

## THE DUAL OF A SPACE WITH THE RADON-NIKODYM PROPERTY

JAMES B. COLLIER

**Two characterizations of a Banach space with the Radon-Nikodym property are proved here. The first shows its equivalence with a condition on the dual space which is somewhat weaker than that of being an Asplund space. This leads to a second characterization by a renorming property.**

A convex function  $f$  on a Banach space  $X$  will be assumed to take its values in  $(-\infty, +\infty]$  and to be finite at some point. The *domain of continuity* of  $f$  is the convex open set of all points at which  $f$  is finite and continuous. The space  $X$  is called an *Asplund space* if each convex function on  $X$  is Fréchet differentiable on a dense  $G_\delta$  subset of its domain of continuity. If  $X$  is the dual of a Banach space  $Y$ , then it will be called a *weak\*-Asplund space* if each weak\* lower semi-continuous ( $w^*$ -lsc) convex function on  $X$  is Fréchet differentiable on a dense  $G_\delta$  subset of its domain of continuity. The terms " $G_\delta$ " and "domain of continuity" here still refer to the norm topology on  $X$ . Thus a dual space which is an Asplund space is also a weak\*-Asplund space. A Banach space may be said to have the *Radon-Nikodym property* (RNP) if each closed bounded convex subset is the closed convex hull of its strongly exposed points [6]. A point  $x$  in a set  $C$  is said to be *strongly exposed* by a linear functional  $y$  if the supremum of  $y$  over  $C$  is finite and attained at  $x$  and  $\|x_i - x\| \rightarrow 0$  whenever  $\{x_i\}$  is a sequence in  $C$  for which  $y(x_i) \rightarrow y(x)$ .

Using the same method as in [3], we characterize the dual of a space with the RNP by its differentiability properties. This allows us to give an alternate proof of a result of Huff and Morris [4] concerning the density of strongly exposing functionals and to observe that weak\*-Asplund spaces enjoy some of the permanence properties that Asplund spaces do.

**THEOREM 1.** *A Banach space  $X$  has the RNP if and only if  $X^*$  is a weak\*-Asplund space.*

*Proof.* Assume  $X$  has the RNP and let  $f$  be a  $w^*$ -lsc convex function on  $X^*$  with nonempty domain of continuity  $D$ . Choose any point  $w \in D$  and an  $\varepsilon > 0$  so that  $f$  is bounded on  $N = \{y: \|y - w\| \leq \varepsilon\}$  and  $N \subseteq D$ . We use the dual norm so that  $N$  is weak\* closed. Define  $g$

on  $X^*$  by  $g(y) = f(y)$  if  $y \in N$  and  $g(y) = +\infty$  otherwise. Then  $g$  is a  $w^*$ -lsc convex function on  $X^*$ , bounded on  $N$ , and whose domain of continuity is the interior of  $N$ . We may assume without loss of generality that the unit ball  $B$  of  $X^*$  is contained in  $N$  and  $-1 \leq g(y) \leq 0$  for all  $y \in N$ . Choose some  $\lambda > 1$  such that  $N \subseteq \lambda B$ .

Define  $p$  on  $X^*$  by  $p(y) = 0$  if  $y \in B$  and  $p(y) = +\infty$  otherwise. Let  $q(y) = p(y/\lambda) - 1$ . Then  $p$  and  $q$  are  $w^*$ -lsc convex functions on  $X^*$ . For any convex function  $h$  on  $X^*$ , the conjugate of  $h$  on  $X$  is  $h^*(x) = \sup\{y(x) - h(y) : y \in X^*\}$  for each  $x \in X$ . Thus  $p^*(x) = \|x\|$  and  $q^*(x) = \lambda\|x\| + 1$ . Since  $q(y) \leq g(y) \leq p(y)$  for all  $y \in X^*$ ,  $p^*(x) \leq g^*(x) \leq q^*(x)$  for all  $x \in X$ , and hence  $\|x\| \leq g^*(x) \leq \lambda\|x\| + 1$  for all  $x \in X$ . This implies that the closed convex set  $C = \{x \in X : g^*(x) \leq 2\}$  is bounded and has nonempty interior.

Let  $\text{epi } g^* = \{(x, r) : x \in X, g^*(x) \leq r\}$ ,  $H = \{(x, r) : x \in X, r \geq 2\}$  and  $K = \text{epi } g^* \cap H$ . Since  $K \subseteq C \times [0, 2]$ ,  $K$  is a closed bounded convex subset of  $X \times \mathbf{R}$  with nonempty interior. It is well-known that the RNP is preserved under products; hence  $X \times \mathbf{R}$  has the RNP and  $K$  must be the closed convex hull of its strongly exposed points. As a consequence, there must be a point  $a$  in the interior of  $C$  such that  $g^*(a) < 2$  and  $(a, g^*(a))$  is strongly exposed as a point of  $K$  by some functional  $(b, -1) \in X^* \times \mathbf{R}$ . Since  $g^*(a) < 2$  and  $\text{epi } g$  is convex,  $(a, g^*(a))$  is also strongly exposed as a point of  $\text{epi } g$  by  $(b, -1)$ . Because  $g$  is  $w^*$ -lsc, Theorem 1 in [2, p. 450] together with the Lemma in [3] implies that  $g$  is Fréchet differentiable at  $b$  with gradient  $a$ . Since  $b$  lies in the interior of  $N$ ,  $f$  is also Fréchet differentiable at  $b$  and  $\|w - b\| < \epsilon$ . Since the choice of  $w \in D$  and  $\epsilon > 0$  was arbitrary, the set  $G$  of points at which  $f$  is Fréchet differentiable is dense in  $D$ . Lemma 6 in [1, p. 43] implies that  $G$  must in fact be a dense  $G_\delta$  subset of  $D$  and therefore  $X^*$  is a weak\*-Asplund space.

