a solution on $[s, T]$. Let u be a solution on $[s, T_1]$, where T_1 is maximal. If $T_1 \neq T$, choose $t_n \to T$. Then $u(t_n) \to u_0 \in X$, because u is Lipschitz continuous. Since $\{A(t)\colon 0 \leqq t \leqq T\}$ satisfies Condition $\mathcal{M}$, $u_0 \in D$. Hence consider the problem

$$u'(t) \in A(t)u(t), \ u(T_1) = u_0.$$

It will have a solution $v(t)$ on $[T_1, T_{u_0})$. Letting

$$f(t) = \begin{cases} u(t) & \text{on} \quad [s, T_1] \\ v(t) & \text{on} \quad [T_1, T_{u_0}) \end{cases},$$

extend the original solution, contradicting the maximality of T_1.

This concludes the proof of Theorem 2.2.

3. A local abstract Cauchy problem. In [1], Brezis and Pazy show that if X^* is uniformly convex, and A is a dissipative, demi-closed operator which satisfies Condition I, then the initial value problem

$$u'(t) \in Au(t), \ u(0) = x$$

has a unique global solution. The techniques of Brezis and Pazy may also be used in solving a local abstract Cauchy problem of a similar nature. Attention is confined to multi-valued operators which satisfy the Condition $\mathcal{S}$ below.

Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of multi-valued operators with common domain D. $\{A(t)\}$ is said to satisfy *Condition* $\mathcal{S}$ if there is a non-decreasing sequence of sets $\{D_n\}$ so that $D = \bigcup_{n=1}^{\infty} D_n$, and $sA_n(t) \equiv A(t)_{|D_n}$ is ω_n-dissipative with $0 \leqq \omega_1 \leqq \omega_2 \leqq \cdots$.

Multi-valued operators satisfying similar conditions are studied in [3].

In the pages which follow, the problem

$$u'(t) \in A_n(t)u(t), \ u(0) = x \in D_1$$

will be denoted by ACP_n, and the problem

$$u'(t) \in A(t)u(t), \ u(0) = x$$

will be denoted by ACP.

Now consider the problem of finding a solution of ACP with $u(0) = x \in D_1$, where $\{A(t)\}$ satisfies Condition $\mathcal{S}$. It is easy to show that if $x \in D_1$, and ACP_n has a solution on $[0, b_n)$ and $b_n \to b$, then ACP has a solution on $[0, b)$. This is proved in Theorem 3.3. With only the Condition $\mathcal{S}$, this solution does not have to be unique, as the following example illustrates.

Define $A: R \times [0, 1] \setminus (0, 1) \times (0, 1) \to R \times R$, by

$$A(x, y) = \begin{cases} (0, 0), & -\infty < x \leqq 0, \ 0 \leqq y \leqq 1 \\ (x, 0), & 0 \leqq x \leqq 1, \ y = 1 \\ (1, 0), & 1 \leqq x < \infty, \ 0 \leqq y \leqq 1 \\ (\sqrt{x}, 0), & 0 \leqq x \leqq 1, \ y = 0. \end{cases}$$

Let $D_n = D(A) \setminus (0, n^{-2}) \times \{0\}$. Then $A(t) = A$ satisfies Condition $\mathcal{S}$. In particular, each A_n is n-dissipative, closed, and satisfies Condition $\mathcal{D}$. Thus it follows from Theorem 2.2 that ACP_n with $x = (0, 0)$ has a unique solution. This solution is, of course, $u_n(t) = (0, 0)$. However, ACP with $x = (0, 0)$ has $u(t) = (t^2/4, 0)$ for $0 \leqq t \leqq 2$ as a solution, together with $u(t) = (0, 0)$. We therefore make the following definition.

Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of multi-valued operators with common domain D, which satisfies Condition $\mathcal{S}$, with $D = \bigcup_{n=1}^{\infty} D_n$; u is said to be *a solution of* ACP *with respect to* $\{D_n\}$ *on* $[0, b)$, if for each $t_0 \in [0, b)$, u is a solution of ACP_n on $[0, t_0)$ for some $n = n(t_0)$.

In the above example, $u(t) = (t^2/4, 0)$ is not a solution of ACP with respect to $\{D_n\}$, while $u(t) = (0, 0)$ is such a solution.

