ON THE STRUCTURE OF ALGEBRAIC ALGEBRAS

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Let A be an infinite dimensional (associative) algebra over a field F. It is shown that A has an infinite dimensional commutative subalgebra C of *one* of the following types:

    (a) C is generated by one element

    (b) $C^2 = \{0\}$

    (c) $C^3 = \{0\}$ and C is an ideal of an ideal of A

    (d) C is generated by mutually orthogonal idempotents

    (e) C is a field.

A necessary and sufficient condition (in terms of the quadratic forms over F) is obtained for the validity of the statement: Every infinite dimensional nil algebra over F has an infinite dimensional subalgebra B with $B^2 = \{0\}$. An ideal $y(A)$ of an algebra A (analogous to the F.C. subgroup in group theory) is defined and several properties of it are obtained.

In §1 of this paper we give a list of notation and definitions. In §2 we prove some preliminary results on rings and algebras. In §3 we obtain several properties of an ideal $y(A)$ of an algebra A (defined in §1). The ideal $y(A)$ is an algebra-theoretic analogue of the F.C. (finite conjugate) subgroup in group theory. The main result of this paper is the Main Theorem, proved in §4, which states:

MAIN THEOREM (4.1). *Let A be an infinite dimensional (associative) algebraic algebra over a field F and assume*

    (a)   *A contains no infinite set of mutually orthogonal idempotents.*

    (b)   *A contains no infinite dimensional subalgebra B with $B^2 = 0$*

    (c)   *$y(A)$ is finite dimensional.*

Then $A = M \oplus N$ where M is finite dimensional and N is the direct sum of finitely many algebraic division algebras.

An immediate consequence of the Main Theorem is the fact, proved in [5], that every infinite dimensional associative algebra over a field has an infinite dimensional commutative subalgebra.

Several results in this paper deal with algebraic algebras which have no infinite dimensional subalgebras with square zero. This condition restricts the structure of an algebraic algebra considerably. For example, if A is a nil algebra over a field and A has no infinite dimensional subalgebra C with $C^2 = 0$, then A is locally finite and A is even necessarily finite dimensional for a large class of fields.

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1. Notation and definitions. Throughout the remainder of the paper the words ring and algebra will mean associative ring and associative algebra, respectively. If A is an algebra over a field and B is a subspace (or subalgebra) of A, A/B denotes the factor space of A by B and, if B is an ideal of A, A/B denotes the corresponding factor algebra.

If X, Y are subspaces or subalgebras of A, $X + Y$ denotes the vector space sum of X, Y. If X or Y is an ideal of A, then $X + Y$ coincides with the algebra sum of X, Y. Also $\dim(X)$ denotes the dimension of X.

$\oplus$ is used to denote the direct sum of algebras.

If S is a nonempty subset of ring R, then ring (S) denotes the subring generated by S. If B is a nonempty subset of an algebra A, alg (B) denotes the subalgebra generated by A.

If S is a nonempty subset of ring R, then

$$\operatorname{Ann}(S) = \{r \in R \mid rs = sr = 0 \text{ for all } s \in S\}.$$

If $S = \{x\}$, write $\operatorname{Ann}(\{x\}) = \operatorname{Ann}(x)$.

If A is an algebra, we write

$$y(A) = \{a \in A \mid \dim(A/\operatorname{Ann}(a)) \text{ is finite}\}$$

An algebra A is called *algebraic* over a field F if each element of A generates a finite dimensional subalgebra A. A is called *locally finite* if each finite nonempty subset of A generates a finite dimensional subalgebra of A.

We recall that by a result of Golod-Shafarevich ([**3**], Theorem (8.1.3)) there exist algebras which are algebraic but not locally finite.

We say that a field F is *square-dependent* if given any sequence $\{a_n\}$ of elements of F, there exists a positive integer $N = N(\{a_n\})$ and elements $\lambda_1, \cdots, \lambda_N \in F$, not all zero, such that

$$a_1 \lambda_1^2 + \cdots + a_N \lambda_N^2 = 0.$$

We note that a formally real field is not square-dependent.

Several results on and examples of square-dependent fields can be found in [**2**].

0 is used to denote both the zero element and the zero algebra.

2. Some general results on algebras. Our first result deals with subalgebras of finite codimension. The group-theoretic analogue of this result is well-known and the ring-theoretic analogue has been obtained by J. Lewin (**[6]**).