Assume now that  $X^*$  is a weak\*-Asplund space and  $C$  is a closed bounded convex set in  $X$ . Define  $f(x) = 0$  if  $x \in C$  and  $f(x) = +\infty$  otherwise. Then  $f^*(y) = \sup\{y(x) - f(x) : x \in X\}$  is a  $w^*$ -lsc convex function on  $X^*$  whose domain of continuity is  $X^*$ . Since  $X^*$  is a weak\*-Asplund space,  $f^*$  is Fréchet differentiable on a dense  $G_\delta$  subset  $G$  of  $X^*$ . From Theorem 1 in [2, p. 450] it follows that each functional in  $G$  strongly exposes a point of  $C$ . The density of  $G$  implies that  $C$  is the closed convex hull of its strongly exposed points and hence  $X$  has the RNP.

The last part of the proof of Theorem 1 actually proves the following result of Huff and Morris [4]:

**COROLLARY 2.** *If  $X$  has the RNP and  $C$  is a closed bounded convex subset, then the set of linear functionals which strongly expose some point of  $C$  is a dense  $G_\delta$  subset of  $X^*$ .*

The convexity restriction on  $C$  which occurs in this proof is easily dropped by observing that a linear functional strongly exposes a point of a closed bounded set  $A$  whenever it strongly exposes a point of the closed convex hull of  $A$ .

Since the dual norm on a dual space is a  $w^*$ -lsc convex function, the following is also an immediate consequence of Theorem 1:

**COROLLARY 3.** *If  $X$  has the RNP, then the dual norm on  $X^*$  is Fréchet differentiable on a dense  $G_\delta$  subset of  $X^*$ .*

The differentiability of the dual norm can be used to characterize spaces with the RNP. Let  $A$  be a nonempty bounded subset of a Banach space  $X$ . A slice of  $A$  will be any set of the form  $S(A, y, \epsilon) = \{x \in A : y(x) + \epsilon > \sup y[A]\}$  where  $y \in X^*$  and  $\epsilon > 0$ . We can show the following:

**THEOREM 4.** *A Banach space  $X$  fails to have the RNP if and only if there is an equivalent norm on  $X$  for which the dual norm on  $X^*$  is Fréchet differentiable nowhere.*

*Proof.* If such a renorming exists, then Corollary 3 implies that  $X$  cannot have the RNP. In order to prove the other direction assume that for each equivalent norm on  $X$ , the dual norm on  $X^*$  is Fréchet differentiable at some point. Let  $C$  be any closed bounded convex subset of  $X$  and let  $B$  be the unit ball. Let  $D$  be the closure of  $C + B$  and let  $E$  be the closure of  $D + (-D)$ , then  $E$  is the unit ball of an equivalent norm on  $X$ .

Define  $f$  on  $X$  by  $f(x) = 0$  if  $x \in E$  and  $f(x) = +\infty$  otherwise, then the conjugate of  $f$ ,  $f^*(y) = \sup\{y(x) - f(x) : x \in X\}$ , is the corresponding dual norm on  $X^*$ . By hypothesis,  $f^*$  is Fréchet differentiable at some point  $b$  with gradient  $a \in X^{**}$ . Since  $f^*$  is  $w^*$ -lsc, Corollary 5 in [2] implies that  $a$  actually belongs to  $X$ . By Theorem 1 in [2]  $f$  is norm rotund at  $a$  relative to  $b$  and therefore  $E$  is strongly exposed at  $a$  by  $b$ . Thus  $\text{diam } S(E, b, \epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . By the construction of  $E$  it follows that  $\text{diam } S(E, b, \epsilon) \geq \text{diam } S(C, b, \epsilon)$  and hence  $\text{diam } S(C, b, \epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$  also. Since each closed bounded convex set  $C$  is dentable in the sense defined by Rieffel [7],  $X$  must have the RNP and the theorem follows.

A number of permanence properties for weak\*-Asplund spaces may be proved using Theorem 1 in very much the same fashion as the permanence properties for Asplund spaces were proved in [5].

**THEOREM 5.** *If  $X^*$  and  $Y^*$  are weak\*-Asplund spaces, then  $X^* \times Y^*$  is weak\*-Asplund.*

*Proof.* Theorem 1 implies that both  $X$  and  $Y$  have the RNP and hence  $X \times Y$  has the RNP. Therefore  $(X \times Y)^*$ , which is isomorphic to  $X^* \times Y^*$ , is weak\*-Asplund.

**THEOREM 6.** *If  $X^*$  is a weak\*-Asplund space and  $M$  is a weak\* closed subspace of  $X^*$ , then  $X^*/M$  is weak\*-Asplund.*

*Proof.* Let  $M^\perp = \{x \in X: y(x) = 0 \text{ for all } y \in M\}$  be a closed subspace of  $X$ . Theorem 1 implies that  $X$  has the RNP and hence  $M^\perp$  has the RNP also. Therefore  $(M^\perp)^*$ , which is isomorphic to  $X^*/M$ , is weak\*-Asplund.

Namioka and Phelps [5] raised the question of whether a Banach space  $X$  is an Asplund space whenever  $X^*$  has the RNP. This may now be restated in the following way: If  $X^{**}$  is a weak\*-Asplund space, is  $X$  an Asplund space? The converse is known to be true. If we consider  $X$  to be a (norm) closed subspace of  $X^{**}$  by the usual embedding, then each continuous convex function defined on an open convex subset of  $X$  is the restriction of a  $w^*$ -lsc convex function on  $X^{**}$ . We note, however, that since the RNP is not preserved under quotients, we cannot expect even a weak\* closed subspace of a weak\*-Asplund space to have good differentiability properties.

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UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES