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ON IRREDUCIBLE SPACES II

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A topological space is said to be *irreducible* if every open covering has an open refinement that covers the space minimally. Irreducibility is a fundamental property related to cardinality conditions for open coverings. In this paper, a constructive proof is presented to establish that the weak $\bar{\theta}$-refinable spaces of Smith are irreducible. Various results concerning cardinality conditions for open coverings follow as corollaries. Some examples are included.

1. Introduction.

Recently Smith [11] introduced the notion of a weak $\bar{\theta}$-refinable space and obtained a variety of applications of this topological structure. Smith defines a space to be *weak $\bar{\theta}$-refinable* if every open covering has a weak θ-refinement [2], $\cup\{\mathcal{G}_i: i \in N\}$ such that $\{\mathcal{G}_i^*: i \in N\}$ is point-finite. For a collection of sets $\mathcal{M}$, $\mathcal{M}^*$ denotes the union of the sets in $\mathcal{M}$. We define a space to be *weakly θ-refinable of finite type* if every open covering has a weak θ-refinement $\cup\{\mathcal{G}_i: i \in N\}$ such that all but finitely many $\mathcal{G}_i$ are empty. Also we call a weakly θ-refinable space which is *not* of finite type, *strictly weakly θ-refinable.* The variations of Bing's Example G which are presented in [2], [6], [7] and [12] as presented in [4], are weakly θ-refinable of finite type. Clearly, metacompact spaces are weakly θ-refinable of finite type. Weakly θ-refinable spaces of finite type and the θ-refinable spaces of Wicke and Worrell [13] (As Smith proves in [11]) are weak $\bar{\theta}$-refinable.

The primary purpose of this paper is to establish that the weak $\bar{\theta}$-refinable spaces of Smith have the fundamental property of irreducibility. A space is *irreducible* if every open covering has an open refinement that covers the space minimally. In [5] a constructive proof that every θ-refinable space is irreducible is presented. Also, the following characterization of irreducibility is established.

THEOREM 1.1. *A nonempty space X is irreducible if and only if for each open covering $\{V_\alpha: \alpha \in A\}$ of X there exists a discrete collection of nonempty closed sets $\{T_\beta: \beta \in B\}$ such that $B \subset A$, $T_\beta \subset V_\beta$ for each $\beta \in B$ and $\{V_\beta: \beta \in B\}$ covers X.*

In this paper the cardinality of set γ will be denoted by $|\gamma|$ and the natural numbers will be denoted by N. Also, if $\mathcal{V} = \{V_\alpha: \alpha \in A\}$ is a collection of subsets of a space where A is well-ordered and a collection of nonempty subsets $\mathscr{L} = \{L_\gamma : \gamma \subset A, \gamma \neq \varnothing\}$ refines $\mathscr{V}$ in such a way that $L_\gamma \subset \cap \{V_\alpha : \alpha \in \gamma\}$, for each $\gamma \subset A$, $\gamma \neq \varnothing$, then we will adopt the following convention in the proof of Theorem 2.1, that $\mathrm{st}(\mathscr{L}, \mathscr{V})$ is the union of the sets V_α where α is the least element in γ for some γ in the indexing set of L. When denoting a collection of sets $\mathscr{V}$ as the range $\{V_\alpha : \alpha \in A\}$ of some function, we mean that this function is a bijection.

The main results of this study are contained in §2. Some examples relating to weak $\bar{\theta}$-refinable spaces are in §3 and §4 is the proof of the main theorem, Theorem 2.1.

2. Main theorem and some corollaries.

THEOREM 2.1. *Every weak $\bar{\theta}$-refinable space is irreducible.*

Proof. Section 4.

A space is said to have *property* (δ) **[11]** if discrete collections in X are countable. The following corollary to Theorem 2.1 is a special case of Theorem 2.5.

COROLLARY 2.2 **[11]**. *In a weak $\bar{\theta}$-refinable space the following are equivalent*:

(a) *property* (δ)
(b) *Lindelöf*
(c) $\aleph_1$*-compact.*

Since the essential property of metacompact spaces used by Arens and Dugundji in **[1]** was irreducibility, we have the following corollary.

COROLLARY 2.3. **[11]** *A weak $\bar{\theta}$-refinable space is compact if and only if it is countably compact.*

Also since irreducibility of spaces with property (δ) implies these spaces are Lindelöf, and regular Lindelöf spaces have the star-finite property [**9**, Theorem 10], Theorem 4.2 of **[11]** can be generalized in the following manner.

COROLLARY 2.4. *Every regular irreducible space with property* (δ) *has the star-finite property.*

The results of Corollary 2.2 can be extended to arbitrary infinite cardinal numbers. For instance, if m is any infinite cardinal we say a space has *property* (m) if every discrete collection has cardinality $\leqq m$. Also, a space is m*-Lindelöf* if every open covering has a subcovering of cardinality $\leqq m$.

THEOREM 2.5. *Let X be an irreducible space. If X has property (m), then X is m-Lindelöf.*

Proof. Suppose X is not m-Lindelöf. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open covering of X such that every subcovering of $\mathcal{U}$ has cardinality $> m$. Since X is irreducible, by Theorem 1.1, there exists a discrete collection of nonempty closed sets $\{T_\beta : \beta \in B\}$ such that $B \subset A$, $T_\beta \subset U_\beta$ for each $\beta \in B$ and $\{U_\beta : \beta \in B\}$ covers X. Since $\{U_\beta : \beta \in B\}$ is a subcovering of $\mathcal{U}$, the cardinality of $\{T_\beta : \beta \in B\}$ is strictly greater than m. Thus X does not have property m. This completes the proof.

Further, for an infinite cardinal m a space is called *m-compact* if every subset of cardinaltiy m has a cluster point. Let m^+ be the first cardinal exceeding m. In the following corollary, which is an extension of Corollary 2.2, the irreducibility of the space is needed only in (a) $\Rightarrow$ (b). The implications (a) $\Leftrightarrow$ (c) and (b) $\Rightarrow$ (a) are easily verified.

COROLLARY 2.6. *In an irreducible T_1-space the following are equivalent:*

(a) *property (m)*
(b) *m-Lindelöf*
(c) *m^+-compact.*

Every discrete collection of subsets of the ordinal space $[0, \Omega)$, where Ω is the first uncountable ordinal, has finite cardinality. Thus $[0, \Omega)$ has property $(\aleph_0)$ ($\equiv$ property (δ)). However, $[0, \Omega)$ is not $\aleph_0$-Lindelöf ($\equiv$ Lindelöf). Thus, (a) does not imply (b), if the space is not irreducible.

3. Examples.

EXAMPLE 3.1. *A space which is weakly θ-refinable of finite type which is not countably θ-refinable.*

This is Example 2 of **[10]**. It is presented here to correct an inconsistency between the statement in **[10]** that this space is θ-refinable but not countably metacompact and the theorem of Gittings **[8]** which establishes that countable metacompactness is equivalent to countable θ-refinability.

Let $G = \{p_i : i \in N\}$ be any countably infinite set of objects which are not real numbers, and let R be the set of real numbers. Let $X = R \cup G$. A neighborhood base at $r \in R$ is a usual neighborhood base in R. A neighborhood base at $p_i \in G$ consists of all sets of the form $\{p_i\} \cup (R - C)$ where C is any closed countable subset of R. Let $\mathcal{U} = \{U_n : n \in N\}$ be any countable open covering of X by basic open sets. Then every open σ-refinement of $\mathcal{U}$, $\mathcal{V} = \cup\{\mathcal{V}_j : j \in N\}$, which covers X on every level must contain on every level j, sets of the form $V_j^i = \{p_i\} \cup (R - C_j^i)$, for each $p_i \in G$. Since $R \cap (\cap\{V_j^i : i, j \in N\}) = R - (\cup\{C_j^i : i, j \in N\}) \neq \emptyset$, there are uncountably many $r \in R$ such that $r \in V_j^i$ for each $i \in N$ and on each level j. Accordingly, X is *not* countably θ-refinable. The space X is T_1, Lindelöf and weakly θ-refinable of finite type. Thus X is irreducible. The closed set G is not a G_δ set, which is precisely the reason that X is not θ-refinable **[2]**.

EXAMPLE 3.2. *A strictly weakly θ-refinable space which is weak $\bar{\theta}$-refinable.*

All examples of weakly θ-refinable, but not θ-refinable spaces that appear in the papers cited in the introduction are of finite type. Thus these examples are trivially weak $\bar{\theta}$-refinable. This example is built on the disjoint union of a countably infinite number of copies of the real line.

