PACIFIC JOURNAL OF MATHEMATICS
Vol. 63, No. 2, 1976

V-LOCALIZATIONS AND V-MONADS II

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For a symmetric monoidal closed category $\mathfrak{V}$ satisfying certain completeness conditions, consider a $\mathfrak{V}$-category $\mathfrak{A}$, a subcategory Σ of $\mathfrak{A}$ which admits a $\mathfrak{V}$-calculus of left fractions, and a $\mathfrak{V}$-monad $\mathfrak{T} = (T, \eta, \mu)$ on $\mathfrak{A}$. Suppose $\mathfrak{T}$ is compatible with Σ so that a $\mathfrak{V}$-monad $\mathfrak{T}'$ is induced on $\mathfrak{A}[\Sigma^{-1}]$ and the canonical projection $\mathfrak{V}$-functor $\Phi: \mathfrak{A} \rightarrow \mathfrak{A}[\Sigma^{-1}]$ induces a $\mathfrak{V}$-functor $L: \mathfrak{A}^T \rightarrow \mathfrak{A}[\Sigma^{-1}]^{T'}$ on the $\mathfrak{V}$-categories of Eilenberg-Moore algebras. Suppose that Σ is conice and $\mathfrak{A}^T$ has coequalizers. We prove that, if L preserves coequalizers (which is true in the case where T preserves coequalizers), then L is the canonical projection for the $\mathfrak{V}$-localization of a subcategory of $\mathfrak{V}^T$ which admits a $\mathfrak{V}$-calculus of left fractions.

In [12] (and again in [13]) we studied the question of the relationship between $\mathfrak{V}$-localizations and $\mathfrak{V}$-monads. The principle concern was with the question of when the process of forming a localization "commutes" with the process of forming the category of algebras for a monad in the following sense: given a monad $\mathfrak{T} = (T, \eta, \mu)$ on a $\mathfrak{V}$-category $\mathscr{A}$ and a $\mathfrak{V}$-localizing subcategory $\Sigma \subseteq \mathscr{A}_0$, when is there an induced monad $\mathfrak{T}'$ on the localization $\mathfrak{A}[\Sigma^{-1}]$ whose algebras are a localization of the category of algebras for $\mathfrak{T}$? In [12] we dealt primarily with the situation in which the canonical localization functor had a left or right adjoint. One result (Proposition 3.11) did deal with the case in which the canonical functor $\Phi: \mathfrak{A} \rightarrow \mathfrak{A}[\Sigma^{-1}]$ did not have an adjoint. This result, however, had many complicating hypotheses and since in many instances of localizations the canonical functor does not have an adjoint, it is desirable to give a more thorough examination of this case. It is for this purpose that this paper was written.

The main result we prove is the following. Suppose $\mathfrak{V}$ is cocomplete and finitely complete, and that any filtered colimits which exist in $\mathfrak{V}$ are such that they commute with equalizers and are preserved by the canonical functor $V: \mathfrak{V} \rightarrow \text{Sets}$. Let $\Sigma \subseteq \mathfrak{A}$ be conice and admit a $\mathfrak{V}$-calculus of left fractions. Let $\mathfrak{T}$ be a $\mathfrak{V}$-monad on $\mathfrak{A}$ such that $\Phi T(s)$ is an isomorphism for all $s \in \Sigma$. Then there is a $\mathfrak{V}$-monad $\mathfrak{T}'$ on $\mathfrak{A}[\Sigma^{-1}]$ and a $\mathfrak{V}$-functor $L: \mathfrak{A}^T \rightarrow \mathfrak{A}[\Sigma^{-1}]^{T'}$ such that if $\mathfrak{A}^T$ has coequalizers which are preserved by L then $S = \{\alpha \in \mathfrak{A}^T \mid \Phi U(\alpha) \text{ is an isomorphism}\}$ admits a $\mathfrak{V}$-calculus of left fractions and $\mathfrak{A}^T[\Sigma^{-1}] \approx \mathfrak{A}[\Sigma^{-1}]^{T'}$.

There are many symmetric monoidal $\mathfrak{V}$ which satisfy the hypothesis. For example, Sets, R-modules for a commutative ring R, the category of semi-simiplicial sets and in general, any category which is finitarily monadic over sets with respect to a commutative monad.

As an immediate consequence of the above result we get that if $\mathfrak{A}$ has $\mathfrak{B}$-coequalizers which are preserved by $\mathfrak{T}$ then $\mathfrak{A}^T$ has coequalizers which are preserved by L and so the conclusions of the above result hold. This removes many of the complicated hypothesis in 3.11 of [12].

In the last section of the paper we give some illustrative examples of the main results.

Throughout we assume that $\mathfrak{B}$ is a symmetric monoidal closed category. We also assume that the reader is familiar with the theory of $\mathfrak{B}$-monads. (See [12] or [3].)