LEMMA (2.1). *Let A be an algebra over a field F and let B be a subalgebra of A with $\dim(A/B) = n < \infty$. Then B contains an ideal I of A with $\dim(A/I) \leqq n(n+1)^2$.*

Proof. Let $x_1 + B, \cdots, x_n + B$ be a basis for A/B. Define subspaces B_{ij} of B as follows:
Let

$$B_{i0} = \{b \in B \mid x_i b \in B\} \qquad (i \geqq 1)$$

$$B_{0j} = \{b \in B \mid bx_j \in B\} \qquad (j \geqq 1)$$

$$B_{ij} = \{b \in B \mid x_i bx_j \in B\} \qquad (i \geqq 1, j \geqq 1).$$

Fix $(i, j) \neq (0, 0)$. The map $b \to x_i b x_j + B \, (b \in B)$ is an F-space homomorphism of B into A/B with kernel B_{ij}. So $\dim(B/B_{ij}) \leqq n$. Let $I = \bigcap_{(i,j) \neq (0,0)} B_{ij}$. Then

$$\dim(A/I) = \dim(A/B) + \dim(B/I) \leqq n + n((n+1)^2 - 1) = n(n+1)^2.$$

It is straightforward to check that I is an ideal of A.

We are grateful to the referee and to H. E. Bell for shortening our original proof of the next result. The proof here is due to H. E. Bell.

LEMMA (2.2). *Let R be a ring. The following statements are equivalent:*
   (1)   *R has an infinite set of commuting idempotents.*
   (2)   *R has an infinite set of mutually orthogonal idempotents.*

Proof. Clearly (2) implies (1). Suppose then that (1) holds. Let X be an infinite set of nonzero commuting idempotents of R. We may assume R is generated by X. For each subring S of R, let $E(S)$ be the set of nonzero idempotents in S. Note that if $e \in E(S)$, then $S = eS + \operatorname{Ann}(e)$ so each $f \in E(S)$ splits as a sum $f_1 + f_2$ of idempotents f_1, f_2. Hence if $E(S)$ is infinite so is at least one of $E(eS)$, $E(\operatorname{Ann}(e))$. Thus starting with $E(R)$ we may construct a sequence $e_1, e_2, \cdots, e_n, \cdots$ of distinct nonzero idempotents such that for each i *either* (a) $e_i e_j = 0$ for all $j > i$ or (b) $e_i e_j = e_j$ for all $j > i$. If (a) holds for infinitely many i, then (2) follows. We may thus assume that (b) holds for all i. Then $\{e_1 - e_2, e_2 - e_3, e_3 - e_4, \cdots\}$ is an infinite set of mutually orthogonal idempotents.

COROLLARY (2.3). *Let R be a ring with no nonzero nilpotent elements. The following are equivalent:*

(i)   *R has an infinite set of mutually orthogonal idempotents.*

(ii)   *R has infinitely many idempotents.*

Our next result gives some information on the structure of algebraic algebras.

LEMMA (2.4). *Let A be an algebraic algebra over a field F and assume that A has no nonzero nilpotent elements. Then one of the following statements holds:*

(i)   *A has an infinite set of mutually orthogonal idempotents.*

(ii)   *A is a finite direct sum of algebraic division algebras.*

Proof. Assume that (i) does not hold. By Corollary (2.3), A has only finitely many idempotents. Let X be the subalgebra generated by the idempotents and let $e_1, e_2, \cdots, e_n$ be the primitive idempotents of X. Then every idempotent in A is a sum of some of the e_i. If $a \in A$, the algebra generated by a is finite dimensional and thus it is a direct sum of finitely many fields. Thus $a = ea$ for some idempotent e. Thus $A = e_1 A + \cdots + e_n A$ and, since each e_i is central and $e_i e_j = 0$ $(i \neq j)$, this sum is direct. Since e_i is the only nonzero idempotent in $e_i A$, $e_i A$ is a division algebra. This completes the proof.

3.   Properties of $y(A)$. In this section we obtain several properties of $y(A)$ for A an algebra over a field F.

We begin with some elementary facts.