Let $X = R \times N = \cup\{R_n : n \in N\}$ where $R_n = \{(r, n) : r \in R\}$. For each $(r, n) \in X$ a basic open neighborhood of (r, n) is any set of the form $U(r, n, \lambda, \mathcal{F}_n) = \{(p, n) : p \in (r - \lambda, r + \lambda)\} \cup \bigcup\{R_j - F_j : j > n\}$, where $F_j \in \mathcal{F}_n$, F_j is any finite subset of R_j and $\lambda > 0$.

This space is hereditarily weak $\bar{\theta}$-refinable, T_1 and Lindelöf. It is not of finite type. That is, it is strictly weakly θ-refinable.

4. Proof of Theorem 2.1.

Proof. Let X be a weak $\bar{\theta}$-refinable space, and let $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ be any open covering of X. Consider A to be well ordered. Recall, all stars of collections of subsets are taken according to the criterion described in the introduction. Let $\mathcal{U} = \cup\{\mathcal{U}_i : i \in N\}$ be a σ-precise weak $\bar{\theta}$-refinement of $\mathcal{V}$, where $\mathcal{U}_i = \{U_\alpha^i : \alpha \in A_i \subset A, U_\alpha^i \neq \emptyset\}$.

The discrete collection of closed sets, indicated in Theorem 1.1, *is constructed by induction with the sets defined in the following paragraph.*

For each $n, k \in N$, let $K_{nk} = \{p \in X : p$ is in exactly k sets in $\mathcal{U}_n\}$ and $Z_n = X - \mathcal{U}_n^*$. Let $\mathcal{F}(n, k) = \{F(n, \gamma) : \gamma \subset A_n, |\gamma| = k\}$, where $F(n, \gamma) = K_{nk} \cap (\cap\{U_\alpha^n : \alpha \in \gamma\})$. Then $\mathcal{F}(n, k)$ is a collection of subsets of $X - (Z_n \cup \bigcup\{\mathcal{F} * (n, j) : j < k\})$, which are closed in this subspace. Also, $\mathcal{F}(n, k)$ is discrete in this same subspace. Let $\mathcal{N}_k = \{\Gamma \subset N : |\Gamma| = k\}$, and let $H_k = \{p \in X : p$ is an element of exactly k sets in $\{\mathcal{U}_i^* : i \in N\}\}$. Let $\mathcal{H}_k = \{H(k, \Gamma) : \Gamma \in \mathcal{N}_k, H(k, \Gamma) \neq \emptyset\}$, where

$$H(k, \Gamma) = H_k \cap (\cap\{\mathcal{U}_j^* : i \in \Gamma\}),$$

for each $\Gamma \in \mathcal{N}_k$. Then $\mathcal{H}_k$ is a collection of subsets of $X - \cup\{\mathcal{H}_j^* : j < k\}$, which are closed in this subspace. Also, $\mathcal{H}_k$ is discrete in this same subspace. Note that: $X = \cup\{\mathcal{H}_k^*: k \in N\}$, and $X = \cup\{\mathcal{F}^*(n,k): n, k \in N\}$.

The desired discrete collection of nonempty closed subsets of X which are constructed by induction is denoted by $\mathcal{T}$ and will consist of unions of the families denoted $\mathcal{C}(n, j, k, \Gamma)$ and $\mathcal{T}(k, \Gamma)$. The verification that $\mathcal{T}$ is a discrete collection of nonempty closed sets is presented at the end of the proof.

We begin the construction in the set $\mathcal{H}_1$. (Every point in $\mathcal{H}_1^*$ is covered by at most a finite number of sets in *one* level of the collection $\mathcal{U}$.) For each $H(1,\Gamma) \in \mathcal{H}_1$, $\Gamma = \{n\}$ for some $n \in N$, and $\mathcal{U}_i^* \cap H(1,\Gamma) = \emptyset$, if $i \neq n$. Let $\mathcal{C}(n,1,1,\Gamma) = \{C(n,\gamma,1,\Gamma)\colon \gamma \subset A_n, |\gamma| = 1, \Gamma = \{n\}\}$, where $C(n,\gamma,1,\Gamma) = F(n,\gamma) \cap H(1,\Gamma)$.