1. Main results. We begin by recalling some definitions from the theory of $\mathfrak{B}$-localizations. Let $\mathfrak{A}$ be a $\mathfrak{B}$-category and $\Sigma \subseteqq \mathfrak{A}_0$ any subcategory of $\mathfrak{A}_0$ with the same objects as $\mathfrak{A}$. By a $\mathfrak{B}$-localization of $\mathscr{A}$ with respect to Σ we mean a $\mathfrak{B}$-category $\mathfrak{A}[\Sigma^{-1}]$ together with a $\mathfrak{B}$-functor $\Phi: \mathfrak{A} \rightarrow \mathfrak{A}[\Sigma^{-1}]$ such that $\Phi(s)$ is an isomorphism for all $s \in \Sigma$ and if $F: \mathfrak{A} \rightarrow \mathfrak{B}$ is a $\mathfrak{B}$-functor such that $F(s)$ is an isomorphism for all $s \in \Sigma$ then there exists a unique $\mathfrak{B}$-functor $\bar{F}: \mathfrak{A}[\Sigma^{-1}] \rightarrow \mathfrak{B}$ such that $\bar{F}\Phi = F$. We shall take Φ to be the identity on objects. If $\mathfrak{A}[\Sigma^{-1}]$ exists for Σ we say that Σ is $\mathfrak{B}$-localizable.

If $\Sigma \subseteqq \mathfrak{A}_0$ is $\mathfrak{B}$-localizable we call it $\mathfrak{B}$-well-localizable if for every pair of $\mathfrak{B}$-functors $F, G: \mathfrak{A}[\Sigma^{-1}] \rightarrow \mathfrak{B}$, the correspondence $\alpha \rightarrow \alpha\Phi$ is one-one and onto between $\mathfrak{B}$-natural transformations α from F to G and $\mathfrak{B}$-natural transformations from $F\Phi$ to $G\Phi$. It can be shown that if $\mathfrak{B}$ has pullbacks then any $\mathfrak{B}$-localization is $\mathfrak{B}$-well-localizable.

Let $\Sigma \subseteqq \mathfrak{A}_0$ be a subcategory and let A be an object of $\mathfrak{A}$. The category A/Σ is the category with objects $s: A \rightarrow B$, $s \in \Sigma$. A morphism from $s: A \rightarrow B$ to $s': A \rightarrow c$ is a morphism $f: B \rightarrow C$ in A such that

$$\begin{array}{ccc} & A & \\ {}^{s}\swarrow & & \searrow^{s'} \\ B & \xrightarrow{\;f\;} & C \end{array}$$

commutes. Let $Q^A: A/\Sigma \rightarrow \mathfrak{A}$ be the obvious projection. Then Σ is said to admit a $\mathfrak{B}$-calculus of left fractions if Σ is $\mathfrak{B}$-localizable and for all A, B,

$$\mathfrak{A}[\Sigma^{-1}](B, A) = \varinjlim \left(A/\Sigma \xrightarrow{Q^A} A \xrightarrow{\mathfrak{A}(B,\,-)} \mathfrak{B}\right) .$$

Now let $\mathscr{T} = (T, \eta, \mu)$ be a $\mathfrak{B}$-monad on $\mathfrak{A}$ and suppose that $\Sigma \subseteqq \mathfrak{A}_0$ is $\mathfrak{B}$-localizable. We say that $\mathfrak{T}$ is compatible with Σ if $\Phi T(s)$ is an isomorphism for all $s \in \Sigma$. It is easy to show that T is compatible with Σ if and only if there exists a $\mathfrak{V}$-monad $\mathfrak{T}'$ on $\mathscr{A}[\Sigma^{-1}]$ such that $\Phi T = T'\Phi$.

PROPOSITION 1.1. *Let $\Sigma \subseteq \mathfrak{A}_0$ be $\mathfrak{V}$-well-localizable and let $\mathfrak{T} = (T, \eta, \mu)$ be compatible with Σ and let $\mathfrak{T}'$ be the induced monad on $\mathfrak{A}[\Sigma^{-1}]$. Then there exists a $\mathfrak{V}$-functor $L: \mathfrak{A}^T \to \mathfrak{A}[\Sigma^{-1}]^{T'}$* ($\mathfrak{A}^T$ is the category of algebras over the monad) *such that*

1. $LF = F'\Phi$ (F is the free functor $F: \mathfrak{A} \to \mathfrak{A}^T$)
2. $U'L = \Phi U$ (U is the underlying functor $U: \mathscr{A}^T \to \mathscr{A}$)
3. *L preserves $\mathfrak{V}$-coequalizers of U split pairs* (see [7])
4. *If $M: \mathfrak{A}^T \to \mathfrak{B}$ is a functor which preserves coequalizers of U split pairs, where B has $\mathfrak{V}$-coequalizers of reflexive pairs, and which inverts the morphisms inverted by L, then there exists a unique $\mathfrak{V}$-functor of $\tilde{M}: \mathfrak{A}[\Sigma^{-1}]^{T'} \to \mathfrak{B}$ with $\tilde{M}L \approx M$* (unique up to equivalence).

Proof. The existence of L follows from the well-known properties about lifting of functors to the algebra category (see [3]). In particular we define the following action σ of T' on $\Phi U^T\sigma$: $T'\Phi U = \Phi T U = \Phi UFU \xrightarrow{\Phi U\varepsilon} \Phi U$. One checks that this is an action. Hence $L(A, \sigma) = (\Phi A, \Phi U\varepsilon)$. Note $LF(A) = L(TA, \mu_A) = (\Phi TA, \Phi U\varepsilon) = (T'\Phi A, \mu') = F'\Phi(A)$.

If $A \rightrightarrows B \to C$ is a coequalizer diagram which is U-split then since $U'L = \Phi U$ applying $U'L$ to the diagram gives a U' split coequalizer. Since U' reflects coequalizers of U' split pairs, L applied to the diagram gives a coequalizer.