LEMMA (3.1). *Let A be an algebra over a field F. Then*

(i)   *$y(A) = A$ if A in finite dimensional.*

(ii)   *$y(y(A)) = y(A)$.*

(iii)   *if $a \in A$, then $a \in y(A)$ if and only if $Aa + aA$ is a finite dimensional subspace of A.*

(iv)   *if B is a subalgebra of A with A/B finite dimensional, then $y(B) = B \cap y(A)$.*

(v)   *if $y(A)$ is finite dimensional, then $y(A/y(A)) = 0$.*

An easy consequence of Lemma (3.1)(iii) is.

LEMMA (3.2). *Let A be an algebra over a field F. Then $y(A)$ is an ideal of A.*

We now prove

LEMMA (3.3). *Let A be an algebra over a field F. The following are equivalent.*

(a) $a \in y(A)$

(b) a is contained in a finite dimensional ideal of A.

Let $a \in y(A)$. Now aA, Aa are finite dimensional and also, since $aA \subseteq y(A)$, $Aa \subseteq y(A)$, AaA is finite dimensional. It follows easily that the smallest ideal containing a is finite dimensional. The converse is trivial.

COROLLARY (3.4). *Let A be an algebra over a field F. Then $y(A)$ is a locally finite ideal of A.*

Our next result gives some information on the size of $y(A)$.

LEMMA (3.5). *Let A be an algebra over a field F and assume that A has no infinite dimensional subalgebra C with $C^2 = 0$. Then $a \in y(A)$ if and only if $a^2 \in y(A)$.*

Proof. Let $a \in A$ be such that $a^2 \in y(A)$. The map $x \to axa$ ($x \in \text{Ann}(a^2)$) is a linear transformation of $\text{Ann}(a^2)$ whose image is a subalgebra of A whose square is zero. Thus

$$U = \{x \in \text{Ann}(a^2) \mid axa = 0\}$$

has finite codimension in $\text{Ann}(a^2)$ and therefore in A. Let $W = \{ax \mid x \in U\}$. Then W is a subalgebra of A whose square is zero. So W is finite dimensional. Thus

$$V = \{x \in U \mid ax = 0\}$$

has finite codimension in U. Using symmetry, we find that $a \in y(A)$.

The converse is trivial.

COROLLARY (3.6). *Let A be an algebra over a field F and assume that A has no infinite dimensional subalgebra whose square is zero. Then $A/y(A)$ has no nonzero nilpotent elements.*

COROLLARY (3.7). *Let A be a nil algebra over a field F and assume that A has no nonzero finite dimensional ideals. Then A has an infinite dimensional subalgebra whose square is zero (unless $A = 0$).*

LEMMA (3.8). *Let A be an algebra over a field F and assume that $y(A)$ is infinite dimensional. Then there exists a sequence $\{X_n \mid n = 1, 2, 3, \cdots\}$ of ideals of A such that*

(a) X_n *is finite dimensional.*

(b) $X_n \subseteq y(A)$

(c) $X_n X_m = 0 \ (m \neq n)$

and

(d) $X_{n+1} \not\subseteq X_1 + \cdots + X_n \ (n \geq 1)$.

Proof. Note if $x \in y(A)$ then the smallest ideal of A containing x is finite dimensional. Having chosen elements $x_1, \cdots, x_n \in y(A)$ such that if X_i is the smallest ideal of A containing x_i, then

(α) $X_{i+1} \not\subseteq X_1 + \cdots + X_i \qquad (i \geq 1)$

(β) $X_i X_j = 0 \qquad (i \neq j)$

let

$$W_n = y(A) \cap \mathrm{Ann}(X_1 + \cdots + X_n).$$

Note that W_n is infinite dimensional and that $X_1 + \cdots + X_n$ is finite dimensional. Let

$$x_{n+1} \in W_n - (X_1 + \cdots + X_n)$$

and let X_{n+1} be the smallest ideal of A containing x_{n+1}. The result follows by induction.

LEMMA (3.9). *Let F be a square dependent field, V a finite dimensional F-space and $v_0, v_1, v_2, \cdots$ an infinite sequence of elements of V. Then there exist sequences $\lambda_0, \lambda_1, \lambda_2, \cdots$ of elements of F and $0 = n(0) < n(1) < n(2) < \cdots$ natural numbers, such that*

(a) *for all j, there exists i with $n(j) \leq i < n(j+1)$ such that $\lambda_i \neq 0$, and*

(b) $\Sigma_{n(j) \leq i < n(j+1)} \lambda_i^2 v_i = 0$.

