

ON FINITE HANKEL TRANSFORMATION OF GENERALIZED FUNCTIONS

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In this paper the finite Hankel transformation of generalized function of a certain space is defined, and an inversion formula for the transformation is established. The inversion formula gives rise to a Fourier-Bessel series expansion of generalized functions. The convergence of the series is interpreted in the weak distributional sense. An operation transform formula is also obtained, which together with the inversion formula, is applied in solving certain distributional differential equations.

1. Introduction. The orthonormal series expansions of distributions of certain classes have been studied by Gelfand and Shilov [3, Vol. 3], Giertz [2], Walter [9] and Zemanian [11]. Some previous works on Fourier and Hermite series expansions of certain classes of distributions are due to Schwartz [8, Vol. II] and Korevaar [5]. The procedures of Giertz, Walter, and Korevaar are appropriate for the fundamental sequence approach to generalized functions (see Lighthill [6] and Korevaar [4]), whereas those of Schwartz, Gelfand and Shilov and Zemanian, including the present work are suitable to functional approach to generalized functions.

Some particular cases of orthonormal series expansions including Fourier-Bessel series expansion of generalized functions have also been studied by Zemanian [11]. The method involved in his work is very much related to Hilbert space techniques. The present goal is to extend the classical inversion theorem for finite Hankel transform [10, p. 591] to a class of generalized functions, which gives rise to the Fourier-Bessel series expansion of the generalized function; the convergence of the series is interpreted in the weak distributional sense. The techniques developed in establishing the present inversion theorem are quite different from those employed in previous works and, at the same time, are quite simple and handy. An operation transform formula is also established and is applied in solving certain distributional differential equations.

The finite Hankel transformation of a function $f(t)$ defined on the interval $(0, 1)$ is defined as

$$(1.1) \quad H(m) = \int_0^1 tf(t)J_\nu(j_m t)dt, \quad m = 1, 2, 3, \dots,$$

where $J_\nu(z)$ is the Bessel function of first kind of order $\nu \geq -\frac{1}{2}$ for any complex z , and j_1, j_2, j_3, \dots are positive zeros of $J_\nu(z)$ arranged in the ascending order. Note that for $\nu > -1$, all the zeros of $J_\nu(z)$ lie on the real x -axis [10, p. 483].

The inversion theorem for the transform (1.1) is given in [10, p. 591] and stated as:

THEOREM 1.1. *Let $f(t)$ be a function defined over the interval $(0, 1)$, and let $\int_0^1 t^{\frac{1}{2}} f(t) dt$ exist and (if it is improper integral) let it be absolutely convergent. Let*

$$a_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 t f(t) J_\nu(j_m t) dt, \quad m = 1, 2, 3, \dots,$$

for $\nu \geq -\frac{1}{2}$. Let x be any internal point of an interval (a, b) such that $0 < a < b < 1$ and that $f(t)$ has total limited fluctuation in (a, b) . Then the series

$$(1.2) \quad \sum_{m=1}^{\infty} a_m J_{\nu}(j_m x)$$

is convergent, and its sum is $\frac{1}{2}\{f(x+0) + f(x-0)\}$, and is $f(x)$ if the function f is continuous at the point x .

In this paper, the above theorem will be extended to a class of generalized functions.

2. The notation and terminology. Throughout this work, j_m , $m = 1, 2, \dots$, will denote the positive zeros of $J_\nu(z)$, arranged in the ascending order. The interval $(0, 1)$ will be denoted by the letter I . The letters t and x will stand for real variables in the interval $(0, 1)$. The symbol $D(I)$ will denote the space of infinitely differentiable functions on $I = (0, 1)$, which have compact support on I . The topology of $D(I)$ is that which makes its dual $D'(I)$ of Schwartz's distribution. $E(I)$ will denote the space of all infinitely differentiable functions on I . Its dual $E'(I)$ is the space of distributions with compact supports. We will use the following operators:

$$(2.1) \quad \Omega_{\nu, x}^k = \left(D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2} \right)^k, \quad k = 0, 1, 2, \dots; \quad \nu \geq -\frac{1}{2}, \quad D_x = \frac{d}{dx}.$$

3. The testing function space $U_{\alpha, \nu}(I)$ and its dual. For each pair of real numbers α and ν such that $\alpha \geq \frac{1}{2}$ and $\nu \geq -\frac{1}{2}$, we define

$U_{\alpha,\nu}(I)$ as the space of all complex-valued functions $\varphi(x)$ on $I = (0, 1)$ such that $\varphi(x)$ is infinitely differentiable and that

$$\gamma_k^{\alpha,\nu}(\varphi) = \sup_{0 < x < 1} |x^\alpha \Omega_{\nu,x}^k [x^{-1} \varphi(x)]| < \infty,$$

for each $k = 0, 1, 2, \dots$.

$U_{\alpha,\nu}(I)$ is a linear space. The topology of $U_{\alpha,\nu}(I)$ is that generated by the collection of seminorms, $\{\gamma_k^{\alpha,\nu}\}$, $k = 0, 1, 2, \dots$. $U'_{\alpha,\nu}(I)$ is the dual of $U_{\alpha,\nu}(I)$, and is equipped with the usual weak topology. We will refer to the members of $U'_{\alpha,\nu}(I)$ as generalized functions. It turns out that $U_{\alpha,\nu}(I)$ is a Fréchet Space.

We now note some further properties of the space $U_{\alpha,\nu}(I)$ and its dual.

(i) $D(I) \subset U_{\alpha,\nu}(I)$ and the topology of $D(I)$ is stronger than that induced on it by $U_{\alpha,\nu}(I)$. Consequently, the restriction of any member of $U'_{\alpha,\nu}(I)$ to $D(I)$ is in $D'(I)$.

(ii) The space $E'(I)$ can be identified as a subspace of $U'_{\alpha,\nu}(I)$.

(iii) For each $f \in U'_{\alpha,\nu}(I)$, there exist a nonnegative integer r and a positive constant C such that, for all $\varphi \in U_{\alpha,\nu}(I)$

$$|\langle f, \varphi \rangle| \leq C \max_{0 \leq k \leq r} \gamma_k^{\alpha,\nu}(\varphi).$$

The proof of this statement follows by the boundedness property of generalized functions.

(iv) Let $f(x)$ be a function defined on the interval $(0, 1)$ such that $\int_0^1 x^{1-\alpha} |f(x)| dx$ exists for $\alpha \geq \frac{1}{2}$. Then $f(x)$ generates a regular generalized function in $U'_{\alpha,\nu}(I)$ defined by

$$(3.1) \quad \langle f, \varphi \rangle = \int_0^1 f(x) \varphi(x) dx, \quad \varphi \in U_{\alpha,\nu}(I).$$

This result easily follows from the inequality

$$|\langle f, \varphi \rangle| \leq \gamma_0^{\alpha,\nu}(\varphi) \int_0^1 x^{1-\alpha} |f(x)| dx < \infty.$$

(v) For each $m = 1, 2, \dots$, and for $\nu \geq -\frac{1}{2}$ the function $xJ_\nu(j_mx)$, $0 < x < 1$, is a member of $U_{\alpha,\nu}(I)$. Indeed, an easy computation leads to

$$(3.2) \quad \left(D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2} \right) [J_\nu(j_mx)] = -j_m^2 J_\nu(j_mx).$$

Using the above operational property, it is quite simple to see that $\gamma_k^{\alpha, \nu} [xJ_\nu(j_mx)] < \infty$ for each $k = 0, 1, 2, \dots$.

4. The generalized finite Hankel transformation. For an arbitrary generalized function $f \in U'_{\alpha, \nu}(I)$, we define its finite Hankel transformation $\mathcal{H}_\nu(f)$ as the application of f to the kernel $xJ_\nu(j_mx)$; i.e.,

$$(4.1) \quad (\mathcal{H}_\nu f)(m) = F(m) = \langle f(x), xJ_\nu(j_mx) \rangle, \quad m = 1, 2, \dots$$

(4.1) is well defined in view of the fact that $xJ_\nu(j_mx) \in U_{\alpha, \nu}(I)$ for each $m = 1, 2, \dots$.

We will often use the expression, $\sum_{m=1}^N 2J_\nu(j_mx)J_\nu(j_mt)/J_{\nu+1}^2(j_m)$ in sequel, so let us denote it for simplicity by the symbol $T_N(t, x)$, for N to be any positive integer and $t, x \in (0, 1)$.