To finish the proof of Proposition 1.1 we need the following general lemma. The proof is an easy consequence of some work of Street [11] so we omit it (see also [9]).

LEMMA. *Let $\mathfrak{S}_1 = (S_1, \eta_1, \mu_1)$ and $\mathfrak{S}_2 = (S_2, \eta_2, \mu_2)$ be $\mathfrak{V}$-monads on $\mathfrak{V}$-categories $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and let $H: \mathfrak{A} \to \mathfrak{B}$ be a $\mathfrak{V}$-functor. If $\mathfrak{B}^{S_2}$ has $\mathfrak{V}$-coequalizers of reflexive pairs* (relative to U_2) *then there exists a one-one correspondence* (up to natural equivalence of functors) *between funcsors $H': \mathfrak{A}^{S_1} \to \mathfrak{B}^{S_2}$ such that $H'F_1 = F_2H$ and which preserve coequalizers of U_1-split pairs and natural transformations $\lambda: HS_1 \to S_2H$ satisfying* (i) $\mu_2H \cdot S_2\lambda \cdot \lambda S_1 = \lambda \cdot H\mu_1$ *and* (ii) $\lambda \cdot H\eta_1 = \eta_2 H$.

The above lemma is utilized in the following way. First we note that since $LF = F'\Phi$, if $s \in \Sigma$ then $F(s)$ is inverted by L. Consequently, $MF(s)$ is an isomorphism for all $s \in \Sigma$. So there exists a unique $\mathfrak{V}$-functor $N: \mathfrak{A}[\Sigma^{-1}] \to \mathfrak{B}$ such that $N\Phi = MF$. Now define $\lambda: NT' \to N$ to be the natural transformation corresponding to $M \in F$: $NT'\Phi = N\Phi T = MFT \to MF = N\Phi$. It is easily checked that λ

satisfies conditions (i) and (ii) of the lemma relative to the monads $\mathfrak{T}'$ on $\mathfrak{A}[\Sigma^{-1}]$ and the identity on $\mathfrak{B}$. So there exists a unique $\tilde{M}: \mathfrak{A}[\Sigma^{-1}]^{T'} \to \mathfrak{B}$ with $\tilde{M}F' = N$. Now $\tilde{M}LF' = \tilde{M}F'\Phi = N\Phi = MF$. Since both $\tilde{M}L$ and M preserve coequalizers of U split pairs and $\tilde{M}LF = MF$, by the lemma we have $\tilde{M}L \approx M$.

Before we prove our main result we recall a result from [14]. A subcategory Σ of $\mathfrak{C}_0$ containing the identities of $\mathfrak{C}_0$ is conice (Almkvist [1, p. 451]) if, for every object $A \in \mathfrak{C}$, there is a set of objects $\mathscr{F}_A$ such that, for every $s: A \to C$ in Σ, there exists $u: C \to D$ with $D \in \mathscr{F}_A$ and $u \cdot s \in \Sigma$.

LEMMA 1.2. *Let $\mathfrak{V}$ be cocomplete with finite limits such that $V: \mathfrak{V} \to$ Sets commutes with filtered colimits and filtered colimits commute with equalizers in $\mathfrak{V}$. Let $\mathfrak{C}$ be a $\mathfrak{V}$-category and $\Sigma \subseteq \mathfrak{C}_0$ be conice such that C/Σ is filtered for all C. Then Σ admits a $\mathfrak{V}$-calculus of left fractions iff there exists a $\mathfrak{V}$-category $\mathfrak{B}$ and a $\mathfrak{V}$-functor $L: C \to B$ such that:*

(1) *$L(s)$ is an isomorphism for each $s \in \Sigma$;*

(2) *L is the identity on objects;*

(3) *For every $A, B \in \mathfrak{B}$, $\mathfrak{B}(A, B) = \varinjlim_{B/\Sigma} \mathfrak{C}(A, Q^B)$ with universal natural transformation given by $\psi_s^{AB} = \mathfrak{B}(A, L(s)^{-1}) \cdot L_{AE}$ where $B \xrightarrow{s} E \in B/\Sigma$.*

Furthermore in this case $\mathfrak{C}[\Sigma^{-1}]_0 = \mathfrak{C}_0[\Sigma^{-1}]$.

Proof. See [14].

The following theorem improves 3.11 in [12].

THEOREM 1.3. *Let $\mathfrak{V}$ be cocomplete with finite limits such that $V: V \to$ Sets commutes with filtered colimits and filtered colimits commute with equalizers in $\mathfrak{V}$. Let $\Sigma \subseteq \mathfrak{A}_0$ be conice such that Σ admits a $\mathfrak{V}$-calculus of left fractions. Let $\mathfrak{T} = (T, \eta, \mu)$ be compatible with Σ. Let $L: \mathfrak{A}^T \to \mathfrak{A}[\Sigma^{-1}]^{T'}$ be the functor of* 1.1. *If $\mathfrak{A}^T$ has coequalizers which are preserved by L, then $\bar{\Sigma} = \{\alpha \in \mathfrak{A}^T \mid \Phi U(\alpha)$ is an isomorphism$\}$ admits a $\mathfrak{V}$-calculus of left fractions and $\mathfrak{A}^T[\bar{\Sigma}^{-1}] \approx \mathfrak{A}[\Sigma^{-1}]^{T'}$.*

Proof. We verify the conditions of Lemma 1.2. First we note that we can assume that Σ is saturated, i.e., if $\Phi(s)$ is an isomorphism then $s \in \Sigma$. For if we take the saturation Σ_Φ of Σ, then A/Σ is a final subcategory of A/Σ_Φ (see [10]), and thus the same conditions hold.

