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STANDARD REGULAR SEMIGROUPS

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We give a structure theorem for a class of regular semigroups. Let S be a regular semigroup, let T denote the union of the maximal subgroups of S, and let $E(T)$ denote the set of idempotents of T. Assume T is a semigroup (equivalently, T is a semilattice Y of completely simple semigroups $(T_y: y \in Y)$). If Y has a greatest element and $e, f, g \in E(T)$, $e \geqslant f$, and $e \geqslant g$ imply $fg = gf$, we term S a standard regular gemigroup. The structure of S is given modulo right groups and an inverse semigroup V in which every subgroup is a single element by means of an explict multiplication. We specialize the structure theorem to orthodox, $\mathscr{L}$-unipotent, and inverse semigroups, and to a class of semigroups with Y an ωY-semilattice.

Finally, we show that S is a regular extension of T by V in the sense of Yamada [19].

Let us first state the structure theorem. Let Y be a semilattice with greatest element. Let V be an inverse semigroup with semilattice of idempotents Y such that each subgroup of V consists of a single element. Let $(I, \circ)$ be a standard regular semilattice Y of left zero semigroups $(I_y: y \in Y)$. Let $(J, *)$ be a standard regular semilattice Y of right groups $(J_y: y \in Y)$. Suppose $I_y \cap J_y = \{e_y\}$, a single idempotent element, and $e_y^* e_z = e_y \circ e_z = e_{yz}$ for all $y, z \in Y$. Let H_y denote the maximal subgroup of J_y containing e_y. Let $i \rightarrow B_i$ be a homomorphism of $(I, \circ)$ into $P(J)$, the semigroup of right translations of $(J, *)$; let $b \rightarrow \beta_b$ be a mapping of V into $\operatorname{End}(J, *)$, the semigroup of endomorphism of $(J, *)$, and let g be a mapping of $V \times V$ into $H = \bigcup (H_y: y \in Y)$, a semilattice Y of groups $(H_y: y \in Y)$ (with respect to the multiplication $*$ in J) such that 1(a) $jB_i \in H_{yz}$ for $i \in I_y$ and $j \in J_z$, (b) $J_r\beta_b \subseteqq H_{b^{-1}rb}$, (c) $g(c, d) \in H_{(cd)^{-1}cd}$. 2(a) $hB_{e_y} = h\beta_y = h^* e_y$ for $h \in J$ and $y \in Y$. (b) if $j \in H_z$ and $i \in I_z$, $jB_i = j$ (c) $g(y, z) = e_{yz}$ for $y, z \in Y$. 3(a)

$$\beta_c\beta_d = \beta_{cd}C_{g(c,d)}(xC_z = z^{-1*}x*z \text{ for } x, z \in H)$$

(b) $g(a, bc)*g(b, c) = g(ab, c)*(g(a, b)\beta_c)$. Let $(Y, I, J, V, B, \beta, g)$ denote $\{(i, a, j): a \in V, i \in I_{aa^{-1}}, j \in J_{a^{-1}a}\}$ under the multiplication (4)

$$(i, a, j)(w, b, v) = (i \circ e_{(ab)(ab)^{-1}}, ab, g(a, b)*jB_w\beta_b*v)\ .$$

We show (Theorem 3.14) that $(Y, I, J, V, B, \beta, g)$ is a standard regular semigroup, and, conversely, every standard regular semigroup is isomorphic to some $(Y, I, J, V, B, \beta, g)$.

If X is a semigroup, $E(X)$ will denote the set of idempotents of X. Let $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, and $\mathscr{D}$ denote Green's relations (notation of [1]).

Using [1, Theorem 2.3], $\mathscr{H}$ is the identity congruence on V. Hence, using a result of Munn [5, Theorem 2.3; see also 6], V is isomorphic to a subsemigroup U of the semigroup X of isomorphisms between principal ideals of $E(V)$ with $E(U) = E(X)$.

In special cases, explicit multiplications for V have been given (see for example [8] and [4]). Probably, the most familiar example of V is the bicyclic semigroup.

The multiplication for J is described by means of "connecting homomorphisms" between the J_y (i.e. if $a \in J_y$, $b \in J_z$, $a*b = a\zeta_{y,yz}*b\zeta_{z,yz}$ where $\zeta_{y,w}(y \geqslant w)$ is a homomorphism of J_y into J_w). The multiplication for I is similarly characterized (see Remarks 1.7, 1.8, and 3.15).

A regular semigroup X is termed locally inverse if $e, f, g \in E(X)$, $e \geqslant f$, and $e \geqslant g$ imply $fg = gf$. (Let X be a regular semigroup and let $e \in E(X)$. Hence, if $a \in eXe$, there exists $y \in X$ such that $a = aya = (ae)y(ea) = a(eye)a$. Thus, the semigroup eXe is also regular. Hence, using [1, Theorem 1.17], a regular semigroup X is locally inverse if and only if eXe is an inverse semigroup for all $e \in E(X)$). Thus, a standard regular semigroup is a locally inverse semigroup such that T is a semigroup and Y has a greatest element.

Following Hall [3], a regular semigroup X is termed orthodox if $E(X)$ is a semigroup. In general, a standard regular semigroup is not orthodox.

Yamada [18, Theorem 2] described the structure of locally inverse orthodox semigroups in terms of inverse semigroups (locally inverse orthodox = generalized inverse in the sense of Yamada [18, Theorem 1]).

A Cliffordian semigroup is a semigroup which is a union of its subgroups. A semigroup S is Cliffordian if and only if S is a semilattice Y of completely simple semigroups $(S_y : y \in Y)$ (Clifford, [1, Theorem 4.6]).

In §1, we show that the multiplication of a locally inverse Cliffordian semigroup S is described by means of connecting homomorphisms between the S_y (Theorem 1.6) and give some consequences of this theorem. The results of this section are applied repeatedly in the sequel.

In §§2 and 3, we prove the converse and direct parts, respectively, of our structure theorem (Theorem 3.14).

Let N denote the nonnegative integers and let Y be a semilattice with greatest element. If $W = N \times Y$ with $(k, \alpha)\Lambda(s, \lambda) = (k, \alpha)$, (s, λ), or $(k, \alpha\Lambda\lambda)$ according to whether $k > s$, $s > k$, or $s = k$, we term W an ωY-semilattice. A regular semigroup S is termed ωY-$\mathscr{L}$-unipotent if $E(S)$ is an ωY-semilattice of right zero semigroups $(E_{(n,\delta)}: (n, \delta) \in$

$N \times Y$) and $f_{(n,\delta)} \mathscr{D} f_{(m,\lambda)} (f_{(n,\delta)} \in E_{(n,\delta)}; f_{(m,\lambda)} \in E_{(m,\lambda)})$ if and only if $\delta = \lambda$. If $E_{(n,\delta)}$ is a single element for each $(n, \delta) \in N \times Y$, we term S an ωY-inverse semigroup.