LEMMA 4.1. *Let $f \in U'_{\alpha, \nu}(I)$. Then for any positive integer N and for an arbitrary $\varphi(x) \in D(I)$,*

$$(4.2) \quad \int_0^1 \langle f(t), tT_N(t, x) \rangle \varphi(x) x dx = \left\langle f(t), \int_0^1 tT_N(t, x) \varphi(x) x dx \right\rangle.$$

Proof. As it is obvious to see that the functions $tT_N(t, x)$ for some fixed $x \in (0, 1)$, and $\int_0^1 tT_N(t, x) \varphi(x) x dx$ are members of the space $U_{\alpha, \nu}(I)$ with t as the variable of testing functions, expressions on both the sides of (4.2) have sense. Using the technique of Riemann sums as used in [7, Th. 2], (4.2) can be easily established.

LEMMA 4.2. *Let a, b be any two real numbers satisfying $0 < a < b < 1$. Then*

$$(4.3) \quad \lim_{N \rightarrow \infty} \int_a^b T_N(t, x) x dx = 1, \quad \text{when } a < t < b.$$

Proof. By interchanging the variables x and t and setting $f(x) = 1$, $0 < x < 1$ in Theorem 1.1, we get

$$\lim_{N \rightarrow \infty} \int_0^1 T_N(t, x) x dx = 1 \quad \text{when } 0 < a < t < b < 1.$$

Now,

$$(4.4) \quad \int_0^1 T_N(t, x) x dx = \int_0^a T_N(t, x) x dx + \int_a^b T_N(t, x) x dx + \int_b^1 T_N(t, x) x dx.$$

By virtue of the analogue of Riemann–Lebesgue Lemma [10, p. 589] for $f(x) = 1$, $0 < x < 1$, the first and third integrals on the righthand side of (4.3) tend to zero as $N \rightarrow \infty$, when $a < t < b$. Therefore, taking limits as $N \rightarrow \infty$ on both sides of (4.4), (4.3) is established.

LEMMA 4.3. *Let $\psi(x)$ be an arbitrary member of $D(I)$, $I = (0, 1)$. Then for $\alpha \geq \frac{1}{2}$*

$$(4.5) \quad t^\alpha \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

uniformly for all $t \in (0, 1)$, where the support of $\psi(x)$ is contained in the interval (a, b) ; $0 < a < b < 1$.

Proof. Let us divide the interval $(0, 1)$ into two mutually disjoint sets $(0, a) \cup (b, 1)$ and $[a, b]$.

For $t \in (0, a) \cup (b, 1)$, $\psi(t) = 0$, as support of $\psi(t)$ is contained in (a, b) . Therefore,

$$\int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx = \int_a^b T_N(t, x) \psi(x) x dx.$$

In view of the analogue of Riemann–Lebesgue Lemma [10, p. 589], for a given $\epsilon > 0$, there exists a positive integer N_0 such that for all $N \geq N_0$,

$$\left| \int_a^b T_N(t, x) \psi(x) dx \right| \leq \frac{8c_1^2 \epsilon}{\pi c_2^2 (2 - t - b) \sqrt{t}},$$

[10, p. 590, line 9] which is again bounded by $8c_1^2 \epsilon / (\pi c_2^2 (1 - b) \sqrt{t})$. Here c_1 and c_2 are some constants. Therefore for all $N \geq N_0$ and for all $t \in (0, a) \cup (b, 1)$.

$$t^\alpha \left| \int_a^b T_N(t, x) \psi(x) dx \right| \leq \frac{8c_1^2 \epsilon}{\pi c_2^2 (1 - b)} t^{\alpha - \frac{1}{2}} \leq \frac{8c_1^2 \epsilon}{\pi c_2^2 (1 - b)}, \quad \text{since } \alpha \geq \frac{1}{2}.$$

Hence, as ϵ is arbitrary,

$$(4.6) \quad t^\alpha \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

uniformly for all $t \in (0, a) \cup (b, 1)$.

Next, we want to show that

$$(4.7) \quad t^\alpha \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

uniformly for all $t \in [a, b]$.

Let $F(t, x) (x^2 - t^2) = x^{-\nu} [\psi(x) - \psi(t)]$ for $0 < x < 1$ and $0 < t < 1$. Now define the function $G(t, x)$ in the square domain $\{0 < t < 1; 0 < x < 1\}$ as:

$$G(t, x) = \begin{cases} F(t, x) & , \quad t \neq x \\ \frac{x^{-\nu} \psi'(x)}{2x} & , \quad t = x. \end{cases}$$

Obviously $G(t, x)$ is a continuous function of t and x in the domain $\{0 < t < 1; 0 < x < 1\}$. Now,

$$\begin{aligned} \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx &= \int_a^b x^{\nu+1} F(t, x) (x^2 - t^2) T_N(t, x) dx \\ &= \int_a^b x^{\nu+1} G(t, x) (x^2 - t^2) T_N(t, x) dx, \end{aligned}$$

as the value of the integral remains unchanged by replacing the expression $F(t, x) (x^2 - t^2)$ by $G(t, x) (x^2 - t^2)$.