Now we show that $\bar{\Sigma}$ is conice. Let $(A, \sigma) \in \mathfrak{A}^T$ let $D \in \mathfrak{C}_A$ and let $s: A \to D$ be in Σ. Let $\mathfrak{S}_s^D = \{(f, g) \mid f \cdot T(s) = g \cdot s \cdot \sigma,\ g \in \Sigma$ and codomain D_g of g in $\mathfrak{F}_A\}$. The nonemptiness $\mathfrak{S}_s^D$ is assured since Σ admits a Sets-calculus of left fractions and $T(s) \in \Sigma$. Furthermore $\mathfrak{S}_s^D$ is a set since $\mathfrak{F}_A$ forms a set. For $(f, g) \in \mathfrak{S}_s$ form the following "commutative" diagram in A^T

$$\begin{array}{ccc} (T^2A, \mu TA) & \xrightarrow{T^2(s)} & (T^2D, \mu_D) \\ T(\sigma)\Big\downarrow\Big\downarrow \mu_A & T(f) & \Big\downarrow\Big\downarrow \mu_{D_g}T^2(g) \\ (TA, \mu_A) & \xrightarrow{T(gs)} & (TD_g^s, \mu_{D_g}) \end{array}$$

Taking coequalizers yields an object $\psi^s_{(f,g)} = (D_\xi, \xi)$ and a morphism $l_g: (A, \sigma) \to \psi^s_{(f,g)}$. Note that $l_g \in \bar{\Sigma}$ since L preserves coequalizers and $LT^2(s)$, $LT(gs)$ are isomorphisms. Let $\bar{\mathscr{F}}_D$ be the union over all $s: A \to D$ in Σ of the sets $\{\psi^s_{(f,g)}(f, g) \in \mathfrak{S}_s^D\}$. Finally set $\bar{\bar{\mathfrak{F}}} = \bigcup_{D \in \mathfrak{F}_A} \bar{\mathfrak{F}}_D$. This is clearly a set.

For any $t: (A, \sigma) \to (B, \tau)$ in $\bar{\Sigma}$, there exists $D \in \mathfrak{F}_A$ and a morphism $g: B \to D$ with $gt \in \Sigma$. Then there exists $s \in \Sigma$ with codomain $D_s \in \mathfrak{F}_A$, and u with $uT(g) = sgt$. Then $uT(gt) = sgt \cdot \sigma$. So $(u, s) \in \mathfrak{S}_{gf}^D$ and by the construction above there exists a map $(B, \tau) \xrightarrow{\omega} \psi^{gf}_{(u,s)}$ such that $\omega \cdot t \in \bar{\Sigma}$. So $\bar{\Sigma}$ is conice.

Now we show that $(A, \sigma)/\Sigma$ is filtered for each (A, σ). It suffices to show that the usual conditions of a calculus of left fractions hold in Σ (see [1] or [6]). The coequalizers condition is clear since $\mathfrak{A}^T$ has coequalizers which are preserved by L. So we need to show that if

$$\begin{array}{ccc} (A, \sigma) & \xrightarrow{t} & (B, \tau) \\ f\Big\downarrow & & \\ (C, \delta) & & \end{array}$$

is a diagram in $\mathfrak{A}^T$ with $t \in \bar{\bar{\mathfrak{F}}}$ then there exists s, m with $s \in \bar{\Sigma}$ such that $sf = mt$.

Since Σ admits a calculus of left fractions there is u, l with $u \in \Sigma$ $u: C \to D$ such that $uf = lt$. Then there exists $u': D \to D'$ and $g: TD \to D'$ such that $u'u\delta = g \cdot T(u)$. Now $T(t) \in \Sigma$ and $g \cdot Tl \cdot T(t) = g \cdot T(lt) = g \cdot T(u) \cdot T(f) = u'u\delta \cdot T(f) = u'u \cdot f\sigma = u'lt\sigma = u'l\tau T(t)$. So there is $\omega: D' \to E$ with $\omega \in \Sigma$ and $\omega \cdot gTl = \omega u'l\tau$. Note that if we apply Φ, we get a commutative diagram $\Phi(u)\Phi(f) = \Phi(l)\Phi(t)$. Since ΦA, ΦB and ΦC all have $\mathfrak{T}'$-algebra structures given by $\Phi(\sigma)\Phi(\tau)$ and $\Phi(\delta)$ respectively, and, $\Phi(u)$ is an isomorphism ΦD has a $\mathfrak{T}'$-algebra structure given by $\Phi(u)\Phi(\delta)T(u)^{-1}$ such that $\Phi(u)$ and $\Phi(l)$ are algebra maps. The following equation holds in $\mathfrak{A}[\Sigma^{-1}]$: $\Phi(u)\Phi(\delta)Tu^{-1} = \Phi(u')^{-1}\Phi(g)$.

Consider the following diagram in $\mathfrak{A}^T$

$$\begin{array}{ccccc}
(T^2C,\ \mu_{TC}) & \xrightarrow{T^2(u)} & (T^2D,\ \mu_D) & \xleftarrow{T^2(l)} & (T^2B,\ \mu_B) \\
T(\delta)\downdownarrows\mu_C & & \mu_E T^2(\omega u')\downdownarrows T(\omega g) & & T(\tau)\downdownarrows\mu_B \\
(TC,\ \mu_C) & \xrightarrow[T(\omega u' u)]{} & (TE,\ \mu_E) & \xleftarrow[T(\omega u' l)]{} & (TB,\ \mu_B)
\end{array}$$

Taking coequalizers yields a $\mathfrak{T}$-algebra (E', γ) and morphisms $\alpha: (C, \delta) \to (E', \gamma)$ and $\beta: (B, \tau) \to (E', \gamma)$ with α in $\bar{\Sigma}$. Then there exists an isomorphism $h: (\Phi D, \Phi(u)\Phi(\gamma)\Phi T(u)^{-1} = \varepsilon) \to L(E', \gamma)$ such that $h \cdot \Phi(u) = L(\alpha)$ and $h \cdot \Phi(l) = L(\beta)$. For consider the following diagram in $\mathfrak{A}[\Sigma^{-1}]^{T'}$

$$\begin{array}{ccccc}
L(T^2D,\ \mu_D) & \overset{LT(\omega g)}{\underset{L\mu_E T^2(\omega u')}{\rightrightarrows}} & L(TE,\ \mu_E) & \xrightarrow{Ld} & L(E',\ \gamma) \\
\Big\| & & \Big\downarrow T'\Phi(\omega u') & & \\
(T'^2D,\ \mu'_{T'D}) & \overset{T'\varepsilon}{\underset{\mu'_D}{\rightrightarrows}} & (T'D,\ \mu_D) & \xrightarrow{\varepsilon} & (D,\ \varepsilon)
\end{array}$$

Since L preserves coequalizers and $T'\Phi(\omega u')$ is an isomorphism we get an isomorphism $h: (D, \varepsilon) \to L(E', \gamma)$ such that $h\varepsilon = Ld \cdot T'\Phi(\omega u')$. The equations $h\Phi(u) = L(\alpha)$ and $h \cdot \Phi(l) = L(\beta t)$ are easily checked.

Consequently we have $L(\alpha f) = L(\beta t)$. Hence the coequalizer m of αf and βt lies in $\bar{\Sigma}$. So $m\alpha f = m\beta t$ and $m\alpha \in \bar{\Sigma}$.

Now let $(A, \sigma) \in \mathfrak{A}^T$; then $\bar{U}$ induces a functor $U_A: (A, \sigma)/\bar{\Sigma} \to A/\Sigma$. This functor has a left adjoint $F_A: A/\Sigma \to (A, \sigma)/\bar{\Sigma}$ defined as follows. Let $A \xrightarrow{s} B \in A/\Sigma$. Consider the diagram

$$\begin{array}{ccccc}
(T^2A,\ \mu_A) & \overset{\mu}{\underset{T\sigma}{\rightrightarrows}} & (TA,\ \mu_A) & \xrightarrow{\sigma} & (A,\ \sigma) \\
 & & T\sigma T(s)\Big\downarrow & & \Big\downarrow F_A(s) \\
 & & (TB,\ \mu_B) & \xrightarrow{G_B} & (\hat{B},\ b)
\end{array}$$

where G_B is the coequalizer of $T(s) \cdot \mu$ and $T(s) \cdot T(\sigma)$. Then there exists a unique $F_A(s): (A, \sigma) \to (\hat{B}, b)$ such that $F_A(s) \cdot \sigma = G_B \cdot T(s)$. Apply L to diagram. Since L preserves coequalizers we get that $L\sigma$ is the coequalizer of L_μ and $LT\sigma$, and LG_B is the coequalizer of $LT(s) \cdot L\mu$ and $LT(s) \cdot LT\sigma$. But $LT(s)$ is an isomorphism since $U'LT(s) = \Phi\, UT(s)$ and $LT(s)$ is an isomorphism. Hence $LF_A(s)$ is an isomorphism and $F_A(s) \in \bar{\Sigma}$. It is clear how F_A is made into a functor.

The unit of the adjunction is defined as follows: Let $A \xrightarrow{s} B$ be in Σ/A. Then we set ${}_A\eta(B) = B \xrightarrow{\eta_B} TB \xrightarrow{G_B} \hat{B}$. Now $G_B \cdot \eta B \cdot s = G_B \cdot T(s) \cdot \eta_A = F_A(s) \cdot \sigma \cdot \eta_A = F_A(s)$. So ${}_A\eta^{(B)}$ is a map in A/Σ.

The counit ${}_A\varepsilon$ is defined as follows: Let $t: (A, \sigma) \to (B, \tau)$ be in $(A, \sigma)/\bar{\Sigma}$. Then the following commutes

$$\begin{array}{ccc} (TA, \mu_A) & \xrightarrow{\sigma} & (A, \sigma_A) \\ \downarrow{\scriptstyle T(t)} & & \downarrow{\scriptstyle t} \\ (TB, \mu_B) & \xrightarrow{\tau} & (B, \tau) \end{array}$$

Hence there exists a unique ${}_A\varepsilon(B, \tau)$: $F_A U_A(B, \tau) = (\hat{B}, b) \to (B, \tau)$ such that ${}_A\varepsilon \cdot G_B = \tau$ and ${}_A\varepsilon \cdot F_A(\tau) = t$.

It is easily checked that the adjunction equations hold. And so $U_A \dashv F_A$. As a consequence we get that U_A is a final functor.

Now since $(A, \sigma)/\bar{\Sigma} \xrightarrow{U_A} A/\Sigma$ is a final functor we have

$$\mathfrak{A}[\Sigma^{-1}](C, A) = \varinjlim_{A/\Sigma} \mathfrak{A}(C, Q_A) = \varinjlim_{(A,\sigma)/\Sigma} \mathfrak{A}(C, Q_A U_A -) .$$

Let $(A, \sigma) \xrightarrow{\alpha} (B, \tau)$ be in $\bar{\Sigma}$; then the following diagram commutes

$$\begin{array}{ccccc} \mathfrak{A}^T((C, S), (B, \tau)) & \xrightarrow{U} & \mathfrak{A}(C, B) & \rightrightarrows & \mathfrak{A}(TC, B) \\ \downarrow{\scriptstyle \bar{\psi}_\alpha^{CB}} & & \downarrow{\scriptstyle \psi_\alpha^{CB}} & & \downarrow{\scriptstyle \psi_\alpha^{TCB}} \\ \mathfrak{A}[\Sigma^{-1}]^{T'}(L(C, \delta), L(A, \sigma)) & \longrightarrow & \mathfrak{A}[\Sigma^{-1}](C, A) & \rightrightarrows & \mathfrak{A}[\Sigma^{-1}](T'C, A) \end{array}$$

where the ψ's are defined in Lemma 1.2. Hence taking colimits over $(A, \sigma)/\bar{\Sigma}$ yields $\varinjlim_{(A,\sigma)/\Sigma^-} \mathfrak{A}^T((C, \delta), Q_{(A,\sigma)}) - = \mathfrak{A}[\Sigma^{-1}]^{T'}(L(C, \alpha), L(A, \sigma))$ since filtered limits commute with equalizers.

Hence if we let $\mathscr{B}$ be the category whose objects are those of $\mathfrak{A}^T$ and such that $\mathscr{B}((A, \sigma), (B, \tau)) = \mathfrak{A}[\Sigma^{-1}]^{T'}(L(A, \sigma), L(B, \tau))$ we get by the lemma that $\mathscr{B} = \mathfrak{A}^T[\bar{\Sigma}^{-1}]$ and that $\bar{\Sigma}$ admits a $\mathfrak{B}$-calculus of left fractions.

To show that $\mathfrak{A}^T[\bar{\Sigma}^{-1}] \approx \mathfrak{A}[\Sigma^{-1}]^{T'}$ we need only show that each $\mathfrak{T}'$ algebra (A, σ) is isomorphic to $L(\hat{A}, \hat{a})$ for some $\mathfrak{T}$-algebra $(\hat{A}, \hat{\sigma})$. Now $\sigma = s^{-1}a$: $T'A \to A$ where $TA \xrightarrow{a} C \xleftarrow{s} A$. Let the following diagram in $\mathfrak{A}^T$

$$(T^2A, \mu_A) \underset{Ts \cdot \mu_A}{\overset{Ta}{\rightrightarrows}} (TC, \mu_C) \xrightarrow{d} (\hat{A}, \hat{a})$$

be a coequalizer. Applying L gives the following coequalizer diagram in $\mathfrak{A}[\Sigma^{-1}]^{T'}$

$$(T'^2A, \mu'_A) \underset{T'\Phi s \cdot \mu'_A}{\overset{T'\Phi a}{\rightrightarrows}} (T'\Phi c, \mu'_c) \xrightarrow{Ld} L(\hat{A}, \hat{\alpha}) .$$

It is clear then that $(A, \sigma) \approx (\hat{A}, \hat{\alpha})$.

COROLLARY 1.4. *Let $\mathfrak{B}$ be as in 1.3. If $\mathfrak{A}$ has $\mathfrak{B}$-coequalizers* *which are preserved by T and Σ admits a $\mathfrak{B}$-calculus of left fractions then $\bar{\Sigma}$ admits a $\mathfrak{B}$-calculus of left fractions and $\mathfrak{A}^T[\bar{\Sigma}^{-1}] \approx \mathfrak{A}[\Sigma^{-1}]^{T'}$.*

Proof. In this case L is easily seen to preserve coequalizers since U' reflects coequalizers.