Munn [5, Theorem 3.3] described the structure of simple ωY-inverse semigroups. In [10, Theorem 4], Warne described the structure of simple ωY-$\mathscr{L}$-unipotent semigroups such that $e \in E_{(n,\delta)}$, $f \in E_{(m,\lambda)}$, and $(n, \delta) < (m, \lambda)$ implies $e < f$. When specialized to inverse semigroups, this result yields a theorem [10, Corollary 5] equivalent to Munn's theorem (see also [15, Lemma 2.1]). In [11, theorem and corollary], we show "simple" may be omitted. In [15, Theorem 6.1], we give a structure theorem for ωY-$\mathscr{L}$-unipotent semigroups.

Let S be a standard regular semigroup such that T is an ωY-semilattice of completely simple semigroups $(T_{(n,\delta)}: (n, \delta) \in N \times Y)$. If $f_{(n,\delta)} \mathscr{D} f_{(m,\lambda)} (f_{(n,\delta)} \in E(T_{(n,\delta)}); f_{(m,\lambda)} \in E(T_{(m,\lambda)}))$ if and only if $\delta = \lambda$, we term S a standard regular semigroup of type ωY.

In §4, we specialize Theorem 3.14 to obtain the structure of standard regular semigroups of type ωY (Theorem 4.2). In Theorem 4.2, the factor terms "$g(c, d)$" are omitted and V is an ωY-inverse semigroup with each subgroup a single element. Hence, an explicit multiplication for V is given by [15, Theorem 2.3]. Further specialization yields the structure of simple and bisimple standard regular semigroups of type ω (T is an ω-chain of completely simple semigroups—no condition of the $\mathscr{D}$-classes).

In §5, we describe the structure of standard orthodox, standard $\mathscr{L}$-unipotent, and standard inverse semigroups (Theorems 5.1, 5.3, and 5.5 respectively). A standard regular semigroup is termed standard orthodox ($\mathscr{L}$-unipotent)(inverse) if T is a semilattice of rectangular groups (right groups)(groups). The structure theorems are obtained by specializing Theorem 3.14. In each of the theorems the term "B", is omitted. In Theorem 5.3 and 5.5, $I_y = \{e_y\}$ for each $y \in Y$. In Theorem 5.5, $J_y = H_y$ for each $y \in Y$.

Warne [9, page 206, paragraph 3] and Munn [5, page 66, paragraph 3] have exhibited inverse semigroups with identity on which $\mathscr{H}$, Green's relation, is not a congruence. Using Lemma 2.13, these semigroups are not standard.

Let S be a standard regular semigroup. If $a \in S$, let $\mathscr{I}(a)$ denote the collection of inverses of a. Let $t = \{(a, b) \in S^2: aa', bb' \in E(T_y)$ and $a'a, b'b \in E(T_z)$ for some $a' \in \mathscr{I}(a)$, $b' \in \mathscr{I}(b)$, and $y, z \in Y\}$.

In §5, we show t is a congruence on S, $S/_t \cong V$, $\ker t = T$, and S is a regular extension of T by $S/_t$ in the sense of Yamada [19].

We use the definitions of Clifford and Preston [1] unless otherwise specified. In particular, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, and $\mathscr{D}$ will denote Green's relations on a semigroup S, i.e., $(a, b) \in \mathscr{R}$ if $a \cup aS = b \cup bS$; $(a, b) \in \mathscr{L}$ if $a \cup Sa = b \cup Sb$; $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$; $\mathscr{D} = \mathscr{R} \circ \mathscr{L}((a, b) \in \mathscr{D}$ if there exists $x \in S$ such that $(a, x) \in \mathscr{R}$ and $(x, b) \in \mathscr{L}$)). R_a will denote the $\mathscr{R}$-class containing $a \in S$. A semigroup consisting of a single $\mathscr{D}$-class is termed a bisimple semigroup. A semigroup S which is a union of a collection of pairwise disjoint subsemigroups $(S_y: y \in Y)$ where Y is a semilattice and $S_yS_z \subseteq S_{y \Lambda z}$ for all $y, z \in Y$ is termed a semilattice Y of the semigroups $(S_y: y \in Y)$. If $Y = N$ with $n \Lambda m = \max(n, m)$, S is termed an ω-chain of the semigroups $(S_n: n \in N)$. A semigroup S is termed regular if $a \in aSa$ for all $a \in S$. If S is a regular semigroup, for each $a \in S$, there exists $y \in S$ such that $aya = a$ and $yay = y$ (for example, if $a = axa$, let $y = xax$ [1, Lemma 1.14]). The element y is termed an inverse of a. A regular semigroup S is termed an inverse semigroup if each $a \in S$ has precisely one inverse. A rectangular band is the algebraic direct product of a left zero semigroup $U(x, y \in U$ implies $xy = x)$ and a right zero semigroup. A rectangular group is the algebraic direct product of a group and a rectangular band. A right group is a semigroup X such that $a, b \in X$ implies there exists a unique $x \in X$ such that $ax = b$. If S is a semigroup we may define a partial order "$\leqslant$" on $E(S)$ by the rule: $e \leqq f$ means $ef = fe = e$. A band is a semigroup S such that $x^2 = x$ for each $x \in S$. If S is a commutative band, $(S, \leqslant)$ is a semilattice with $a \Lambda b = ab$ and, conversely, every semilattice is a commutative band with $ab = a \Lambda b$ [1, Theorem 1.12]. A semigroup S is termed simple if S is its only ideal. If, furthermore, $e, f \in E(S)$ and $e \leqslant f$ imply $e = f$, S is termed completely simple. The structure of such S is known modulo groups by theorem of Rees [1, Theorem 3.5].

1. Locally inverse Cliffordian semigroups.

In this section, we give a characterization of locally inverse Cliffordian semigroups (Theorem 1.6) and related results to be used in the sequel.

In the remainder of this section, S will denote a locally inverse Cliffordian semigroup, i.e. S is a locally inverse semilattice Y of completely simple semigroups $(S_y: y \in Y)$.

LEMMA 1.1. *If $E \in E(S_y)$ and $y \geqslant z$, there exists precisely one $e \in E(S_z)$ such that $E \geqslant e$. Furthermore, $S_zL_E \subseteqq L_e$ and $R_ES_z \subseteqq R_e$.*

Proof. If $y = z$, take $e = E$. Suppose $y > z$. Using the proof of [7, Theorem], there exists $e \in E(S_z)$ such that $E > e$. Let $g, h \in E(S_z)$ with $g \leqq E$ and $h \leqq E$. Hence, since S is locally inverse, $gh = hg$. Thus, $(hg)(hg) = hhgg = hg$, and, hence, $hg \in E(S_z)$. Furthermore, $g(hg) = hg = (hg)g$. Thus, $hg \leqslant g$. Thus, since S_z is completely simple, $hg = g$. Similarly, $hg = h$ and, thus, $g = h$. The proof of the second sentence of the lemma is contained in the proof of [7, Theorem] for $y > z$. If $y = z$, apply the Rees theorem [1, Theorem 3.5].