Let us now divide the interval $a \leq x \leq b$ into p equal parts by the points $a = x_0, x_1, x_2, \dots, x_p = b$, and write, $G(t, x) \equiv G(t, x_{m-1}) + W_m(t, x)$, for $x_{m-1} \leq x \leq x_m$, $a \leq t \leq b$ so that $|W_m(t, x)| \leq U_m - L_m$, where U_m and L_m are respectively the supremum and infimum of $G(t, x)$ taken over $\{x_{m-1} \leq x \leq x_m; a \leq t \leq b\}$, $m = 1, 2, \dots, p$.

Using uniform continuity of the function $G(t, x)$ over the region $\{a \leq t \leq b; a \leq x \leq b\}$ and following exactly on the same lines as in the proof of the analogue of Riemann Lebesgue Lemma [10, p. 589], for an arbitrary $\epsilon > 0$, we get a positive integer N_1 such that

$$\left| \int_a^b x^{\nu+1} G(t, x) (x^2 - t^2) T_N(t, x) dx \right| \leq \frac{4c_1^2 \epsilon}{\pi c_2^2 (1-b) \sqrt{t}}$$

for $N \geq N_1$, where c_1 and c_2 are some constants. Hence for $t \in [a, b]$,

$$\left| t^\alpha \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx \right| \leq \frac{4c_1^2 \epsilon}{\pi c_2^2 (1-b)} t^{\alpha-\frac{1}{2}} \leq \frac{4c_1^2 \epsilon}{\pi c_2^2 (1-b)},$$

for $N \geq N_1$. This proves (4.7), as ϵ is arbitrary. Combining (4.6) and (4.7), the lemma is proven.

5. The main result. We will now prove the following inversion theorem for our generalized finite Hankel transform.

THEOREM 5.1 (inversion). *Let f be an arbitrary generalized function in the space $U'_{\alpha,\nu}(I)$ where $\nu \geq -\frac{1}{2}$, and let $F(m)$ be the finite Hankel transform of f as defined by (4.1). Then in the sense of convergence in $D'(I)$,*

$$(5.1) \quad f(t) = \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} F(m) J_{\nu}(j_m x).$$

Proof. Let $\varphi(x)$ be an arbitrary member of $D(I)$. We wish to show that

$$(5.2) \quad \left\langle \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} F(m) J_{\nu}(j_m x), \varphi(x) \right\rangle \rightarrow \langle f(t), \varphi(t) \rangle$$

as $N \rightarrow \infty$.

Since $\varphi(x) \in D(I)$ iff $x\varphi(x) \in D(I)$, (5.2) will be equivalent to showing

$$(5.2)' \quad \left\langle \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} F(m) J_{\nu}(j_m x), x\varphi(x) \right\rangle \rightarrow \langle f(t), t\varphi(t) \rangle$$

as $N \rightarrow \infty$.

As $\varphi(x) \in D(I)$, let us assume that the support of $\varphi(x)$ is contained in the interval (a, b) , where $0 < a < b < 1$. Now, the theorem will be proven by justifying the steps in the following manipulations:

$$(5.3) \quad \left\langle \sum_{m=1}^N \frac{2F(m)}{J_{\nu+1}^2(j_m)} J_{\nu}(j_m x), x\varphi(x) \right\rangle$$

$$(5.4) \quad = \int_a^b \sum_{m=1}^N \frac{2F(m)}{J_{\nu+1}^2(j_m)} J_{\nu}(j_m x) \varphi(x) x \, dx$$

$$(5.4)' \quad = \int_a^b \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} \langle f(t), tJ_{\nu}(j_m t) \rangle J_{\nu}(j_m x) \varphi(x) x \, dx$$

$$(5.5) \quad = \int_a^b \left\langle f(t), t \sum_{m=1}^N \frac{2J_{\nu}(J_m t) J_{\nu}(j_m x)}{J_{\nu+1}^2(j_m)} \right\rangle \varphi(x) x \, dx$$

$$(5.5)' \quad = \int_a^b \langle f(t), tT_N(t, x) \rangle \varphi(x) x \, dx$$

$$(5.6) \quad = \left\langle f(t), t \int_a^b T_N(t, x) \varphi(x) x \, dx \right\rangle$$

$$(5.7) \quad \rightarrow \langle f(t), t\varphi(t) \rangle \quad \text{as } N \rightarrow \infty.$$

(5.3) equals (5.4) in view of the fact that the function $\Sigma_{m=1}^N (2F(m)/J_{\nu+1}^2(j_m))J_{\nu}(j_m x)$ is locally integrable over the interval $(0, 1)$, and $\varphi(x)$ is in $D(I)$ with support contained in (a, b) . (5.4)' leads to (5.5) by the linearity of the functional. That (5.5)' equals (5.6) is proved in Lemma 4.1. To prove that (5.6) converges to (5.7), we need show that for each $k = 0, 1, 2, \dots$

$$(5.8) \quad t^{\alpha} \Omega_{\nu, t}^k \left[\int_a^b T_N(t, x) \varphi(x) x dx - \varphi(t) \right] \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

uniformly for all $t \in (0, 1)$.

In view of the operational relation (3.2), one can easily see that

$$(5.9) \quad \Omega_{\nu, t} [T_N(t, x)] = \Omega_{\nu, x} [T_N(t, x)].$$

Since the order of differentiation and integration in (5.8) is interchangeable,

$$\begin{aligned} \Omega_{\nu, t} \int_a^b T_N(t, x) \varphi(x) x dx &= \int_a^b \Omega_{\nu, t} [T_N(t, x)] \varphi(x) x dx \\ &= \int_a^b \Omega_{\nu, x} [T_N(t, x)] \varphi(x) x dx, \quad (\text{by (5.9)}) \\ &= \int_a^b T_N(t, x) \Omega_{\nu, x} [\varphi(x)] x dx, \\ &\quad (\text{by integration by parts}). \end{aligned}$$

Operating by the operator $\Omega_{\nu, t}$ successively and using the integration by parts, it can be shown that,

$$\Omega_{\nu, t}^k \int_a^b T_N(t, x) \varphi(x) x dx = \int_a^b T_N(t, x) \Omega_{\nu, x}^k [\varphi(x)] x dx.$$

Hence, in the light of Lemma 4.2, we have as $N \rightarrow \infty$,

$$\begin{aligned} \Omega_{\nu, t}^k \left[\int_a^b T_N(t, x) \varphi(x) x dx - \varphi(t) \right] &= \int_a^b T_N(t, x) [\Omega_{\nu, x}^k \varphi(x) - \Omega_{\nu, t}^k \varphi(t)] x dx \\ &= \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx, \end{aligned}$$

where $\psi(x) = \Omega_{\nu, x}^k \varphi(x)$, which is obviously a member of $D(I)$ with support contained in (a, b) .

Hence it suffices to show that

$$t^\alpha \int_a^b T_N(t, x)[\psi(x) - \psi(t)]x \, dx \text{ converges to zero}$$

as $N \rightarrow \infty$, uniformly for all $t \in (0, 1)$, which is true in view of Lemma 4.3. This completes the proof of the theorem.

THEOREM 5.2 (The Uniqueness Theorem). *Let $f, g \in U'_{\alpha, \nu}(I)$ and let finite Hankel transforms of f and g be $F(m)$ and $G(m)$ respectively, as defined by (4.1). If $F(m) = G(m)$ for each $m = 1, 2, \dots$, then $f = g$ in the sense of equality in $D'(I)$.*

Proof. By Theorem 5.1, in the sense of convergence in $D'(I)$,

$$\begin{aligned} f - g &= \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} [F(m) - G(m)] J_\nu(j_m x) \\ &= 0, \text{ as } F(m) = G(m) \text{ for each } m = 1, 2, \dots \end{aligned}$$

Hence $f = g$ in the sense of equality in $D'(I)$.