Then for each $k > 1$, let $\mathcal{C}(n,k,1,\Gamma) = \{C(n,\gamma,1,\Gamma)\colon \gamma \subset A_n, |\gamma| = k, \Gamma = \{n\}\}$, where

$$C(n,\gamma,1,\Gamma) = (F(n,\gamma) \cap H(1,\Gamma)) - st\left(\bigcup_{j<k} \mathcal{C}(n,j,1,\Gamma), \mathcal{V}\right).$$

Hence $\mathcal{T}_1 = \cup\{\mathcal{T}(1,\Gamma)\colon H(1,\Gamma) \in \mathcal{H}_1\}$, where $\mathcal{T}(1,\Gamma) = \cup\{\mathcal{C}(n,j,1,\Gamma)\colon \Gamma = \{n\}, j \in N\}$ is a discrete collection of closed subsets of X and $\mathcal{T}_1^* \subset \mathcal{H}_1^* \subset st(\mathcal{T}_1, \mathcal{V})$. This completes the construction for points in one set in $\{\mathcal{U}_i^*: i \in N\}$.

Suppose this process has been continued for the sets $\mathcal{H}_j$, $j < k$. By this we mean, discrete collections of closed subsets of X, $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_{k-1}$ have been defined such that $\mathcal{T}_j^* \subset \mathcal{H}_j^* \subset st(\bigcup_{i \leqq j} \mathcal{T}_i, \mathcal{V})$ for each $j \leqq k-1$.

Now consider the collection $\mathcal{H}_k$. Let $H(k,\Gamma) \in H_k$. Say $\Gamma = \{n(1), n(2), \cdots, n(k)\}$, where $n(1) < n(2) < \cdots < n(k)$. Let $M(k,\Gamma) = H(k,\Gamma) - st(\cup\{\mathcal{T}_j\colon j = 1, 2, \cdots, k-1\}, \mathcal{V})$, for each $H(k,\Gamma) \in \mathcal{H}_k$. Note that $\{M(k,\Gamma)\colon \Gamma \in \mathcal{N}_k\}$ is a discrete collection of closed subsets of X and every point in $M(k,\Gamma)$ is covered by a finite number of sets on at least one of the levels $n(1), n(2), \cdots, n(k)$.

The construction of the discrete collections of closed subsets of $M(k,\Gamma)$ will proceed in the following order:

$$\mathcal{F}(n(1),1), \mathcal{F}(n(2),1), \cdots, \mathcal{F}(n(k),1),$$
$$\mathcal{F}(n(1),2), \cdots, \mathcal{F}(n(k),2), \cdots,$$
$$\mathcal{F}(n(1),m), \mathcal{F}(n(2),m), \cdots, \mathcal{F}(n(k),m), \cdots.$$

Let $\mathcal{C}(n(1),1,k,\Gamma) = \{C(n(1),\gamma,k,\Gamma)\colon \gamma \subset A_{n(1)}, |\gamma| = 1\}$, where

$$C(n(1), \gamma, k, \Gamma) = F(n(1), \gamma) \cap M(k, \Gamma).$$

Also let $\mathscr{C}(n(2), 1, k, \Gamma) = \{C(n(2), \gamma, k, \Gamma)\colon \gamma \subset A_{n(2)}, |\gamma| = 1\}$, where $C(n(2), \gamma, k, \Gamma) = F(n(2), \gamma) \cap M(k, \Gamma) - st(\mathscr{C}(n(1), 1, k, \Gamma), \mathscr{V})$. For $2 < j \leqq k$, let $\mathscr{C}(n(j), 1, k, \Gamma) = \{C(n(j), \gamma, k, \Gamma)\colon \gamma \subset A_{n(j)}, |\gamma| = 1\}$, where

$$C(n(j), \gamma, k, \Gamma) = F(n(j), \gamma) \cap M(k, \Gamma) - st(\cup \{\mathscr{C}(n(i), 1, k, \Gamma)\colon i = 1, 2, \cdots, j-1\}, \mathscr{V}).$$

This completes the steps through $\mathscr{F}(n(k), 1)$.