Let $A \in S_y$ and suppose that $A \in R_E \cap L_F$ where $E, F \in E(S_y)$. Let e and f denote the unique idempotents of $S_z(y \geqslant z)$ under E and F respectively. We define $A\zeta_{y,z} = eAf$. It is shown in the proof of [7, Theorem] that $\zeta_{y,z}$ is well defined (i.e. $\zeta_{y,z}$ does not depend on the selection of E and F).

LEMMA 1.2. *For $y>z$, $\zeta_{y,z}$ is a homomorphism of S_y into S_z. Let $A \in S_y$ and $B \in S_z$. If $y > z$, $AB = A\zeta_{y,z}B$. If $z > y$, $AB = A(B\zeta_{z,y})$.*

Proof. The proof of Lemma 1.2 is contained in the proof of [7, Theorem].

LEMMA 1.3. *If $y \in Y$, $\zeta_{y,y}$ is the identity mapping of S_y.*

Proof. Let $A \in S_y$. Hence, $A \in R_E \cap L_F$ for some $E, F \in E(S_y)$. Let e and f denote the unique idempotents of S_y under E and F respectively. Hence, since S_y is completely simple, $e = E$ and $f = F$, Thus, $A\zeta_{y,y} = eAf = EAF = A$.

Let $yz = y \wedge z$ in the semilattice Y.

LEMMA 1.4. *If $A \in S_y$ and $B \in S_z$, $AB = A\zeta_{y,yz}B\zeta_{z,yz}$.*

Proof. Let $A \in L_F$ and $F \in E(S_y)$. Thus, utilizing Lemma 1.2 or 1.3, $AB = A(FB) = A\zeta_{y,yz}(FB) = A\zeta_{y,yz}F\zeta_{y,yz}B = (AF)\zeta_{y,yz}B = A\zeta_{y,yz}B = A\zeta_{y,yz}B\zeta_{z,yz}$.

LEMMA 1.5. *For $x \geqslant y \geqslant z$, $\zeta_{x,y}\zeta_{y,z} = \zeta_{x,z}$.*

Proof. Let $A \in S_x$ and suppose that $A \in R_E \in L_F$ for some $E, F \in E(S_x)$. Let e and f denote the idempotents of S_y under E and F respectively. Hence, $A\zeta_{x,y} = eAf$. By Lemma 1.1, $eA \in S_yL_F \subseteqq L_f$ and $Af \in R_ES_y \subseteqq R_e$. Let e' and f' denote the unique idempotents of S_z under e and f respectively. Hence, $A\zeta_{x,y}\zeta_{y,z} = e'eAff' = e'Af'$. However, $E \geqslant e \geqslant e'$ and $F \geqslant f \geqslant f'$. Hence, by Lemma 1.1, $A\zeta_{x,z} = e'Af' = A\zeta_{x,y}\zeta_{y,z}$.

THEOREM 1.6. *Let $\{S_y : y \in Y\}$ be a collection of pairwise disjoint completely simple semigroups indexed by the semilattice Y. For each $y, z \in Y$ with $y \geqslant z$, let $\zeta_{y,z}$ be a homomorphism of S_y into S_y such that*

(1) *$\zeta_{y,y}$ is the identity automorphism of S_y.*

(2) $\zeta_{x,y}\zeta_{y,z} = \zeta_{x,z}$ *for* $x \geqslant y \geqslant z$ *in* Y.

Let $S = \bigcup (S_y\colon y \in Y)$ *and define a product on* S *by the rule*

(3) $A \circ B = A\zeta_{y,yz}B\zeta_{z,yz}$

where the right hand product is taken in S_{yz}. *Then,* $(S, \circ)$ *is a locally inverse Cliffordian semigroup.*

Conversely, let $(S, \circ)$ *be a locally inverse Cliffordian semigroup. Then,* S *is the union of a collection of pairwise disjoint completely simple semigroups* $(S_y\colon y \in Y)$ *indexed by a semilattice* Y. *For each* $y, z \in Y$ *with* $y \geqslant z$, *there exists a homomorphism* $\zeta_{y,z}$ *of* S_y *into* S_z *such that* (1) *and* (2) *are valid and the multiplication is given by* (3).

Proof. The converse is a consequence of [1, Theorem 4.6] and Lemmas 1.2-1.5. Let us now establish the direct part. Let $xy = x\Lambda y$. Let $A \in S_x$, $B \in S_y$, and $C \in S_z$. Hence, uzing (3), the fact $\zeta_{yz,xyz}$ is a homomorphism, and (2).

$$A \circ (B \circ C) = A \circ (B\zeta_{y,yz}C\zeta_{z,yz}) = A\zeta_{x,xyz}((B\zeta_{y,yz}C\zeta_{z,yz})\zeta_{yz,xyz})$$
$$= A\zeta_{x,xyz}(B\zeta_{y,xyz}C\zeta_{z,xyz}) .$$

Similarly, $(A \circ B) \circ C = (A\zeta_{x,xyz}B\zeta_{y,xyz})C\zeta_{z,xyz}$. Hence $(A \circ B) \circ C = A \circ (B \circ C)$ by associativity in S_{xyz}. By (1) and (3), S_y is a completely simple subsemigroup of S for all $y \in Y$. Thus, S_y is a Cliffordian semigroup for each $y \in Y$ by [1, Theorem 2.52]. Hence, S is Cliffordian. Clearly, S is a regular semigroup. Finally, let $E \in E(S_x)$, $f \in E(S_y)$, and $g \in E(S_z)$ and suppose that $E \geqslant f$ and $E \geqslant g$. Hence, $x \geqslant y$ and $x \geqslant z$. Thus, using (3) and (1), $f = E \circ f = E\zeta_{x,y}f = f \circ E = f(E\zeta_{x,y})$. Hence, $E\zeta_{x,y} \geqslant f\,(E\zeta_{x,y},\ f \in E(S_y))$. Thus, $f = E\zeta_{x,y}$ since S_y is a completely simple semigroup. Similarly, $g = E\zeta_{x,z}$. Hence, $f \circ g = f\zeta_{y,yz}g\zeta_{z,yz} = E\zeta_{x,y}\zeta_{y,yz}E\zeta_{x,z}\zeta_{z,yz} = E\zeta_{x,yz}E\zeta_{x,yz} = E\zeta_{x,yz}$. Similarly, $g \circ f = E\zeta_{x,yz}$. Thus, $g \circ f = f \circ g$, and hence, S is locally inverse.

REMARK. In Theorem 1.6, we term $\{\zeta_{y,z}\colon y, z \in Y\}$ the collection of structure homomorphisms of S.

REMARK 1.7. In the statement of Theorem 1.6, we may replace "completely simple semigroup" by "left zero semigroup" and "Cliffordian semigroup" by "semilattice of left zero semigroups". Using Theorem 1.6, a band E is left normal [17] if and only if E is a locally inverse semilattice of left zero semigroups. Hence, we have obtained the Yamada-Kimura characterization of left normal bands [17, Theorem 1].

