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SCALAR SPECTRAL OPERATORS, ORDERED l^p-DIRECT SUMS, AND THE COUNTEREXAMPLE OF KAKUTANI — MC CARTHY

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Contrary to the situation on Hilbert space, the sum and product of two commuting scalar spectral operators on a Banach space X need not be spectral, even if X is reflexive. This has been shown by Kakutani and Mc Carthy. In this note, order-theoretic methods are used to discuss Mc Carthy's construction. To this end, a special class of scalar spectral operators is introduced.

In what follows, X always denotes a Banach space over $\mathbf{R}$, $B(X)$ the Banach algebra of bounded linear operators on X, $\sigma(T)$, $\rho(T)$ and $R(z, T) = (T - zI)^{-1}$ for $T \in B(X)$ the spectrum, the resolvent set and the resolvent operator for $z \in \rho(T)$ (taken as usual with respect to the complexification of X).

In the first section we introduce a class $\mathcal{S}$ of C^m-selfadjoint operators ([2]) defined on an ordered Banach space X. In Theorem 1, conditions are given which assure that the elements of $\mathcal{S}$ are, in fact, scalar spectral operators. In the counterexample of Mc Carthy ([1]), which improves an example constructed by Kakutani ([4]), the underlying Banach space X is the l^p-direct sum of finite-dimensional spaces. So in the second section, some order properties of l^p-direct sums of ordered Banach spaces are considered. The last section is devoted to Mc Carthy's construction. It is shown that the natural order of the space he used is not normal. This is remarkable since this order is induced by the natural $\mathbf{R}^n$-order of the summands. Therefore, a direct proof of the nonnormality is added.

1. A class of scalar spectral operators.

A proper convex cone $K \subset X$ induces an ordering on $B(X)$ by

$$T \geqq 0 \quad \text{iff} \quad TK \subset K.$$

Then $\mathcal{K} = \{T \in B(X),\ T \geqq 0\}$ is a convex cone. $\mathcal{K}$ is proper if K generates X, that is, if $K - K = X$.

Let $\mathscr{S} \subset B(X)$ consist of all operators A with the following properties:

1. $\sigma(A) \subset \mathbf{R}$.
2. There exist constants $m, M \in \mathbf{R}$ with

$$\|R(z, A)\| \leqq M \cdot |\operatorname{Im} z|^{-m}$$

for all $z \in \mathbf{C}$ with $\operatorname{Im} z \neq 0$.

3. $R(z, A)R(\bar{z}, A) \in \mathscr{K}$ for every $z \in \rho(A)$ ($\bar{z}$ denotes the complex conjugate of z).

If $A \in B(X)$ obeys 1 and 2, A is a $C^{m+1}(I)$-selfadjoint operator in the sense of Colojoara and Foias **[2]** for every interval $I \subset \mathbf{R}$ which contains $\sigma(A)$ in its interior. According to Tillmann **[6]**, the uniquely determined $C^{m+1}(I)$-spectral function

$$U: C^{m+1}(I) \rightarrow B(X)$$

for A is given by

$$U(f) = \lim_{\epsilon \rightarrow 0} (2\pi i)^{-1} \int_I f(x)\{R(x + i\epsilon, A) - R(x - i\epsilon, A)\}\, dx.$$

From the resolvent equation

$$R(z, A) - R(z', A) = (z - z')R(z, A)R(z', A)$$

it follows that

$$U(f) = \lim_{\epsilon \rightarrow 0} \epsilon\pi^{-1} \cdot \int_I f(x)\{R(x + i\epsilon, A)R(x - i\epsilon, A)\}\, dx. \tag{1}$$

U is a continuous homomorphism from $C^{m+1}(I)$ into $B(X)$, if $C^{m+1}(I)$ is endowed with the topology of uniform convergence of all derivatives up to order $m+1$. Now let M_λ for $\lambda \in \mathbf{R}$ denote the interval $(-\infty, \lambda)$. We set

$$C_0^\infty(M_\lambda) := \{f \in C^\infty(\mathbf{R}),\ \operatorname{supp} f \subset M_\lambda \text{ compact}\}.$$