THEOREM 3.1. *Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of multi-valued operators with common domain D. If $\{A(t)\}$ satisfies Condition $\mathcal{S}$ with*

$D = \bigcup_{n=1}^{\infty} D_n$, then ACP *with $x \in D_1$ has at most one solution with respect to $\{D_n\}$ on the interval* $[0, b)$.

Proof. Suppose that u and v are solutions of ACP with respect to $\{D_n\}$ on $[0, b)$. Choose $t_0 \in [0, b)$. Then $u(t) \in D_n$ and $u'(t) \in A_n(t)u(t)$ for some n and almost all $t \in [0, b)$. Also, $v(t) \in D_m$ and $v'(t) \in A_m(t)v(t)$ for some m and almost all $t \in [0, b)$. Suppose that $m < n$. Then

$$\begin{aligned} &\|u(t) - v(t)\| \frac{d}{dt} \|u(t) - v(t)\| \\ &= \langle u'(t) - v'(t), f(t) \rangle \\ &\leqq \omega_n \|u(t) - v(t)\|^2 \end{aligned}$$

because

$$v'(t) \in A_m(t)v(t) \subset A_n(t)v(t),$$

and

$$f(t) \in F(u(t) - v(t)).$$

Hence, if $t \in [0, t_0)$ then

$$\|u(t) - v(t)\| \leqq \omega_n \int_0^t \|u(s) - v(s)\| \, ds,$$

and it follows that $u(t) = v(t)$. Since t_0 was arbitrary, $u(t) = v(t)$ for all $t \in [0, b)$.

REMARK. An interesting question whose answer is unknown to the author is: If u is a solution of ACP with respect to $\{D_n\}$ and v is a solution of ACP with respect to $\{E_n\}$, where $D = \bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} E_n$, does $u = v$?

THEOREM 3.2. *Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of multi-valued operators with common domain D, which also satisfies Condition $\mathcal{S}$. If u_n is a solution of* ACP_n *on $[0, b_n)$, and if $0 < b_1 \leqq b_2 \leqq \cdots$, then $u_n = u_{n+1|[0,b_n)}$.*

Proof. The proof is straightforward and will not be given.

THEOREM 3.3. *Let $\{A(t): 0 \leqq t \leqq T\}$ be a family of multi-valued operators with common domain D, which satisfies Condition $\mathcal{S}$. If u_n is a solution of* ACP_n *on $[0, b_n)$, with $\{b_n\}$ increasing and $b_n \to b$, then*

$$u(t)=\begin{cases} u_1(t) & \text{on} \quad [0, b_1) \\ u_{n+1}(t) & \text{on} \quad [b_n, b_{n+1})\ n=1,2,\cdots \end{cases}$$

is a solution of ACP *with respect to* $\{D_n\}$ *on* $[0, b)$.

The proof is omitted.

THEOREM 3.4. *Let* X^* *be uniformly convex. Let* $\{A(t)\colon 0 \leqq t \leqq T\}$ *be a family of multi-valued operators with common domain D. Suppose that* $\{A(t)\}$ *is* ω*-dissipative and satisfies Conditions* $\mathcal{D}$, $\mathcal{K}$, *and* $\mathcal{C}$,. *Let* $\tilde{A}(t)$ *denote the smallest extension of* $A(t)$ *for which* $\{\tilde{A}(t)\colon 0 \leqq t \leqq T\}$ *satisfies Condition* $\mathcal{M}$. *Let* $\tilde{\tilde{A}}(t)x$ *denote the closed convex hull of* $\tilde{A}(t)x$ *for* $x \in D(\tilde{A}(t))$. *Then there is a number* $b > 0$ *and a unique function* $u\colon [0, b) \to X$ *such that* $u'(t) \in \tilde{\tilde{A}}(t)u(t)$ *a.e. on* $[0, b)$, *and* $u(0) = x \in D$.