REMARK 1.8. In the statement of Theorem 1.6, we may replace "completely simple semigroup" by "right group" and "Cliffordian semigroup" by "semilattice of right groups".

The following result will be used in the sequel.

PROPOSITION 1.9. *Let S be a locally inverse Cliffordian semigroup. Then, $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$ are congruence relations on S.*

Proof. Let S be a locally inverse Cliffordian semigroup. We first show that $\mathscr{L}$ is a congruence relation on S. We will apply Theorem 1.6 and its notation. Let $(x, y) \in \mathscr{L}$ in S. Hence, $(x, y) \in \mathscr{L}$ in S_u for some $u \in Y$. Let $z \in S_v$. Since $\zeta_{u,uv}$ is a homomorphism of S_u into S_{uv}, $x\zeta_{u,uv} \mathscr{L} y\zeta_{u,uv}$ in S_{uv}. Using the Rees theorem [1, Theorem 3.5], $\mathscr{L}$ is a congruence relation on S_{uv}. Hence,

$$z \circ x = z\zeta_{v,uv}x\zeta_{u,uv}\mathscr{L}z\zeta_{v,uv}y\zeta_{u,uv} = z \circ y \ .$$

Thus, since $\mathscr{L}$ is a right congruence on any semigroup, $\mathscr{L}$ is a congruence on S. Similarly, $\mathscr{R}$ is a congruence relation on S.

2. Structure theorem for standard regular semigroups (proof of converse). In this section, we will use a sequence of twenty-one lemmas to establish the converse of our structure theorem for standard regular semigroups (Theorem 2.22).

Let S be a standard regular semigroup and let T denote the union of the maximal subgroups of S. Hence, T is a semilattice Y of completely simple semigroups $(T_y: y \in Y)$ where Y has a greatest element y_0. Let $\{\zeta_{y,z}: y, z \in Y\}$ denote the set of structure homomorphisms of T. Let $E_y = E(T_y)$. Select and fix $e_{y_0} \in E_{y_0}$. For each $y \in Y$, define $e_y = e_{y_0}\zeta_{y_0,y}$. Let $S_0 = e_{y_0}Se_{y_0}$.

LEMMA 2.1. $E(S_0) = \{e_y: y \in Y\}$.

Proof. Since $e_{y_0}e_y = e_{y_0}\zeta_{y_0,y}e_y = e_y$, and, similarly, $e_ye_{y_0} = e_y$, $e_y \in E(S_0)$ for all $y \in Y$. Suppose $f \in E(S_0)$ and $f \in E_y$, say. Hence, $f \leqq e_{y_0}$ and $e_y \leqslant e_{y_0}$ implies $fe_y = e_yf$. Thus, since T_y is completely simple, $f = e_y$.

LEMMA 2.2. *$y \rightarrow e_y$ defines an isomorphism of Y onto $E(S_0)$.*

Proof. Let $y, z \in Y$. Hence,

$$e_ye_z = e_y\zeta_{y,yz}e_z\zeta_{z,yz} = (e_{y_0}\zeta_{y_0,y}\zeta_{y,yz})(e_{y_0}\zeta_{y_0,z}\zeta_{z,yz}) = e_{yz} \ .$$

LEMMA 2.3. $S_0 = \bigcup (R_{e_y} \cap L_{e_z}: y, z \in Y)$.

Proof. Let $x \in R_{e_y} \cap L_{e_z}$ where $y, z \in Y$. Using Lemma 2.2, $e_{y_0}x =$

$e_{y_0}e_yx = e_yx = x$ and, similarly, $xe_{y_0} = x$. Conversely, if $x \in S_0$, let x^{-1} denote the unique inverse of x in S_0. Thus, using Lemma 2.1, $xx^{-1} = e_y$ and $x^{-1}x = e_z$ for some $y, z \in Y$. Hence, $x \in R_{e_y} \cap L_{e_z}$.

By the Rees theorem [1, Theorem 3.5], for $y \in Y$, $T_y = G_y \times M_y \times N_y$ where G_y is a group and M_y and N_y are sets under the multiplication $(g, i, j)(h, p, q) = (gf_y(j, p)h, i, q)$ where $(j, p) \to f_y(j, p)$ is a mapping of $N_y \times M_y$ into G_y. We note $e_y = (f_y(j_y, i_y)^{-1}, i_y, j_y)$, say, where $f_y(j_y, i_y)^{-1}$ is the inverse of $f_y(j_y, i_y)$ in the group G_y. Let I_y denote the set of idempotents of the $\mathscr{L}$-class of T_y containing e_y and let J_y denote the $\mathscr{R}$-class of T_y containing e_y. Hence, $I_y = \{(f_y(j_y, i)^{-1}, i, j_y); i \in M_y\}$ and $J_y = \{(g, i_y, j): g \in G$ and $j \in N_y\}$. Let $I = \bigcup (I_y: y \in Y)$ and $J = \bigcup (J_y: y \in Y)$.

LEMMA 2.4. *I is a standard regular semilattice Y of left zero semigroups $(I_y: y \in Y)$.*

Proof. Let $a \in I_y$ and $b \in I_z$. Thus, $a \mathscr{L} e_y$ and $b \mathscr{L} e_z$. Using Proposition 1.9 and Lemma 2.2, $ab \mathscr{L} e_{yz}$. Furthermore, $abab = abe_{yz}ab = abe_ze_yab = abe_ze_yb = abe_{yz}b = abb = ab$. Thus, $ab \in I_{yz}$.

LEMMA 2.5. *J is a standard regular semilattice Y of right groups $(J_y: y \in Y)$.*

Proof. Apply Proposition 1.9 and Lemma 2.2.

The next two lemmas are special cases of left-right duals of [14, Lemmas 1.3 and 1.4]. Note our arbitrary representations of the "e_y" requires a slight modification in the proof of [14, Lemma 1.3].

LEMMA 2.6. *Every element of T may be uniquely expressed in the form $x = ij$ where $i \in I_y$ and $j \in J_y$ for some $y \in Y$.*

If X is a set, T_X will denote the semigroup (iteration) of mappings of X into X.

LEMMA 2.7. *There exists a mapping $j \to A_j$ of J into T_I and a mapping $p \to B_p$ of I into T_J such that $I_yA_j \subseteqq I_{yz}$ for $j \in J_z$ and $J_yB_p \subseteqq J_{yz}$ for $p \in I_z$. If $j \in J$ and $p \in I$, $jp = pA_jjB_p$. Furthermore, $jp \mathscr{R} pA_j (\varepsilon T)$ and $jp \mathscr{L} jB_p (\varepsilon T)$.*

LEMMA 2.8. *$iA_j = e_{rs}$ for $i \in I_r$ and $j \in J_s$.*

Proof. First, we show that $A_j = A_{e_s}$ for $j \in J_s$. Since $\mathscr{R}$ is a congruence relation on T, $(j, e_s) \in \mathscr{R}$ implies $(ji, e_si) \in \mathscr{R}$ for all

$i \in I$. Hence, using Lemma 2.7, $(iA_j, iA_{e_s}) \in \mathscr{R}$, for all $i \in I$. Thus, $iA_j = iA_{e_s}$ for $i \in I$. Let $i \in I_r$, say. Thus, since $e_s i \in I_{rs}$, we utilize Lemma 2.7 to obtain $(e_s i)e_{sr} = e_s i = iA_{e_s} e_s B_i$. Thus, by Lemmas 2.7 and 2.6, $iA_{e_s} = e_s i$. Since $i \mathscr{L} e_r$, $i\zeta_{r,rs} \mathscr{L} e_r \zeta_{r,rs} = e_{rs}$. Hence, $i\zeta_{r,rs} \in I_{rs}$. Hence, using Theorem 1.6, $e_s i = e_{sr}(i\zeta_{r,sr}) = e_{sr}$. Thus, $iA_j = iA_{e_s} = e_{sr}$.

DEFINITION [1, p. 10]. A transformation ρ of a semigroup S is a right translation of S if $(ab)\rho = a(b\rho)$ for all $a, b \in S$.

LEMMA 2.9. *For each $i \in I$, B_i is a right translation of J.*

Proof. Let $r \in J_u$, $s \in J_v$ and $x \in I_p$, say. Hence, utilizing Lemmas 2.5 and 2.7, $(rs)x = xA_{rs}(rs)B_x$ while $r(sx) = r(xA_s sB_x) = xA_s A_r(rB_{xA_s} sB_x)$. However, using Lemma 2.8, $rB_{xA_s} = rB_{e_{vp}}$. Using Lemma 2.5, $re_{vp} \in J_{uvp}$. Hence, $e_{uvp}(re_{vp}) = re_{vp} = e_{vp}A_r rB_{e_{vp}}$. Hence, using Lemmas 2.7 and 2.6, $rB_{e_{vp}} = re_{vp}$. Thus, using Lemmas 2.7 and 2.6, $(rs)B_x = re_{vp}sB_x = r(sB_x)$.

LEMMA 2.10. *$i \rightarrow B_i$ is a homomorphism of I into $P(J)$, the semigroup of right translations of J.*

Proof. Let $r, s \in I$ and $x \in J$. Thus, proceeding as in the proof of Lemma 2.9, $x(rs) = (rs)A_x xB_{rs}$ and

$$(xr)s = rA_x(xB_r s) = (rA_x sA_{xB_r})(xB_r B_s) .$$

Thus $xB_{rs} = xB_r B_s$.

For each $y \in Y$, let H_y denote the maximal subgroup of S containing e_y.

LEMMA 2.11. *If $i \in I_y$ and $j \in J_z$, $jB_i \in H_{yz}$. If $j \in H_y$, $jB_i = j$.*

Proof. Let $i \in I_y$ and $j \in J_z$. Since $j \mathscr{R} e_z$, $j\zeta_{z,yz} \mathscr{R} e_z \zeta_{z,yz} = e_{yz}$ and $j\zeta_{z,yz} \in J_{yz}$. Thus, $e_{yz} j i e_{yz} = e_{yz} j\zeta_{z,yz} i\zeta_{y,yz} e_{yz} = j\zeta_{z,yz} i\zeta_{y,yz} = ji$. Hence, since $ji \in T_{yz}$, $ji \in H_{yz}$. However, using Lemmas 2.7 and 2.8, $ji = iA_j jB_i = e_{yz} jB_i = jB_i$. Thus, $jB_i \in H_{yz}$. If $j \in H_y$, $jB_i = ji = je_y i = je_y = j$.

LEMMA 2.12. *If $j \in J$, $jB_{e_n} = je_n$.*

Proof. Utilize the proof of Lemma 2.11.

LEMMA 2.13. *Let X be an inverse semigroup such that the union* *of the maximal subgroups of X is a subsemigroup. Then, $\mathscr{H}$ is a congruence relation on X.*

Proof. Let $H = \bigcup (H_e: e \in E(X)))$. Using [1, Theorem 1.7], H is an inverse semigroup which is a union of groups. Hence, $E(H)$ is contained in the center of H by [1, Lemma 4.8]. Let $(a, b) \in \mathscr{H}$. Hence, $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$. Let $c \in S$. Thus,

$$\begin{aligned}(bc)(bc)^{-1} &= bcc^{-1}b^{-1} = ba^{-1}acc^{-1}a^{-1}ab^{-1} = ba^{-1}(acc^{-1}a^{-1})ab^{-1} \\ &= ba^{-1}ab^{-1}acc^{-1}a^{-1} = aa^{-1}acc^{-1}a^{-1} = (ac)(ac)^{-1}\end{aligned}$$

while $(bc)^{-1}bc = c^{-1}b^{-1}bc = c^{-1}a^{-1}ac = (ac)^{-1}ac$. Hence, $(ac, bc) \in \mathscr{H}$. Similarly, $(ca, cb) \in \mathscr{H}$.

LEMMA 2.14. *$\mathscr{H}$ is a congruence relation on S_0.*

Proof. Using Lemma 2.3, $\{H_y: y \in Y\}$ is the collection of maximal subgroups of S_0. Let $a \in H_y$ and $b \in H_z$, say. Since $a\mathscr{H}e_y$, $a\zeta_{y,yz}\mathscr{H}e_{yz}$ and, thus, $a\zeta_{y,yz} \in H_{yz}$. Similarly, $b\zeta_{z,yz} \in H_{yz}$. Thus, $ab = a\zeta_{y,yz}b\zeta_{z,yz} \in H_{yz}$. Hence, using Lemma 2.13, $\mathscr{H}$ is a congruence relation on S_0.

LEMMA 2.15. *There exists a homomorphism ϕ of S_0 onto an inverse semigroup V where $E(V) = Y$ and each $\mathscr{H}$-class of V consists of a single element. Furthermore, $(a, b) \in \mathscr{H}(\varepsilon S_0)$ if and only if $a\phi = b\phi$. Thus, if $h_c = c\phi^{-1}$, $\{h_c: c \in V\}$ is the collection of $\mathscr{H}$-classes of S_0.*

Proof. Using Theorem 2.14 and [1, Theorem 7.36], $S_{0/\mathscr{H}}$ is an inverse semigroup. Let $a \to \bar{a}$ denote the natural homomorphism of S_0 onto $S_{0/\mathscr{H}}$. Suppose $(\bar{a}, \bar{b}) \in \mathscr{H}(\varepsilon S_{0/\mathscr{H}})$. Hence, $\bar{a}\bar{a}^{-1} = \bar{b}\bar{b}^{-1}$ and $\bar{a}^{-1}\bar{a} = \bar{b}^{-1}\bar{b}$. Thus, $\overline{aa^{-1}} = \overline{bb^{-1}}$ and $\overline{a^{-1}a} = \overline{b^{-1}b}$. Hence, $(aa^{-1}, bb^{-1}) \in \mathscr{H}(\varepsilon S_0)$ and $(a^{-1}a, b^{-1}b) \in \mathscr{H}(\varepsilon S_0)$. Thus, $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$. Hence, $(a, b) \in \mathscr{H}$, and, thus, $\bar{a} = \bar{b}$. Thus, each $\mathscr{H}$-class of $S_{0/\mathscr{H}}$ consists of a single element. Using Lemma 2.2 and [1, Lemma 7.34], $\bar{e}_y \to y$ defines an isomorphism of $E(S_{0/H})$ onto the semilattice Y. Hence, we may extend this isomorphism to an isomorphism λ of $S_{0/\mathscr{H}}$ onto a semigroup V with $E(V) = Y$. For $a \in S_0$, define $a\phi = \bar{a}\lambda$.

For each $c \in V$, select a representative element $v_c \in h_c$. For $y \in Y$, let $v_y = e_y$. Hence, using Lemma 2.15 and its proof, $v_c v_c^{-1} = e_{cc^{-1}}$ and $v_c^{-1}v_c = e_{c^{-1}c}$ for $c \in V$.

LEMMA 2.16. *Every element of S may be uniquely expressed in the form iv_cj where $i \in I_{cc^{-1}}$ and $j \in J_{c^{-1}c}$.*

Proof. Let $x \in S$. Hence, $x \in R_e \cap L_f$ where $e \in E_y$ and $f \in E_z$ for some $y, z \in Y$. Thus, $(e, i) \in \mathscr{R}$ for some $i \in I_y$ and hence $(x, i) \in \mathscr{R}$. We note that $e_y \mathscr{D} e \mathscr{D} x \mathscr{D} f \mathscr{D} e_z$. Hence, $R_{e_y} \cap L_{e_z} \neq \Box$. Thus, using Lemmas 2.3 and 2.15, $R_{e_y} \cap L_{e_z} = h_c$ for some $c \in V$. Thus, $v_c v_c^{-1} = e_y$ and $v_c^{-1} v_c = e_z$. Hence, $x = ix = ie_y x = iv_c(v_c^{-1}x)$. We will show that $v_c^{-1}x \in T_{c^{-1}c}$. Using the proof of [1, Theorem 2.18], there exists an inverse x' of x such that $x' \in R_f \cap L_e$, $xx' = e$, and $x'x = f$. Let $xx' = (f_y(s, r)^{-1}, r, s)$ for some $r \in M_y$ and $s \in N_y$ and let $z = (f_y(j_y, r)^{-1}, r, j_y)$. Using the Rees theorem, $e_y xx'z = e_y$. Hence, $e_z = v_c^{-1}v_c = v_c^{-1}e_y v_c = v_c^{-1}e_y xx'zv_c = v_c^{-1}xx'zv_c \in v_c^{-1}xS$. However, $v_c^{-1}x = e_z v_c^{-1}x \in e_z S$. Again, using the Rees theorem, $ze_y xx' = xx'$. Hence, $x = ze_y x = zv_c v_c^{-1}x$. Thus, $f = x'x = x'zv_c v_c^{-1}x \in Sv_c^{-1}x$. However, $v_c^{-1}x \in Sx = Sf$. Hence, $v_c^{-1}x \in R_{e_z} \cap L_f \subseteqq T_z$. Since $e_z v_c^{-1}x = v_c^{-1}x$, $v_c^{-1}x \in J_z$. Since $e_z = v_c^{-1}v_c$, $z = e_z\phi = (v_c\phi)^{-1}v_c\phi = c^{-1}c$. Thus, $v_c^{-1}x \in J_{c^{-1}c}$. Hence, $x = iv_c j$ where $i \in I_{cc^{-1}}$ and $j \in J_{c^{-1}c}$. Suppose $iv_c j = rv_d s$ where $r \in I_{dd^{-1}}$ and $s \in J_{d^{-1}d}$. Since $j \mathscr{R} e_{c^{-1}c}$, $v_c j \mathscr{R} v_c e_{c^{-1}c} = v_c \mathscr{R} e_{cc^{-1}}$. Hence, $iv_c j \in \mathscr{R} ie_{cc^{-1}} = i$. Similarly, $iv_c j \mathscr{L} j$. Hence, $iv_c j \in R_i \cap L_j$. Thus, $i \mathscr{R} r$ and $j \mathscr{L} s$. Hence, $cc^{-1} = dd^{-1}$, $i = r$, and $c^{-1}c = d^{-1}d$. Thus, $c \mathscr{H} d (\varepsilon V)$ and, hence, $c = d$. Thus, $iv_c j = iv_c s$. Therefore, $e_{cc^{-1}}iv_c j = e_{cc^{-1}}iv_c s$. Hence, $v_c j = v_c s$. Thus, $j = e_{c^{-1}c}j = v_c^{-1}v_c j = v_c^{-1}v_c s = e_{c^{-1}c}s = s$.

LEMMA 2.17. *If $c, d \in V$, $v_c v_d = v_{cd}g(c, d)$ where g is a function of $V \times V$ into $H = \bigcup (H_y: y \in Y)$ such that $g(c, d) \in H_{(cd)^{-1}cd}$. If $y, z \in Y$, $g(y, z) = e_{yz}$.*

Proof. Using Lemma 2.16, $v_c v_d = iv_x j$ where $i \in I_{xx^{-1}}$ and $j \in J_{x^{-1}x}$. We first show $x = cd$. By the proof of Lemma 2.16, $iv_x j \in R_i \cap L_j$. However, $(v_c v_d(v_c v_d)^{-1})\phi = (cd)(cd)^{-1} = e_{(cd)(cd)^{-1}}\phi$. Hence, using Lemma 2.15, $(v_c v_d)(v_c v_c)^{-1} = e_{(cd)(cd)^{-1}}$. Similarly, $(v_c v_d)^{-1}(v_c v_d) = e_{(cd)^{-1}(cd)}$. Thus, $xx^{-1} = (cd)(cd)^{-1}$ and $x^{-1}x = (cd)^{-1}cd$. Thus, $(x, cd) \in \mathscr{H}(\varepsilon V)$. Hence, using Lemma 2.15, $x = cd$. Thus, $v_c v_d = iv_{cd}j$. Let $j = g(c, d)$. Hence, using Lemma 2.16, g is a function of $V \times V$ into J and $g(c, d) \in J_{(cd)^{-1}(cd)}$. Furthermore, $v_c v_d = e_{(cd)(cd)^{-1}}v_c v_d = e_{(cd)(cd)^{-1}}iv_{cd}g(c, d) = v_{cd}g(c, d)$. We note that $g(c, d) = e_{(cd)^{-1}(cd)}g(c, d) = v_{cd}^{-1}v_{cd}g(c, d) = v_{cd}^{-1}v_c v_d \in S_0$. Thus, using Lemma 2.3, $g(c, d) \in H_{(cd)^{-1}cd}$. To obtain the last statement of the theorem, utilize Lemma 2.2

If $u \in J$ and $s \in V$, define $u\beta_s = v_s^{-1}uv_s$.