If χ_λ denotes the characteristic function of M_λ, we can choose a monotonely increasing sequence $\{f_{\lambda, n}\}_{n \in \mathbf{N}}$ of functions in $C_0^\infty(M_\lambda)$ such that the sets

$$M_{\lambda, n} = \{x \in \mathbf{R},\ f_{\lambda, n}(x) = 1\}$$

exhaust M_λ and $\{f_{\lambda,n}\}_n$ converges pointwise to χ_λ. Then for every $g \in C_0^\infty(M_\lambda)$ with $g \leqq \chi_\lambda$ there is a $n_0 \in \mathbf{N}$ with $g \leqq f_{\lambda,n}$ for every $n \in \mathbf{N}$ with $n \geqq n_0$.

Further for every n there exists a m with

$$f_{\lambda,n}^2 \leqq f_{\lambda,n} \leqq f_{\lambda,m}^2. \tag{2}$$

Clearly the order of functions is taken pointwise.

If in addition A obeys 3, substituting $f_{\lambda,n}$ into (1) we get a monotonely increasing sequence $\{U(f_{\lambda,n})\}_n$ of operators with

$$0 \leqq U(f_{\lambda,n}) \leqq I$$

for every $n \in \mathbf{N}$.

If $\{U(f_{\lambda,n})x\}_n$ converges for every $x \in X$, then

$$E_\lambda := \lim_n U(f_{\lambda,n})$$

is a continuous linear operator. E_λ is positive and idempotent by (2). If K is closed it follows that

$$E_\lambda = \sup_n U(f_{\lambda,n}) \qquad \text{(Schaefer [5])}.$$

The cone K is called normal if there exists a norm $\|\cdot\|^*$ on X equivalent to the initial norm $\|\cdot\|$ such that $0 \leqq x \leqq y$ implies that $\|x\|^* \leqq \|y\|^*$. Such a norm is said to be monotone.

Now we can state

THEOREM 1. *Let X be weakly sequentially complete and ordered by a normal closed and generating cone K. If $A \in \mathcal{S}$, then for every $\lambda \in \mathbf{R}$ the sequence $\{U(f_{\lambda,n})\}_{n\in\mathbf{N}}$ converges in the strong operator topology to a continuous positive operator E_λ. Moreover, we have*

1. $E_\lambda = \sup_n U(f_{\lambda,n})$.
2. $E_\lambda^2 = E_\lambda$.
3. $E_\lambda = \sup\{U(f),\ f \in C^\infty(\mathbf{R}),\ \operatorname{supp} f \subset M_\lambda,\ f \leqq \chi_\lambda\}$.
4. $E_\lambda = 0$ *for* $\lambda < -r(A)$ *($r(A)$ denotes the spectral radius of A).*
5. $E_\lambda = I$ *for* $\lambda > r(A)$.
6. $E_\lambda = \lim_{\mu<\lambda} E_\mu$.
7. $A = \int_{\mathbf{R}} \lambda \, dE_\lambda$ *(where the integral exists in the strong operator topology).*

Since E_λ is a left continuous spectral family, A is a scalar spectral operator in the sense of Dunford-Schwartz **[3]**. *The continuous homomorphism*

$$f \to f(A) = \int f(\lambda)\, dE_\lambda$$

of $C(\mathbf{R})$ into $B(X)$ is an extension of the $C^{m+1}(I)$-spectral function U of A.