REMARK. In certain situations it is possible to eliminate the hypothesis that Σ be conice in Theorem 1.3. The place where this is used is in verifying that $\mathfrak{A}[\Sigma^{-1}]^{T'}(L(C, \delta), L(A, \sigma))$ is the proper direct limit. In the case $\mathfrak{B} =$ Sets or $\mathfrak{B} =$ Abelian groups, for example, a functor $\Phi\colon \mathfrak{C} \to \mathfrak{B}$ is a left fractional category of $\mathfrak{C}$ with respect to Σ if: (i) $\Phi(s)$ is an isomorphism for all s in Σ; (ii) every morphism f in $\mathfrak{B}$ can be written as $f = \Phi(s)^{-1}\Phi(g)$ for $s \in \Sigma$; and (iii) $\Phi(f) = \Phi(g)$ if and only if there is an $s \in \Sigma$ such that $sf = sg$ ([1], [6]). So in the situation of 1.3 with $\mathfrak{B} =$ Sets or Abelian groups we need to verify condition (ii) for the functor $L\colon \mathfrak{A}^T \to \mathfrak{B}$. Let $f\colon L(A, \sigma) \to L(B, \tau)$ be a morphism. Then $U'(f) = \Phi(s)^{-1}\phi(a)$ where $A \xrightarrow{a} C \xleftarrow{s} B$. Since s is an isomorphism ΦC has a $\mathfrak{T}'$-algebra structure given by $\alpha = \Phi(s)\cdot\Phi(\tau)\Phi(Ts)^{-1}$ for which both $\Phi(a)$ and $\Phi(s)$ are $\mathfrak{T}'$-algebra maps. First we note that since Σ admits a calculus of left fractions there exists $s'\colon C \to C'$ and $e\colon TC \to C'$ with $s' \in \Sigma$ and $s's\tau = eTs$. Consequently $\Phi(s)\Phi(\tau)\Phi(Ts)^{-1} = \Phi(s')^{-1}\Phi(e)$. Now since $\Phi(a)$ is an algebra map $\Phi(a)\cdot\Phi(\sigma) = \Phi(s')^{-1}\Phi(e)\Phi(T(a))$. Now consider the following diagram in $\mathfrak{A}^T$

$$\begin{array}{ccccc}
(T^2A, \mu TA) & \xrightarrow{T^2(a)} & (T^2C, \mu C) & \xleftarrow{T^2 s} & (T^2B, \mu TB) \\
{\scriptstyle \mu A}\downarrow\downarrow{\scriptstyle T\sigma} & & {\scriptstyle \mu_D T^2(ts')}\downarrow\downarrow{\scriptstyle T(te)} & & {\scriptstyle \mu B}\downarrow\downarrow{\scriptstyle T(\tau)} \\
(TA, \mu A) & \xrightarrow{T(ts'a)} & (TD, \mu D) & \xrightarrow{T(ts's)} & (TB, \mu B)
\end{array}$$

This diagram "commutes." Taking coequalizers in $\mathfrak{A}^T$ gives maps $(A, \sigma) \xrightarrow{\hat{a}} (\hat{C}, \hat{\alpha}) \xleftarrow{s} (B, \tau)$. Applying L gives

$$L(A, \sigma) \xrightarrow{L\hat{a}} L(\hat{C}, \hat{\alpha}) \xleftarrow{Ls} L(B, \tau)$$

in $\mathfrak{A}[\Sigma^{-1}]^{T'}$. Now $L\hat{s}$ is an isomorphism since L preserves coequalizers, and LT^2s and $LT(ts's)$ are isomorphisms. Now one verifies that $f = L\hat{s}^{-1}\cdot L\hat{a}$.

2. Examples. In this section we present some examples of the results in §1. In the following, when we refer to the dual of a result we mean to dualize categories, functors, etc. over $\mathfrak{B}$ and not $\mathfrak{B}$ itself, just as in ordinary set-based categories when one dualizes a statement one does not also dualize the base category sets.

2.1. Let $\mathfrak{V}$ be any symmetric monoidal closed category which satisfies the conditions of 1.3 and such that 1 is terminal (e.g., Cat or Sets). Let A be any $\mathfrak{V}$-category with finite $\mathfrak{V}$-colimits. Let D be a fixed object of $\mathfrak{A}$. Then we can form the $\mathfrak{V}$-monad $\mathfrak{T} = (- \amalg D, \eta, \mu)$ where $(- \amalg D)A = A \amalg D$; $\eta(A)$ is the canonical map into the coproduct and $\mu(A)$: $(A \amalg D) \amalg D \to A \amalg D$ is $A \amalg \nabla$ where ∇ is the codiagonal. The category of algebras for T is the $\mathfrak{V}$-category of objects under D. For any $\Sigma \subseteq \mathfrak{A}$ which admits a $\mathfrak{V}$-calculus of left fractions we have that T is compatible with Σ since if Φ: $\mathfrak{A} \to \mathfrak{A}|\Sigma^{-1}|$ is the canonical functor, Φ will preserve finite coproducts. So if $f \in \Sigma$, $\Phi(f \amalg D) = \Phi f \amalg \Phi D$ is an isomorphism. Since $- \amalg D$ preserves $\mathfrak{V}$-coequalizers we have that 1.4 applies to any conice Σ admitting a $\mathfrak{V}$-calculus of left fractions. In this case the induced monad $\mathfrak{T}'$ on $\mathfrak{A}[\Sigma^{-1}]$ is $- \amalg \Phi D$ and $\mathfrak{A}[\Sigma^{-1}]^{T'}$ is the $\mathfrak{V}$-category of objects under D.

Dually, one could consider a $\mathfrak{V}$ category $\mathfrak{A}$ with finite limits and for a fixed object $D \in \mathfrak{A}$ form the $\mathfrak{V}$-comonad $\mathfrak{G} = (-xD, \varepsilon, \delta)$ where $(-xD)A = A \times D$; ε is the projection onto the first factor; and δA: $A \times D \to (A \times D) \times D$ is $A \times \Delta$. The category of coalgebras is the $\mathfrak{V}$-category of objects over D. Then, for any $\Sigma \subseteq \mathfrak{A}$ which admits a $\mathfrak{V}$-calculus of right fractions and is nice, G will be compatible with Σ and the dual of 1.4 applies.