LEMMA 2.18. *For $s \in V$, $\beta_s \in \text{End } J$ and $J_r\beta_s \subseteqq H_{s^{-1}rs}$. Furthermore, if $j \in J$ and $s \in Y$, $j\beta_s = je_s$*

Proof. Let $j_1, j_2 \in J$. Suppose $j_1 \in J_k$ and $j_2 \in J_r$. Hence, using Lemma 2.5, $j_1\beta_s j_2\beta_s = (v_s^{-1}j_1 v_s)(v_s^{-1}j_2 v_s) = v_s^{-1}j_1 e_{ss^{-1}}(j_2 e_{ss^{-1}})v_s = v_s^{-1}j_1 j_2 v_s = (j_1 j_2)\beta_s$. Hence, β_s is a homomorphism of J into S. Since is a J

union of groups, β_s is a homomorphism of J into T. Since $e_r\beta_s = v_s^{-1}e_rv_s = e_{s^{-1}rs}$, $j\mathscr{R}e_r$ implies $j\beta_s\mathscr{R}e_{s^{-1}rs}$. Hence, $J_r\beta_s \subseteqq J_{s^{-1}rs}$. Thus, $\beta_s \in \text{End } J$. Let $j \in J_r$. Thus, $v_s^{-1}jv_s = v_s^{-1}jv_s \cdot e_{s-1s} = v_s^{-1}jv_s \cdot e_{s-1s}\zeta_{s^{-1}s,s^{-1}rs} = v_s^{-1}jv_s \cdot e_{s^{-1}rs}$. Thus $J_r\beta_s \subseteqq H_{s^{-1}rs}$. If $s \in Y$ and $j \in J_r$, $j\beta_s = e_sje_s = je_s$.

If $j \in H$ and $z \in H$, define $jC_z = z^{-1}jz$. (Using Proposition 1.9 and Lemma 2.2, H is a semilattice Y of the groups $(H_y: y \in Y)$.)

LEMMA 2.19. $\beta_c\beta_d = \beta_{cd}C_{g(c,d)}$.

Proof. Utilizing Lemma 2.17, $j\beta_c\beta_d = v_d^{-1}v_c^{-1}jv_cv_d = (v_cv_d)^{-1}jv_cv_d = (v_{cd}g(c, d))^{-1}jv_{cd}g(c, d) = j\beta_{cd}C_{g(c,d)}$.

LEMMA 2.20. $g(a, bc)g(b, c) = g(ab, c)(g(a, b)\beta_c)$.

Proof. Using Lemmas 2.17 and 2.5,

$$\begin{aligned}(v_av_b)v_c &= v_{ab}g(a, b)v_c = v_{ab}g(a, b)e_{cc^{-1}}v_c \\ &= v_{ab}e_{cc^{-1}}(g(a, b)e_{cc^{-1}})v_c = v_{ab}v_c(v_c^{-1}g(a, b)v_c) \\ &= v_{abc}g(ab, c)(g(a, b)\beta_c) = e_{(abc)(abc)^{-1}}v_{abc}g(ab, c)(g(a, b)\beta_c) .\end{aligned}$$

Using Lemmas 2.17 and 2.18, $g(ab, c)(g(a, b)\beta_c) \in J_{(abc)^{-1}abc}$. However,

$$v_a(v_bv_c) = v_a(v_{bc}g(b, c)) = e_{(abc)(abc)^{-1}}v_{abc}g(a, bc)g(b, c) .$$

We note that $g(a, bc)g(b, c) \in J_{(abc)^{-1}abc}$. Hence, using Lemma 2.16, $g(a, bc)g(b, c) = g(ab, c)(g(a, b)\beta_c)$.

If $a, b \in J$, define $a^*b = ab$. If $a, b \in I$, define $a \circ b = ab$.

LEMMA 2.21. *$S \cong \{(i, a, j): a \in V,\ i \in I_{aa^{-1}}$, and $j \in J_{a^{-1}a}\}$ under the multiplication $(i, a, j)(u, b, z) = (i \circ e_{(ab)(ab)^{-1}}, ab, g(a, b)^*jB_u\beta_b^*z)$.*

Proof. Let $i \in I_{aa^{-1}}$, $j \in J_{a^{-1}a}$, $u \in I_{bb^{-1}}$, and $z \in J_{b^{-1}b}$. Using Lemmas 2.7, 2.8, and 2.2,

$$\begin{aligned}(iv_aj)(uv_bz) &= iv_auA_jjB_uv_bz \\ &= iv_ae_{bb^{-1}a^{-1}a}jB_uv_bz \\ &= iv_ae_{a^{-1}a}e_{bb^{-1}}jB_uv_bz \\ &= iv_av_bv_b^{-1}jB_uv_bz \\ &= iv_{ab}g(a, b)jB_u\beta_bz \\ &= (i \circ e_{(ab)(ab)^{-1}})v_{ab}(g(a, b)^*jB_u\beta_b^*z) .\end{aligned}$$

Using Lemmas 2.4, 2.5, 2.7, 2.17 and 2.18, $i \circ e_{(ab)(ab)^{-1}} \in I_{(ab)(ab)^{-1}}$ and $g(a, b)^*jB_u\beta_b^*z \in J_{(ab)^{-1}ab}$. Hence, using Lemma 2.16, $(iv_aj)\delta = (i, a, j)$ defines an isomorphism of S onto the groupoid given in the statement of the lemma.

THEOREM 2.22. *Let S be a standard regular semigroup. Then, S is isomorphic to $(Y, I, J, V, B, \beta, g)$ for some Y, I, J, V, B, β, g.*

Proof. Utilize the remark before the proof of Lemma 2.21 and Lemmas 2.2, 2.4, 2.5, 2.10-2.12, 2.15, and 2.17-2.21.

REMARK 2.23. A semilattice Y is said to be directed from above if $y, z \in Y$ implies there exists $w \in Y$ such that $w \geqslant y$ and $w \geqslant z$. Theorem 2.22 is valid if we replace the condition "Y" has a greatest element by "Y is directed from above" and "for each $y \in Y$, there exists $e_y \in E_y$ such that $e_y e_z = e_{yz}$ for all $y, z \in Y$". Just replace S_0 by the inverse semigroup $\bigcup (e_y S e_y \colon y \in Y) = \bigcup (R_{e_y} \cap L_{e_z} \colon y, z \in Y)$ and note that for $z \leqslant y$, $e_y \zeta_{y,z} = e_z$.

3. Structure theorem for standard regular semigroups (proof of direct half). In this section, we show that $(Y, I, J, V, B, \beta, g)$ is a standard regular semigroup and establish other results to be used later in the sequel.

For brevity, let $S = (Y, I, J, V, B, \beta, g)$.