6. Illustration of the inversion theorem by means of a numerical example. Consider the Dirac delta function $\delta(t - k)$ concentrated at a point $k, 0 < k < 1$. Since $\delta(t - k) \in E'(I)$ and $E'(I)$ is a subspace of $U'_{\alpha, \nu}(I)$ therefore $\delta(t - k) \in U'_{\alpha, \nu}(1)$. The finite Hankel transform of $\delta(t - k)$ is given as

$$\begin{aligned} \mathcal{H}_\nu(\delta(t - k))(m) &= \langle \delta(t - k), t J_\nu(j_m t) \rangle \\ &= k J_\nu(j_m k), \quad m = 1, 2, \dots \end{aligned}$$

Now for any $\varphi(x) \in D(I)$,

$$\begin{aligned} &\left\langle \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} k J_\nu(j_m k) J_\nu(j_m x), x \varphi(x) \right\rangle \\ &= k \int_0^1 \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} J_\nu(j_m k) J_\nu(j_m x) \varphi(x) x \, dx \\ &= k \int_0^1 T_N(k, x) \varphi(x) x \, dx \\ &\rightarrow k \varphi(k), \text{ as } N \rightarrow \infty, \end{aligned}$$

in view of the Hankel inversion theorem [10, p. 591]. But $k \varphi(k) = \langle \delta(t - k), t \varphi(t) \rangle$ and therefore the inversion theorem is illustrated.

7. Applications. Our finite Hankel transformation generates

an operation transform formula which together with the inversion theorem proved in §5, is applied in solving certain differential equations involving generalized functions.

An operation transform formula. For $\nu \geq -\frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, we define a generalized operator $\Omega_{x,\nu}^*$ on $U'_{\alpha,\nu}(I)$ as the adjoint of the operator $x\Omega_{x,\nu}x^{-1}$ on $U_{\alpha,\nu}(I)$. More specifically, for arbitrary $\varphi(x)$ in $U_{\alpha,\nu}(I)$ and for arbitrary f in $U'_{\alpha,\nu}(I)$

$$(7.1) \quad \langle \Omega_{x,\nu}^* f(x), \varphi(x) \rangle = \langle f(x), x\Omega_{x,\nu}x^{-1}\varphi(x) \rangle,$$

where as before, $\Omega_{x,\nu} = D_x^2 + (1/x)D_x - \nu^2/x^2$.

The right-hand side of (7.1) has a sense because $x\Omega_{x,\nu}x^{-1}\varphi(x) \in U_{\alpha,\nu}(I)$ when $\varphi(x) \in U_{\alpha,\nu}(I)$. Since the mapping $\varphi(x) \rightarrow x\Omega_{x,\nu}x^{-1}\varphi(x)$ is linear and continuous on $U_{\alpha,\nu}(I)$, it follows that $\Omega_{x,\nu}^*$ is linear and continuous on $U'_{\alpha,\nu}(I)$.

It can also be seen inductively that for any integer k

$$(7.2) \quad \langle (\Omega_{x,\nu}^*)^k f(x), \varphi(x) \rangle = \langle f(x), x\Omega_{x,\nu}^k x^{-1}\varphi(x) \rangle,$$

and $(\Omega_{x,\nu}^*)^k$ is linear and continuous on $U'_{\alpha,\nu}(I)$.

Therefore,

$$\begin{aligned} \langle (\Omega_{x,\nu}^*)^k f(x), xJ_\nu(j_mx) \rangle &= \langle f(x), x\Omega_{x,\nu}^k J_\nu(j_mx) \rangle \\ &= (-1)^k j_m^{2k} \langle f(x), xJ_\nu(j_mx) \rangle, \end{aligned}$$

i.e.,

$$(7.3) \quad \mathcal{H}_\nu[(\Omega_{x,\nu}^*)^k f] = (-1)^k j_m^{2k} \mathcal{H}_\nu(f), \quad m = 1, 2, \dots,$$

which gives an operation transform formula.

Note that if f is a regular distribution in $U'_{\alpha,\nu}(I)$ generated by elements of $D(I)$, it can be easily seen by using integration by parts in (7.1) that

$$(7.4) \quad \Omega_{x,\nu}^* f = \Omega_{x,\nu} f,$$

in which case $\Omega_{x,\nu}^*$ can be replaced by $\Omega_{x,\nu}$ in (7.3). (7.4) also holds if we take f as a regular generalized function in $U'_{\alpha,\nu}(I)$ and put some suitable conditions on it so that the limit terms in integration by parts in (7.1) vanish.

Now consider the operational equation

$$(7.5) \quad P(\Omega_{x,\nu}^*)u = g \quad 0 < x < 1$$

where g is a given member of $U'_{\alpha,\nu}(I)$, P is a polynomial such that $P(-j_m^2) \neq 0$, $m = 1, 2, \dots$, and u is unknown generalized function but required to be in $U'_{\alpha,\nu}(I)$.

By applying generalized finite Hankel transformation to (7.5) and using (7.4), we obtain

$$P(-j_m^2)U(m) = G(m), \quad m = 1, 2, \dots$$

so that

$$(7.6) \quad U(m) = \frac{G(m)}{P(-j_m^2)},$$

where $U(m)$ and $G(m)$ are generalized finite Hankel transformations of u and g respectively. Applying the inversion theorem 5.1 to (7.6) we get

$$(7.7) \quad u(x) = \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} \frac{G(m)}{P(-j_m^2)} J_{\nu}(j_m x),$$

with equality in the sense of $D'(I)$, which is a solution to (7.5). This solution is in fact a restriction of $u \in U'_{\alpha,\nu}(I)$ to $D(I)$, and is unique in view of Theorem 5.2.

One can easily verify that u as determined in (7.7) is also a solution to the distributional differential equation

$$(7.8) \quad P(\Omega_{x,\nu})u = g.$$

Now observe that $J_{\nu}(axi)$, for any real number a satisfies the distributional differential equation

$$(7.9) \quad (\Omega_{x,\nu} - a^2)u = 0.$$

Using the variation of parameters one can show that the general solution to (7.9) in $D'(I)$ is given by

$$u(x) = J_{\nu}(axi) \left[c \int_{1/2}^x [tJ_{\nu}^2(ati)]^{-1} dt + d \right], \quad 0 < x < 1,$$

where c and d are arbitrary constants.

Hence for a polynomial $P(x) = (x - a_1^2)(x - a_2^2) \cdots (x - a_n^2)$, where a_i 's are distinct real numbers, the general solution of the distributional differential equation (7.8) in $D'(I)$ is given by

$$u(x) = \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} \frac{G(m)}{P(-j_m^2)} J_{\nu}(j_m x) \\ + \sum_{k=1}^n J_{\nu}(axi) \left[c_k \int_{1/2}^x [tJ_{\nu}^2(a_k t)]^{-1} dt + d_k \right],$$

where c_k and d_k are arbitrary constants.

A Dirichlet problem in cylindrical co-ordinates. This part is devoted to an application of the present theory to a Dirichlet problem in cylindrical co-ordinates having some generalized function boundary conditions. We wish to find a function $u(r, z)$ on the domain $\{(r, \theta, z): 0 < r < 1, 0 \leq \theta \leq 2\pi, 0 < z < \infty\}$, where $u(r, z)$ does not depend upon θ and satisfies the following differential equation:

$$(7.10) \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{\nu^2}{r^2} u + \frac{\partial^2 u}{\partial z^2} = 0 \quad (\nu \geq 0),$$

with the following boundary conditions:

- (i) As $z \rightarrow \infty$, $u(r, z)$ converges in the sense of $D'(I)$ to zero.
- (ii) As $z \rightarrow 0+$, $u(r, z)$ converges in the sense of $D'(I)$ to a generalized function $f(r) \in U'_{\alpha, \nu}(I)$.
- (iii) As $r \rightarrow 0+$, $U(r, z) = O(r^{\nu})$ uniformly on $\eta \leq z < \infty$ for each $\eta > 0$.
- (iv) As $r \rightarrow 1-$, $u(r, z)$ converges to zero uniformly on $\eta \leq z < \infty$ for each $\eta > 0$.

Now, equation (7.10) can be written as

$$(7.11) \quad \Omega_{\nu, r} u + \frac{\partial^2 u}{\partial z^2} = 0.$$

By applying the generalized finite Hankel transform \mathcal{H}_{ν} to (7.11), and formally interchanging \mathcal{H}_{ν} with $\partial^2/\partial z^2$, we convert (7.11) into

$$-j_m^2 U(m, z) + \frac{\partial^2}{\partial z^2} U(m, z) = 0, \quad m = 1, 2, \dots,$$

where $U(m, z) = \mathcal{H}_{\nu}[u(r, z)] = \langle u(r, z), rJ_{\nu}(j_m r) \rangle$.

Thus,

$$(7.12) \quad U(m, z) = A(m)e^{-Imz} + B(m)e^{Imz},$$

where $A(m)$ and $B(m)$ are constants depending on m .

In view of boundary conditions (i) and (ii), we formally take

$\lim_{z \rightarrow \infty} U(m, z) = 0$ and $\lim_{z \rightarrow 0^+} U(m, z) = \mathcal{H}_\nu(f) = F(m)$ respectively which suggest $B(m) = 0$ and $A(m) = F(m)$ in (7.12). Therefore

$$U(m, z) = F(m)e^{-j_m z}.$$

Applying inversion theorem 5.1 to the above equation we get

$$u(r, z) = \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{2}{J_{\nu+1}^2(j_m)} F(m)e^{-j_m z} J_\nu(j_m r) \quad \text{in } D'(I).$$

For each $\varphi \in D(I)$, one can show that

$$\langle u(r, z), \varphi(r) \rangle = \int_0^1 \sum_{m=1}^{\infty} \frac{2}{J_{\nu+1}^2(j_m)} F(m)e^{-j_m z} J_\nu(j_m r) \varphi(r) dr,$$

so that $u(r, z)$, as a classical function is obtained as follows:

$$(7.13) \quad u(r, z) = \sum_{m=1}^{\infty} \frac{2}{J_{\nu+1}^2(j_m)} F(m)e^{-j_m z} J_\nu(j_m r).$$

We now verify that (7.13) is truly a solution to (7.10). It is an easy consequence of note (iii) of §3 that, as $m \rightarrow \infty$, $F(m) = O(j_m^{2n-\frac{1}{2}})$ for some nonnegative integer n .

Moreover,

$$j_m \sim \pi \left(m + \frac{1}{2} \nu - \frac{1}{4} \right), \quad \text{as } m \rightarrow \infty.$$

$$J_{\nu+1}(j_m) \sim \sqrt{\frac{2}{\pi j_m}}, \quad \text{as } m \rightarrow \infty$$

$$e^{-j_m z} = O(e^{-\pi \eta m}) \quad \text{uniformly on } \eta \leq z < \infty$$

for each fixed $\eta > 0$, and $J_\nu(j_m r)$ is uniformly bounded for each $m = 1, 2, \dots$ on $0 < r < 1$. Using these facts, we can easily verify that the series (7.13) and the series obtained by applying the operators $\Omega_{r, \nu}$ and D_z^2 separately under the summation sign of (7.13) converge uniformly on $0 < r < 1$ and $\eta \leq z < \infty$ ($\eta > 0$). Thus applying $\Omega_{r, \nu} + D_z^2$ term by term to (7.13) and using the fact $\Omega_{r, \nu}[J_\nu(j_m r)] = -j_m^2 J_\nu(j_m r)$, we see that (7.13) satisfies the differential equation (7.10).

The uniform convergence of (7.13) leads to the verification of boundary conditions (iii) and (iv) by taking the limits as $r \rightarrow 0^+$ and $r \rightarrow 1^-$ respectively under the summation sign.

The boundary condition (ii) is verified by justifying the following steps. For any $\varphi(r) \in D(I)$,

$$(7.14) \quad \lim_{z \rightarrow 0^+} \left\langle \sum_{m=1}^{\infty} \frac{2}{J_{\nu+1}^2(j_m)} F(m) e^{-j_m z} J_{\nu}(j_m r), \varphi(r) \right\rangle \\ = \lim_{z \rightarrow 0^+} \int_0^1 \sum_{m=1}^{\infty} \left\{ \frac{2}{J_{\nu+1}^2(j_m)} F(m) e^{-j_m z} J_{\nu}(j_m r) \varphi(r) \right\} dr$$

$$(7.15) \quad = \int_0^1 \sum_{m=1}^{\infty} \frac{2}{J_{\nu+1}^2(j_m)} F(m) J_{\nu}(j_m r) \varphi(r) dr$$

$$(7.16) \quad = \langle f, \varphi \rangle.$$

The step (7.14) is straightforward. The uniform convergence of the series in (7.14) on $0 < r < 1$ and $\eta \leq z < \infty$ for any $\eta > 0$ allows us to take the limit as $z \rightarrow 0^+$ under the integral sign and then under the summation sign, and leads to (7.15). Finally, (7.15) equals (7.16) in view of Theorem 5.1.

The boundary condition (i) can be verified exactly in the same way as (ii). This completes our verification of (7.13) as a solution to (7.10). This solution is unique in the sense of equality over $D'(I)$ in view of Theorem 5.2.

ACKNOWLEDGEMENT. The author is thankful to Dr. J. N. Pandey for his valuable suggestions in preparation of this paper.

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Received December 10, 1974. This work was supported by the NRC Grant No. 240–123.