Let $m > 1$. Suppose discrete collections of closed subsets of X, $\mathscr{C}(n(j), t, k, \Gamma)$ have been defined for $1 \leqq j \leqq k$ and $t < m$. Let $\mathscr{C}(n(1), m, k, \Gamma) = \{C(n(1), \gamma, k, \Gamma)\colon \quad \gamma \subset A_{n(1)}, \quad |\gamma| = m\}$, where $C(n(1), \gamma, k, \Gamma) = F(n(1), \gamma) \cap M(k, \Gamma) - st(\cup \{\mathscr{C}(n(j), t, k, \Gamma)\colon 1 \leqq j \leqq k, t < m\}, \mathscr{V})$. For $n(k) \geqq n(r) > n(1)$ and $\gamma \subset A_{n(r)}$ such that $|\gamma| = m$, let $C(n(r), \gamma, k, \Gamma) = F(n(r), \gamma) \cap M(k, \Gamma) - st(\cup \{\mathscr{C}(n(j), t, k, \Gamma)\colon 1 \leqq j \leqq k, t < m \text{ or } t = m \text{ if } j < r\}, \mathscr{V})$. Then let $\mathscr{C}(n(r), m, k, \Gamma) = \{C(n(r), \gamma, k, \Gamma)\colon \gamma \subset A_{n(r)}, \quad |\gamma| = m\}$. If $\mathscr{T}(k, \Gamma) = \cup \{\mathscr{C}(n(j), n, k, \Gamma)\colon \quad j = 1, 2, \cdots, k, n \in N\}$, then $\mathscr{T}^*(k, \Gamma) \subset M(k, \Gamma) \subset st(\mathscr{T}(k, \Gamma), \mathscr{V})$.

Since $H(k, \Gamma)$ was any set in $\mathscr{H}_k$, there is a discrete collection of closed sets

$$\mathscr{T}_k = \cup \{\mathscr{T}(k, \Gamma)\colon H(k, \Gamma) \in \mathscr{H}_k\} \quad \text{such that} \quad \mathscr{T}_k^* \subset \mathscr{H}_k^* \subset st(\mathscr{T}_k, \mathscr{V}).$$

Thus the discrete collections of closed sets $\mathscr{T}_k$ are defined for each $k \in N$, consisting of sets of the type $C(n, \gamma, k, \Gamma)$. Let

$$\mathscr{T} = \{C(n, \gamma, k, \Gamma) \in \mathscr{T}_k\colon C(n, \gamma, k, \Gamma) \neq \varnothing,\ k \in N\}.$$

$\mathscr{T}$ is a discrete collection of nonempty closed sets. It is easly verified that for each $n, k \in N$, $\mathscr{F}(n, k)$ is a collection of subsets of $X - (Z_n \cup \bigcup \{\mathscr{F}^*(n, j); j < k\})$ which is discrete in this subspace and the sets $F(n, \gamma) \in \mathscr{F}(n, k)$ are closed in this subspace. Also, for each $k \in N$, $\mathscr{H}_k$ is a collection of subsets of $X - \cup \{\mathscr{H}_j^*\colon j < k\}$ which is discrete in this subspace and the sets $H(k, \Gamma) \in \mathscr{H}_k$ are closed in this subspace. Since $X = \cup \{\mathscr{H}_k^*\colon k \in N\}$, $\cup \{\mathscr{H}_j^*\colon j < k\} \subset st(\cup \{\mathscr{T}_j\colon j < k\}, \mathscr{V})$ and $M(k, \Gamma) = H(k, \Gamma) - st(\cup \{\mathscr{T}_j\colon j < k\}, \mathscr{V})$ for each $H(k, \Gamma) \in \mathscr{H}_k$, $\{M(k, \Gamma)\colon H(k, \Gamma) \in \mathscr{H}_k, k \in N\}$ is a discrete collection of closed sets in X. Further, since for each $|\gamma'| < m$, $M(k, \Gamma) \cap F(n, \gamma') \subset st(\cup \{\mathscr{C}(n', t, k, \Gamma)\colon t < m \text{ or } t = m \text{ if } n' < n, \ n' \in \Gamma\}, \ \mathscr{V})$, $\{M(k, \Gamma) \cap F(n, \gamma) - st(\cup \{\mathscr{C}(n', t, k, \Gamma)\colon \ n' < n, n' \in \Gamma, t < m \text{ or } t = m \text{ if } n' < n\}, \mathscr{V})\colon |\gamma| = m\}$ is a discrete collection of closed sets in X. Since $\mathscr{F}(n, k)$ covers $H(k, \Gamma)$ for $k \in N$ and $n \in \Gamma$, it follows from the definition of $C(n, t, k, \Gamma)$ that $\mathscr{T}$ is a discrete collection of

nonempty closed sets in X since its members are the intersections of closed sets from two discrete collections.

Let $B = \{\beta \in A : \beta$ is the least element in γ for some $C(n, \gamma, k, \Gamma) \in \mathscr{T}\}$. By the convention in the introduction for forming the star of a collection of subsets, and the use of the stars in the construction, $\{V_\beta : \beta \in B\}$ covers X. By Theorem 1.1, X is irreducible.

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