Proof. Let b be as in Lemma 1.4. Let $u_n\colon [0, b) \to X$, and $v_n\colon [0, b) \to X$, be as in Lemma 2.3. Then by Condition $\mathcal{M}$, as in the proof of Lemma 2.3, if $t \in [0, b)$ and $u_n(t) \to u(t)$, then

$$u(t) \in D(\tilde{A}(T)) \subset D(\tilde{\tilde{A}}(t)).$$

Now as in the proof of Theorem 2.2 it follows that $u'(t)$ exists a.e. on $(0, b)$, and $u_n' \rightharpoonup u'$ in $L_p(0, b; X)$ for $1 < p < \infty$. Again, as in the proof of Theorem 2.2 it follows that the set of weak cluster points of $\{v_n'(t)\}$, denoted by $V(t)$, is contained in $\tilde{A}(t)u(t)$. Finally, using Lemma 2.5,

$$\begin{aligned} u'(t) &\in \text{closed convex hull of } V(t) \\ &\subset \text{closed convex hull of } \tilde{A}(t)u(t) \\ &= \tilde{\tilde{A}}(t)u(t) \end{aligned}$$

for almost all $t \in (0, b)$. This concludes the proof of Theorem 3.4.

THEOREM 3.5. *Let* X^* *be uniformly convex. Let* $\{A(t)\colon 0 \leqq t \leqq T\}$ *be a family of multi-valued operators with common domain D. Suppose that* $\{A(t)\}$ *satisfies Condition* $\mathcal{S}$ *with* $D = \bigcup_{n=1}^{\infty} D_n$, *and each* $\{A_n(t)\colon 0 \leqq t \leqq T\}$ *satisfies Conditions* $\mathcal{D}$, $\mathcal{K}$, *and* $\mathcal{C}$, *and that*

$$\tilde{\tilde{A}}_n(t) \subset A_{n+1}(t)$$

(*see Theorem* 3.4) *for each positive integer n and each* $t \in [0, T]$. *Then there is a number* $b > 0$ *and a unique function* $u\colon [0, b) \to X$ *so that u is a solution of* ACP *with respect to* $\{D_n\}$ *on* $[0, b)$.

Proof. The family $\{A_n(t): 0 \leqq t \leqq T\}$ satisfies Conditions $\mathscr{D}$, $\mathscr{H}$, and $\mathscr{C}$, so by Theorem 3.4 there exists a number $b_n > 0$ and a function $u_n: [0, b_n) \to X$ such that $u_n'(t) \in \tilde{\tilde{A}}_n(t)u_n(t)$ a.e. on $[0, b_n)$ for $n = 1, 2, 3, \cdots$. Thus, $u_n'(t) \in A_{n+1}(t)u_n(t)$ a.e. on $[0, b_n)$. If the sequence $\{b_n\}$ is bounded, let $b = \sup_n \{b_n\}$; otherwise, let $b = +\infty$. In either instance, if $b_k = b$ for some k, define $u: [0, b) \to X$ by $u(t) = u_k(t)$. If $b_n \neq b$ for all n, then select an increasing subsequence $\{b_{n_k}\}$ of $\{b_n\}$ so that $b_{n_k} \to b$ and define $u: [0, b) \to X$ as in Theorem 3.3, and the result follows from that theorem.

The uniqueness follows from Theorem 3.1.

Now consider alternative approaches to solving the problem

$$u'(t) \in A(t)u(t), \; u(0) = x,$$

where $\{A(t)\}$ satisfies Condition $\mathscr{S}$. In order to facilitate the discussion, consider the specific case in which $X = R$, $A(t)x = Ax = x^2$ for $x \in \{X \in R: x \geqq 0\} = D(A)$, and $D_n = [0, n)$. Letting $b_n = (n - x)/nx$ for $x \in D_n$, one gets $u_n(t) = x/(1 - tx)$ as solutions of ACP$_n$ on $[0, b_n)$.

Theorem 3.1 of [1] does not apply to ACP$_n$ because A_n is not closed, and it does not apply directly to $u' \in \tilde{A}_n u$, or $u' \in \tilde{\tilde{A}}_n u$ because the operators $\tilde{A}_n$ and $\tilde{\tilde{A}}_n$ do not satisfy Condition I of [1].

In Theorem 3.5, this problem is solved by using the methods (but not the results) of [1] because $\tilde{\tilde{A}}_n \subset A_{n+1}$.

It should be noted that Theorem 3.1 of [1] can be applied in a different way, at least in this example. Let

$$A_n^* x = \begin{cases} A_n x & \text{if} \quad 0 \leqq x < n \\ [0, n^2] & \text{if} \quad x = n \end{cases}$$

Then A_n^* is closed and satisfies Condition I. Furthermore, the solution v_n of $v' \in A_n^* v$ agrees with u_n on $[0, b_n)$ and is constant thereafter.