Proof. We only prove the convergence of $\{U(f_{\lambda,n})\}_n$; then the other assertions can be shown following Tillmann **[6]**. Since K is normal, the dual cone

$$K' := \{x' \in X',\ x'(x) \geqq 0 \text{ for every } x \in K\}$$

generates X'. If $x \in K$, $\{x'(U(f_{\lambda,n})x)\}_n$ is monotonely increasing and bounded by $x'(x)$ and consequently convergent for every $x' \in K'$. Since K and K' generate X and X' and since X is weakly sequentially complete $\{U(f_{\lambda,n})\}_n$ thus converges weakly to an operator E_λ. Moreover, since K is normal, the convergence follows in the strong operator topology from the generalized theorem of Dini (**[5]**).

To see that for general Banach spaces and generating closed cones K Theorem 1 need not be true, let $X = C^k[0, 1]$, $K = \{f \in X,\ f(x) \geqq 0$ for all $x \in [0, 1]\}$ and define $A \in B(X)$ by

$$(Af)(t) := t \cdot f(t), \qquad t \in [0, 1].$$

Then

1. $\sigma(A) = [0, 1]$.
2. $$\|R(z, A)f\| = \sup_{0 \leqq i \leqq k} \sup_{t \in [0,1]} \left| \left(\frac{f(t)}{z - t}\right)^{(i)} \right|$$
$$\leqq \|f\| \cdot M \cdot |\operatorname{Im} z|^{-k} \quad \text{for all} \quad z \in \rho(A) \quad \text{and} \quad f \in C^k[0, 1],$$
if M is appropriately chosen.
3. $$(R(z, A)R(\bar{z}, A)f)(t) = (z - t)^{-1}(\bar{z} - t)^{-1} f(t)$$
$$= |z - t|^{-2} f(t) \geqq 0, \quad \text{if} \quad f(t) \geqq 0.$$

Thus we have $A \in \mathcal{S}$, but it is well known that A is not a spectral operator.

2. Ordered l^p-direct sums. Let I be an index set, let $(X_i, \|\cdot\|_i)$ be Banach spaces for $i \in I$, and define for $1 \leqq p < \infty$

$$X^p := l^p((X_i, \|\cdot\|_i)_{i \in I} := \left\{x = \{x_i\}_i,\ x_i \in X_i,\ \sum_i \|x_i\|_i^p < \infty\right\}.$$

Then X^p is a Banach space with respect to the norm

$$\|x\|^p = \left(\sum_i \|x_i\|_i^p\right)^{1/p}.$$

X^p is reflexive for $p > 1$, if the X_i are reflexive.

If for every $i \in I$ X_i is ordered by a convex cone K_i, a natural order on X^p is defined by the cone K with

$$x = \{x_i\}_i \in K \quad \text{iff} \quad x_i \in K_i \quad \text{for every} \quad i \in I.$$

We recall that a cone K in a normed space X is called M-generating if for every $x \in X$ there is a decomposition $x = x^+ - x^-$ with $x^+, x^- \in K$ and $\|x^+\|, \|x^-\| \leqq M \cdot \|x\|$.

Then one can easily prove the following

PROPOSITION 2.

1. *K is a proper cone iff K_i is proper for every $i \in I$.*
2. *If K generates X then K_i generates X_i for every $i \in I$.*
3. *K is M-generating iff K_i is M-generating for every $i \in I$.*
4. *K is normal iff for every $i \in I$ K_i is normal and there is a monotone norm $\|\cdot\|_i^*$ on X_i equivalent to $\|\cdot\|_i$ such that*

$$m \cdot \|\cdot\|_i^* \leqq \|\cdot\|_i \leqq M \cdot \|\cdot\|_i^*$$

with constants $m, M > 0$ independent of i.

3. The counterexample of Kakutani–Mc Carthy. The sum and product of two commuting scalar spectral operators defined on a Banach space X need not be spectral as was shown by an example of Kakutani in **[4]**. Mc Carthy's modification of this example in **[1]** led to a counterexample even on a separable reflexive Banach space, namely the l^2-direct sum of finite-dimensional spaces. This space a natural order can be given by the summands. We start this section by showing that from Mc Carthy's result the nonnormality of this ordering follows. Since we want to give some explicit calculations we have to recall in short the construction of Mc Carthy.

For $n = 1, 2, \cdots$ be $Y_n = Y'_n = \mathbf{R}^{2^n}$ with the sup-norm, and X_n the space of all $2^n \times 2^n$-matrices with real coefficients. On X_n the "projective tensor norm" q_n is defined by

$$q_n(x) := \inf \sum_{i=1}^{l} \|y_i\|_\infty \|z_i\|_\infty,$$

where we take the infimum over all representations of $x \in X_n$ of the form

$$x(i,j) = \sum_{k=1}^{l} y_k(i) z_k(j), \qquad i,j = 1,2,\cdots,2^n,$$

with

$$\begin{aligned} y_k &= (y_k(1),\cdots,y_k(2^n)) \in Y_n, \\ z_k &= (z_k(1),\cdots,z_k(2^n)) \in Y_n', \end{aligned} \qquad k = 1,\cdots,l,\ l \in \mathbf{N}.$$

We define matrices $s_n \in X_n$ by setting

$$s_n(i,j) := (-1)^{\sum_{k=1}^{n} e_k(i) e_k(j)}, \qquad i,j = 1,\cdots,2^n, \tag{3}$$

where the $e_k(i) \in \{0, 1\}$ are uniquely determined by

$$i = e_1(i)2^{n-1} + e_2(i)2^{n-2} + \cdots + e_{n-1}(i)2 + e_n(i) + 1. \tag{4}$$

According to **[1]** and **[4]**, for every $u \in X_n$ we have

$$q_n(u) \geqq 2^{n/2} \cdot \frac{1}{2^{2n}} \cdot \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} s_n(i,j) u(i,j). \tag{5}$$

Now we put $X = l^2((X_n, q_n)_{n \in \mathbf{N}})$. X is a reflexive and therefore weakly sequentially complete Banach space.

We define $S, T \in B(X)$ by

$$T(\{x_n\}_n) := \{T_n x_n\}_n \quad \text{with} \quad (T_n x_n)(i,j) := 2^{-2^n} 3^{-i} x_n(i,j),$$
$$S(\{x_n\}_n) := \{S_n x_n\}_n \quad \text{with} \quad (S_n x_n)(i,j) := 5^{-j} x_n(i,j).$$

T and S are commuting scalar spectral operators, but neither TS nor $T + S$ is spectral.

We show that TS belongs to the family $\mathcal{S}$ introduced in the first chapter. Indeed,

1. $\sigma(TS)$ is the closure of the set of eigenvalues $2^{-2^n} 3^{-i} 5^{-j}$, $1 \leqq i$, $j \leqq 2^n$, $n \in \mathbf{N}$, and is therefore contained in $\mathbf{R}$.

2. T and S are commuting scalar spectral operators. By **[2]**, TS (and $T + S$) is a generalized scalar operator and thus the resolvent of TS obeys a growth condition.

3. If X_n is given its natural order, defined by

$$x_n \in K_n \quad \text{iff} \quad x_n(i,j) \geqq 0 \quad \text{for} \quad i,j = 1,\cdots,2^n,$$

each of the cones K_n is proper, closed and generates X_n. Moreover, K_n is normal since q_n is equivalent to a monotone norm on X_n, e.g. to the sup-norm. If $K \subset X$ is defined as above, K is also a proper and closed cone. K generates X, since if $x_n \in X_n$ is represented by

$$x_n(i,j) = \sum_{k=1}^{l} y_k(i) z_k(j)$$

with $y_k \in Y_n$, $z_k \in Y'_n$, we put

$$y_k^+(i) = \begin{cases} y_k(i), & \text{if } y_k(i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$y_k^-(i) = \begin{cases} -y_k(i), & \text{if } y_k(i) < 0 \\ 0 & \text{otherwise} \end{cases},$$

and define z_k^+, z_k^- similarly.