2.2 Let $\mathfrak{V}$ be the category of abelian groups. Then $\mathfrak{V}$ satisfies the conditions of 1.3. Let $\mathfrak{A}$ be any abelian category. A nonempty class $\mathfrak{C}$ of objects of $\mathfrak{A}$ is called a Serre class if for every exact sequence $0 \to A \to B \to C \to 0$, $A, C \in \mathfrak{C}$ if and only if $B \in \mathfrak{C}$. If Σ_C is the class of all morphisms with kernel and cokernel in $\mathfrak{C}$, then it is well-known that Σ_c admits a $\mathfrak{V}$-calculus of left and right fractions. For any $\mathfrak{V}$-monad $\mathfrak{T} = (T, \eta, \mu)$ on $\mathfrak{A}$ such that T is exact and $T(C) \in \mathfrak{C}$ for all $C \in \mathfrak{C}$ we have that T is compatible with Σ. Consequently 1.4 applies. So $\bar{\Sigma}_c$ admits a calculus of left fractions and $\mathfrak{A}^T[\bar{\Sigma}^{-1}] \approx \mathfrak{A}[\Sigma^{-1}]^{T'}$.

For some explicit examples, let $\mathfrak{A}$ be the category of abelian groups. Let R be a ring with 1 and let $\mathfrak{T} = (- \otimes R, \eta, \mu)$ where ηA: $A \to A \otimes R$ is defined by $\eta A(a) = a \otimes 1$ and μA: $(A \otimes R) \otimes R \to A \otimes R$ is defined by $\mu A((a \otimes r_i) \otimes r_2) = a \otimes (r_1 \cdot r_2)$. Then $\mathfrak{T}$ is a $\mathfrak{V}$-monad whose algebras are the R-modules. Suppose that R is torsion-free as an abelian group. Then $T = - \otimes R$ is exact. So for any class $\mathfrak{C}$ such that $C \otimes R \in \mathfrak{C}$ for all $C \in \mathfrak{C}$ will satisfy the property that T is compatible with Σ_C. This the the case, for example, if $\mathfrak{C}$ is the class of finitely generated groups and R is a ring whose underlying group is finitely generated and torsion-free; or if $\mathfrak{C}$ is the class of groups C such that Card $(C) \leqq \aleph_\alpha$ for some fixed infinite cardinal $\aleph_\alpha$ and R a ring whose underlying group is torsion-free and such that $\operatorname{Card} R \leqq \aleph_\alpha$; or, if $\mathscr{C}$ is any complete class (i.e., $C \in \mathfrak{C}$ and $A \in \mathfrak{A}$ imply $C \otimes A \in \mathfrak{C}$) and R any ring (see [12] for proof that $- \otimes R$ is compatible with Σ).

2.3. Let $\mathfrak{V}$ be a symmetric monidal category which satisfies the conditions of Theorem 1.3. Let $\mathfrak{A}$ be a V-category which has V-coequalizesr. Let $\mathfrak{T} = (T, \eta, \mu)$ be an idempotent $\mathfrak{V}$-monad on $\mathfrak{A}$. Now since the algebras for $\mathfrak{T}$ are a $\mathfrak{V}$-full reflective subcategory, we have that $\mathfrak{A}^T$ has $\mathfrak{V}$-coequalizers. Now if Σ admits a $\mathscr{V}$-calculus of left fractions (and is conice) and T is compatible with Σ then the conditions of Theorem 1.3 will be satisfied.

For further remarks about this example see [13].

2.4. Let $\mathfrak{V}$ be the category of abelian groups. Let R be a commutative integral domain with unit and let $\mathfrak{A}$ be the category of R-modules. Let $\alpha: R \to S$ be a ring homomorphism where S is a ring with unit which is finitely generated as an R-module and such that the image of α lies in the center of S. Let $\mathfrak{G} = (\operatorname{Hom}_R(S, -), \varepsilon, \delta)$ be the comonad defined by $\operatorname{Hom}_R(S, -)(A) = \operatorname{Hom}_R(S, A)$; $\varepsilon A: \operatorname{Hom}_R(S, A) \to A$ is defined by $\varepsilon A(f) = f(1)$; and $\delta A: \operatorname{Hom}_R(S, A) \to \operatorname{Hom}_R(S, \operatorname{Hom}_R(S, A))$ is defined by $\delta A(f)(s)(s_1) = f(ss_1)$. The category of coalgebras is the category of S-modules.

Now let Σ be the essential monomorphisms in R-mod. Then R-mod$[\Sigma^{-1}]$ is the spectral category of R-mod denoted by Spec R ([5]). Furthermore G is compatible with Σ. For if $i: N \to M$ is an essential monomorphism we must show that if $h: S \to M$ is an R-homomorphism then there is an $r \in R$, $r \neq 0$ such that rh factors through i. But this is clear since i is essential and S is finitely generated as an R-module. Consequently the dual of 1.4 applies. Hence $\bar{\Sigma}$ admits a calculus of right fractions and S-mod$[\bar{\Sigma}^{-1}] \approx (\operatorname{Spec} R)_{G'}$ where $(\operatorname{Spec} R)_{G'}$ is the category of coalgebras over G'. Since $\bar{\Sigma}$ consists of all those S-morphisms which when considered as R-homomorphisms are essential monomorphisms we have that $\bar{\Sigma}$ is contained in the collection of essential monomorphisms of S-mod.

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Received August 4, 1975 and in revised form December 31, 1975.

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