It is not clear that this method of finding closed extensions of A_n which satisfy Condition I can be generalized. This process may sometimes be possible, at least in Hilbert spaces, by using methods like those in [2]. However, even if such extensions can be found, it is not clear how a solution of ACP can be constructed from the solutions v_n of $v' \in A_n^* v$. This is because one would not know in general that $v_n'(t) \in Av_n(t)$, or even that $v_n(t) \in D$, for small positive t. It can happen, for example, that $A_n v$ is *properly* contained in $A_n^* v$ for $v \in D_n$, and D_n is not open in $D(A_n^*)$. See Example 2 in [7].

REFERENCES

1. H. Brezis and A. Pazy, *Accretive sets and differential equations in Banach spaces*, Israel J. Math., **8** (1970), 367–383.
2. ———, *Semi-groups of nonlinear contractions on convex sets*, J. Functional Analysis, **6** (1970), 237–281.
3. J. Chambers and S. Oharu, *Semi-groups of local lipschitzians in a Banach space*, Pacific J. Math., **39** (1971), 89–112.
4. M. Crandall and T. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math., **93** (1971), 265–298.
5. M. Crandall and A. Pazy, *Semi-groups of nonlinear contractions and dissipative sets*, J. Functional Analysis, **3** (1969), 376–418.
6. ———, *Nonlinear evolution equations in Banach spaces*, MRC Technical Summary Report, No. 1191.
7. J. Dorroh, *Semigroups of nonlinear transformations with decreasing domain*, J. Math. Analysis and Applications, **34** (1971), 396–411.
8. N. Dunford and J. Schwartz, *Linear Operators*, Part I. Interscience, New York, 1958.
9. W. Fitzgibbon, *Time dependent nonlinear Cauchy problems in Banach spaces*, Proc. Amer. Math. Soc., **36** (2) (1972), 525–530.
10. ———, *Approximations of nonlinear evolution equations*, J. Math. Soc. Japan, **25** (1973), 211–221.
11. E. Hille and R. Phillips, *Functional analysis and semi-groups*, revised ed., Amer. Math. Soc. Colloquium Publications, vol. 31, 1957.
12. T. Kato, *Integration of the equation of evolution in Banach spaces*, J. Math. Soc. Japan, **5** (2) (1953), 208–235.
13. ———, *Nonlinear semi-groups and evolution equations*, J. Math. Soc. Japan, **19** (1967), 508–520.
14. ———, *Accretive operators and nonlinear evolution equations in Banach spaces*, Proc. Symposium Pure Math., **18** (1969), Part I, Amer. Math. Soc., Providence, Rhode Island, 138–161.
15. R. Martin, Jr., *A global existence theorem for autonomous differential equations in a Banach space*, Proc. Amer. Math. Soc., **27** (1970), 307–314.
16. ———, *Differential equations in closed subsets of a Banach space*, Trans. Amer. Math. Soc., **179** (1973), 399–414.
17. ———, *Generating an evolution system in a class of uniformly convex spaces*, J. Functional Analysis, **11** (1972), 62–76.
18. I. Miyadera, *On the convergence of nonlinear semi-groups*, Tohuku Math. J., **21** (1969), 221–236.
19. ———, *On the convergence of nonlinear semi-groups II*, J. Math. Soc. Japan, **21** (1969), 403–412.
20. I. Miyadera and S. Oharu, *Approximation of semi-groups of nonlinear operators*, Tohuku Math. J., **22** (1970), 24–27.
21. S. Oharu, *On the generation of semi-groups of nonlinear contractions*, J. Math. Soc. Japan, **22** (1970), 526–549.
22. A. Plant, *The product integral method for nonlinear evolution equations*, (to appear in Amer. J. Math.).
23. G. Webb, *Nonlinear evolution equations and product integration in Banach spaces*, Trans. Amer. Math. Soc., **148** (1970), 273–282.
24. ———, *Product integral representation of time dependent nonlinear evolution equations in Banach spaces*, Pacific J. Math., **32** (1970), 269–281.
25. ———, *Nonlinear evolution equations and product stable operators in Banach spaces*, Trans. Amer. Math. Soc., **155** (1971), 409–426.

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