We obtain

$$\begin{aligned} x_n(i,j) &= \left(\sum_{k=1}^{l} (y_k^+(i) z_k^+(j) + y_k^-(i) z_k^-(j)) \right) \\ &\quad - \left(\sum_{k=1}^{l} (y_k^-(i) z_k^+(j) + y_k^+(i) z_k^-(j)) \right) \\ &= x_n^+ - x_n^- \end{aligned}$$

with x_n^+, $x_n^- \in K_n$, $q_n(x_n^+)$, $q_n(x_n^-) \leqq 2\, q_n(x_n)$.

Thus the cones K_n are 2-generating in X_n and by proposition 2 the same is true for K in X.

The assertion that $x \in K$ implies $R(z, TS)R(\bar{z}, TS)x \in K$ for $z \in \rho(TS)$ can now be proven in the equivalent form, that

$$(TS - zI)(TS - \bar{z}I)x \in K \quad \text{implies that} \quad x \in K.$$

Indeed, we have

$$(TS - zI)(TS - \bar{z}I)x = \{(T_nS_n - zI)(T_nS_n - \bar{z}I)x_n\}_n$$

and

$$\begin{aligned} ((T_nS_n - zI)(T_nS_n - \bar{z}I)x_n)(i,j) &= ((2^{-2^n} 3^{-i} 5^{-j})^2 + |z|^2 \\ &\quad - 2 \cdot 2^{-2^n} 3^{-i} 5^{-j} \cdot \operatorname{Re} z) x_n(i,j) \geqq 0 \end{aligned}$$

gives $x_n(i,j) \geqq 0$ since the scalar factor at the right side is positive.

Since all assumptions of Theorem 1 except the normality of K are thus fulfilled, K cannot be normal.

We now want to give a direct proof of this remarkable order theoretic fact (we recall that the cones K_n are all normal). For this purpose we use the matrices s_n defined above (3).

LEMMA 3. $2^{n-1}(2^n+1)$ *of the cofficients of* s_n *are* $+1$.

Proof. This is obviously true for $n=1$. s_{n+1} has the form

$$s_{n+1} = \left(\begin{array}{c|c} s_n & s_n \\ \hline s_n & -s_n \end{array}\right),$$

as can be easily deduced from the definition of the $e_k(i)$ (4). Therefore s_{n+1} contains

$$3\cdot 2^{n-1}(2^n+1) + (2^{2n} - 2^{n-1}(2^n+1)) = 2^n(2^{n+1}+1)$$

coefficients of value $+1$.

We now define matrices t_n, $v_n \in X_n$ by

$$t_n(i,j) = \begin{cases} 1, & \text{if } s_n(i,j)=1 \\ 0 & \text{otherwise} \end{cases}$$

$$v_n(i,j) = 1 \quad \text{for every} \quad i,j = 1,\cdots,2^n.$$

Thus

$$0 \leqq t_n \leqq v_n$$

and

$$q_n(v_n) = 1 \qquad \text{for every } n \in \mathbf{N}.$$

On the other hand, according to (5),

$$q_n(t_n) \geqq 2^{n/2}\cdot\frac{1}{2^{2n}}\cdot 2^{n-1}(2^n+1) \tag{6}$$
$$= 2^{n/2}(2^{-1}+2^{-n-1}) \nearrow \infty.$$

Thus K cannot be normal, since otherwise, applying Proposition 2 we could find constants $m, M>0$ independently of n and monotone norms $\|\cdot\|_n$ on X_n with

$$m\cdot\|x_n\|_n \leqq q_n(x_n) \leqq M\cdot\|x_n\|_n \quad \text{for every} \quad x_n \in X_n.$$

This would imply

$$\| t_n \|_n \leqq \| v_n \|_n \leqq \frac{1}{m} q_n(v_n) = \frac{1}{m}$$

and therefore $q_n(t_n) \leqq M/m$ in contrary to (6).

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