Proof. If dim $V = 1$, the result is immediate from the definition of a square-dependent field. Suppose dim $V = n > 1$ and let W, spanned by w, be a one-dimensional subspace. Using induction we may assume that there exist sequences $\lambda_0, \lambda_1, \lambda_2, \cdots$ of elements of F and $0 = n(0) < n(1) < n(2) < \cdots$ of natural numbers such that

(a) for all j, there exists i with $n(j) \leq i < n(j+1)$ such that $\lambda_i \neq 0$

and

(b) $\Sigma_{n(j) \leq i < n(j+1)} \lambda_i^2 v_i = \alpha_j w$

for some $\alpha_j \in F$.

Again, by the definition of a square-dependent field, there exist sequences $\mu_0, \mu_1, \mu_2, \cdots$ of elements of F and $0 = m(0) < m(1) < m(2) < \cdots$ of natural numbers such that

(c) for all j, there exists i with $m(j) \leqq i < m(j+1)$ such that $\mu_i \neq 0$

and

(d) $\Sigma_{m(j) \leqq i < n(j+1)} \mu_i^2 \alpha_i = 0$.

Hence

$$\sum_{m(k) \leqq j < m(k+1)} \sum_{n(j) \leqq i < n(j+1)} \lambda_i^2 \mu_j^2 v_i = 0$$

and the result follows.

THEOREM (3.10). *Let A be an algebra over a field F and assume that $y(A)$ is infinite dimensional. Then one of the following statements holds:*
  (a) $y(A)$ *contains an infinite set of mutually orthogonal idempotents.*
  (b) *A has an ideal I such that*
    (i) $I^3 = \{0\}$
    (ii) *I has an infinite dimensional commutative ideal C.*
If, in addition, F is square-dependent, statement
  (b)(ii) *can be improved to*
  (b)(ii)' *$y(A)$ contains an ideal I of A such that I has an infinite dimensional ideal C with $C^2 = 0$.*

Proof. Let $\{X_n\}_1^\infty$ be an infinite sequence of ideals of A satisfying the conclusion of Lemma (3.8). If infinitely many of the X_i contain nonzero idempotents, then conclusion (a) of the theorem holds. Assume then that only finitely many of the X_i contain nonzero idempotents. By omitting these and relabelling, if necessary, we may assume that no X_i contains a nonzero idempotent and further that X_i is nilpotent for all i.

Let k_i be the least positive integer for which $X_i^{k_i} = 0$ and let $h_i = k_i/2$ (if k_i is even), $h_i = (k_i + 1)/2$ (if k_i is odd). Let $H_i = X_i^{h_i}$. Now H_i is an ideal of A and $H_i^2 = 0$. Also $H_iH_j = 0$. Let H be the algebra generated by $\{H_i\}$. Then $H^2 = 0$, H is an ideal of A and, if H is infinite dimensional, conclusion (b) of the theorem holds with $I = C = H$.

Assume from now on that H is finite dimensional. There exists a positive integer m such that for all $n > m$,

$$H_n \subseteq H_1 + \cdots + H_m.$$

In particular $X_nH_n = H_nX_n = 0$. Thus $X_n^3 = 0$ for all $n > m$. Let I be the ideal generated by $\{X_n + H \mid n > m\}$. Then $I^3 = 0$ and conclusion (b)(i) of the theorem holds.

Let $v_n \in X_n - (X_{m+1} + \cdots + X_{n-1} + H)$ $(n > m)$ and let C_n be the algebra generated by $\{v_n, H_n\}$. (We note that v_n exists because of our choice of m). Since $X_nH = HX_n = 0$, C_n is commutative. Also C_n is an

ideal of I. Thus conclusion (b) (ii) of the theorem holds with $C = \text{alg}(\{C_n\})$.

Suppose now that F is a square-dependent field. Let $v_n^2 = h_n \in H$. Since H is finite dimensional, Lemma (3.9) provides us with a sequence $\lambda_0, \lambda_1, \cdots$ of elements of F and $0 = n(0) < n(1) < n(2) < \cdots$ of natural numbers such that

(1)   for all j, there exists i with $n(j) \leqq i < n(j+1)$ such that $\lambda_i \neq 0$

and

(2) $\sum_{n(j) \leqq i < n(j+1)} \lambda_i^2 h_i = 0$.

Let $w_j = \sum_{n(j) \leqq i < n(j+1)} \lambda_i v_i$.

Then $w_j w_k = 0$ for all j, k and $\{w_j\}$ is linearly independent. Also the algebra generated by $H \cup \{w_j\}$ is an ideal of I. Thus conclusion (b) (ii)′ of the theorem holds. The proof is now complete.

EXAMPLE (3.11).   Let F be a field which is not square-dependent (i.e., there exists an infinite sequence $\{a_n\}_1^\infty$ of elements of F such that

$$a_1 \lambda_1^2 + \cdots + a_m \lambda_m^2 = 0$$

for $\lambda_i \in F$ and $m \geqq 1$ implies $\lambda_1 = \cdots = \lambda_m = 0$). Let A be the algebra generated over F by

$$u_0, u_1, \cdots, u_n, \cdots$$

subject to the relations

$$u_0^2 = 0, \quad u_i^2 = a_i u_0 \qquad (i \geqq 1)$$
$$u_i u_j = 0 \qquad (i \neq j).$$

Then $y(A) = A$ and Au_0 is the only subalgebra of A whose square is zero. Thus the conclusion (b)(ii) cannot in general be modified to $C^2 = 0$ for fields which are not square-dependent.

Theorem (3.10) and Example (3.11) yield the following characterization of square-dependent fields.

THEOREM (3.12).   *Let F be a field. The following are equivalent*
(i)   *F is square dependent*
(ii)   *every infinite dimensional nil algebra over F has an infinite dimensional subalgebra C with $C^2 = 0$.*

4.   The main results.   We first prove the Main Theorem.

Proof.   Assume that A satisfies (a), (b) and (c). By Lemma (3.1) (v), $y(A/y(A)) = 0$. By Corollary (3.6) and the fact that (b) holds,

$A/y(A)$ has no nonzero nilpotent elements. Let $H = \mathrm{Ann}(y(A))$ and $L = y(A) \cap H$. Then H/L is isomorphic to a subalgebra of $A/y(A)$, so H/L has no nonzero nilpotent elements. Next we show that H/L does not have an infinite set of mutually orthogonal idempotents. For suppose that $\{f_n + L\}$ is such a set. Then, since $HL = LH = 0$, $\{f_n^2\}$ is an infinite set of mutually orthogonal idempotents of A, against our assumptions.

Now Lemma (2.4) implies that H/L is a finite direct sum of division algebras. In particular H/L has an identity $e + L$. Let

$$K = \{w \in H \mid we^2 = e^2w = w\}.$$

Since $we^2 - w$ and $e^2w - w$ are elements of L for all $w \in H$, we see that K has finite codimension in H. Also $f = e^2$ is the identity of K. Let

$$N_0 = \{a \in A \mid af = fa = a\}.$$

We note that A/N_0 is finite dimensional. Let $a \in A$ and consider $af - a$. Then $(af - a)K_1 = K_1(af - a) = 0$ where K_1 is the maximal ideal of A contained in K. Thus $af - a \in y(A)$. Similarly $fa - a \in y(A)$. Now

$$a = faf + (af - faf) + (a - af) \in N_0 + y(A).$$

Let N be the maximal ideal of A contained in N_0. Again, by Lemma (2.1), A/N is finite dimensional. Since $f \neq 0$ and $fy(A) = 0$, $N_0 \cap y(A) = 0$. So N is isomorphic to a subalgebra of H/L. In particular N is a finite direct sum of division algebras. So N has an identity, g say. Now if $u \in A$, $ug - u$ is annihilated by N. Thus $ug - u \in y(A)$. Thus $u \in y(A) + N = y(A) \oplus N$.

This completes the proof for general F.

Note that if F is square-dependent, (c) is a consequence of (a), (b), by Theorem (3.12).

Since a maximal subfield of an infinite dimensional division algebra is infinite dimensional ([4], p. 165), Theorem (4.1) implies

COROLLARY (4.2). *Every infinite dimensional algebra over a field F has an infinite dimensional commutative subalgebra.*

This is the main result of [5]. We note that Theorem (4.1) shows that such a commutative subalgebra C can be found of one of the following types:

(a) C is generated by one element.

(b) $C^2 = 0$.

(c) C is generated by mutually orthogonal idempotents.

(d) $C^3 = 0$ and C is an ideal of an ideal of A.

(e) C is a field.

We now prove

THEOREM (4.3).   *Let A be an algebraic algebra over a field F with F finite or algebraically closed.   If A is not locally finite, then A has an infinite dimensional subalgebra C with $C^2 = 0$.*

Proof.   Suppose that the theorem is false and let A be a counterexample. By Corollary (3.6), $A/y(A)$ has no nonzero nilpotent elements. Suppose first that F is finite.   Let $x \in A/y(A)$.   The ring generated by x is finite (since A is algebraic) and has no nonzero nilpotent elements, so $x^{p^k} = x$ for some $k \geqq 1$ where p is the characteristic of F.   Then $A/y(A)$ is commutative by Jacobson's theorem ([3], Theorem (3.1.2)) and thus it is locally finite.   Hence A is locally finite, since $y(A)$ is locally finite.

Suppose then that F is algebraically closed.   Each element $x \in A/y(A)$ is then an F-linear combination of idempotents all of which are central, since $A/y(A)$ has no nonzero nilpotent elements.   So $A/y(A)$ is commutative and the result follows.

Our next result is a consequence of the proof of Theroem (3.10), and the Main Theorem.

THEOREM (4.4).   *Let A be an algebraic algebra over a field F and assume that A has no infinite dimensional subalgebra C with $C^2 = 0$. Then A is Artinian if and only if A is Noetherian.*

Proof.   If A is Noetherian, then the proof of Theorem (4.1) shows that conclusion (iv) of Theorem (4.1) must occur.   Thus A is Artinian.

Suppose now that A is Artinian and that A is not Noetherian. Clearly conclusion (i) or (iv) of Theorem (4.1) cannot occur, so conclusion (iii) must occur.   Thus $y(A)$ is infinite dimensional. Choose the sequence $\{X_n\}$ as in Lemma (3.8).   Then the sequence

$$\text{Ann}(X_1), \quad \text{Ann}(X_1 + X_2), \cdots, \text{Ann}(X_1 + \cdots + X_n), \cdots$$

does not terminate.

We give two final results on how the assumption that an algebra A has no infinite dimensional subalgebra whose square is zero restricts the structure of A.   These results are elementary.

THEOREM (4.5). *Let A be an algebra over a field F and assume that A has no infinite dimensional subalgebra B with $B^2 = 0$. Then*

    (a) *every left invertible element of A is right invertible,*

    (b) *if A has an element which is a left zero-divisor but is not a right zero-divisor, then A has a finite dimensional nonzero ideal I with $I^2 = 0$.*

Proof. (a) Suppose x, $y \in A$ with $yx = 1$, $xy \neq 1$. Suppose that $\{x^k(xy - 1) \mid k \geqq 1\}$ is linearly dependent. Let $\lambda_0, \cdots, \lambda_k \in F$, $\lambda_k \neq 0$, be such that

$$\lambda_0(xy - 1) + \lambda_1 x(xy - 1) + \cdots + \lambda_k x^k(xy - 1) = 0.$$

Multiplying on the left by y^k, we get $xy - 1 = 0$, which is impossible.

(b) Let $z \in A$ be such that $zx = 0$ for some $0 \neq x \in A$ and that $uz = 0$ for $u \in A$ implies $u = 0$. The subalgebra C generated by the elements $xz^k (k = 1, 2, \cdots)$ satisfies $C^2 = 0$, so it is finite dimensional. Then since z is not a right zero-divisor it follows that $\{xz^k \mid k = 0, 1, 2, \cdots\}$ is linearly dependent. Hence there exist $a_0, a_1, \cdots, a_m \in F$, not all zero, such that

$$a_0 x + a_1 xz + \cdots + a_m xz^m = 0.$$

We may assume that m is the least such integer. Then $a_0 \neq 0$. Hence $x^2 = 0$. By Corollary (3.6), $x \in y(A)$. The same argument shows that $w^2 = 0$ for all w in the smallest right ideal of A containing x. Thus, for $c \in A$,

$$0 = (x + xc)^2 = x^2 + x^2 c + xcx + (xc)^2 = xcx.$$

Hence $(Ax)^2 = (xA)^2 = 0$. The result follows.

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