LEMMA 3.1. *S is a semigroup.*

Proof. Utilizing (4) and (1), closure is easily established. Let $(i, a, j)_1 = i$ and $(i, a, j)_3 = j$. Let $x = (i, a, j)$, $y = (u, b, z)$, and $w = (p, c, q)$. Using the fact $e_r \circ e_s = e_{rs}$ for all $r, s \in Y$ and (4), $((xy)w)_1 = (x(yw))_1$. Utilizing (4), the facts $\beta_c \in \operatorname{End}(J, {}^*)$ and $i \to B_i$ is a homomorphism of $(I, \circ)$ into $P(J)$, 3(b), 3(a), 1(c), the fact $e_r^* e_s = e_{rs}$, 1(b), and 2(a),

$$
\begin{aligned}
((xy)w)_3 &= g(ab, c)^*(g(a, b)^* j B_u \beta_b^* z) B_p \beta_c^* q \\
&= g(ab, c)^*(g(a, b)\beta_c)^*(j B_u \beta_b^* z) B_p \beta_c^* q \\
&= g(a, bc)^* g(b, c)^* j B_u \beta_b \beta_c^* z B_p \beta_c^* q \\
&= g(a, bc)^* g(b, c)^* (g(b, c))^{-1} * (j B_u \beta_{bc})^* g(b, c)^* z B_p \beta_c^* q \\
&= g(a, bc)^* e_{(bc)^{-1} bc}^* j B_u \beta_{bc}^* g(b, c)^* z B_p \beta_c^* q \\
&= g(a, bc)^* j B_u \beta_{bc}^* g(b, c)^* z B_p \beta_c^* q \\
&= g(a, bc)^* j B_u \beta_{bc}^* e_{(bc)(bc)^{-1}} \beta_{bc}^* g(b, c)^* z B_p \beta_c^* q \\
&= g(a, bc)^* (j B_u^* e_{(bc)(bc)^{-1}}) \beta_{bc}^* g(b, c)^* z B_p \beta_c^* q \\
&= g(a, bc)^* j B_u B_{e(bc)(bc)^{-1}} \beta_{bc}^* g(b, c)^* z B_p \beta_c^* q \\
&= g(a, bc)^* j B_{u \circ e_{(bc)(bc)^{-1}}} \beta_{bc}^* g(b, c)^* z B_p \beta_c^* q \\
&= (x(yw))_3 \,.
\end{aligned}
$$

Hence, $(xy)w = x(yw)$.

LEMMA 3.2. $E(S) = \{(i, a, j): a \in Y \text{ and } jB_i = e_a\}$.

Proof. Let $(i, a, j) \in E(S)$. Using 4, 2(a), 1(a), and 2(c), $a \in Y$ and $jB_i^* j = j$. Since J_a is a right group, there exists $y \in J_a$ such that $j^* y = e_a$. Hence, using 1(a), $e_a = j^* y = jB_i^* j^* y = jB_i^* e_a = jB_i$. Conversely, if $a \in Y$ and $jB_i = e_a$, using (4), 2(c), and 2(a), $(i, a, j) \in E(S)$.

LEMMA 3.3. *Let $(i, a, j), (u, b, v) \in S$. Then, $(i, a, j)\mathscr{R}(u, b, v)$ if and only if $i = u$.*

Proof. First suppose that $i = u$. Hence, $aa^{-1} = bb^{-1}$. Let $x \in I_{(a^{-1}b)(a^{-1}b)^{-1}}$. Thus, using (1) and the fact $J_{b^{-1}b}$ is a right group, there exists $y \in J_{b^{-1}b}$ such that $(g(a, a^{-1}b)^* jB_x\beta_{a^{-1}b})^* y = v$. Thus, using (4), $(i, a, j)(x, a^{-1}b, y) = (i, b, v)$. Similarly, there exists $p \in I_{(b^{-1}a)(b^{-1}a)^{-1}}$ and $q \in J_{a^{-1}a}$ such that $(i, b, v)(p, b^{-1}a, q) = (i, a, j)$. Thus, $(i, a, j)\mathscr{R}(i, b, v)$. Conversely, suppose that $(i, a, j)\mathscr{R}(u, b, v)$. Using Thus, (4), $i \circ u = u$ and $u \circ i = i$. Hence, $i = u$.

LEMMA 3.4. *S is a regular semigroup.*

Proof. Let $(i, a, j) \in S$. Using Lemma 3.3, $(i, aa^{-1}, e_{aa^{-1}})\mathscr{R}(i, a, j)$. By 2(b), $e_{aa^{-1}}B_i = e_{aa^{-1}}$. Hence, using Lemma 3.2, $(i, aa^{-1}, e_{aa^{-1}}) \in E(S)$. Thus each $\mathscr{R}$-class of S contains an idempotent.

LEMMA 3.5. *Let $(i, a, j), (w, b, z) \in S$. Then, $(i, a, j)\mathscr{L}(w, b, z)$ if and only if $a^{-1}a = b^{-1}b$ and $(j, z) \in \mathscr{H}(\varepsilon J_{b^{-1}b})$.*

Proof. We first show that $(i, a, j)\mathscr{L}(w, b, z)$ if $a^{-1}a = b^{-1}b$ and $(j, z) \in \mathscr{H}(\varepsilon J_{b^{-1}b})$. Since $(j, z) \in \mathscr{H}(\varepsilon J_{b^{-1}b})$, there exists $y \in H_{a^{-1}a}$ such that $y^* j = z$. Since $g(a^{-1}, a) \in H_{a^{-1}a}$ by 1(c), there exists $x \in H_{a^{-1}a}$ such that $x^* g(a^{-1}, a)^* j = z$. Using 1(b), 2(b), 3(a), 2(a), and 1(c), $g(a^{-1}, a)^* x\beta_{a-1}B_i\beta_a^* j = x^* g(a^{-1}, a)^* j = z$. Thus, $(e_{a^{-1}a}, a^{-1}, x\beta_{a^{-1}})(i, a, j) = (e_{a^{-1}a}, a^{-1}a, z)$. However, $(w, b, (g(b, b^{-1}b))^{-1})(e_{b^{-1}b}, b^{-1}b, z) = (w, b, z)$. Hence, $((w, b, (g(b, b^{-1}b))^{-1})(e_a^{-1}{}_a, a^{-1}, x\beta_{a-1}))(i, a, j) = (w, b, z)$. Similarly, there exists $p \in H_{b^{-1}b}$ such that $((i, a, (g(a, a^{-1}a))^{-1})(e_{b^{-1}b}, b^{-1}, p\beta_{b^{-1}}))(w, b, z) = (i, a, j)$. Hence, $(i, a, j)\mathscr{L}(w, b, z)$. Conversely, suppose that $(i, a, j)\mathscr{L}(w, b, z)$. Using (4) and (1), $a^{-1}a = b^{-1}b$ and $(j, z) \in \mathscr{H}(\varepsilon J_{b^{-1}b})$.

LEMMA 3.6. *Let $(i, a, j), (w, b, z) \in S$. Then, $(i, a, j)\mathscr{H}(w, b, z)$ if and only if $i = w$, $a = b$, and $(j, z) \in \mathscr{H}(\varepsilon J_{b^{-1}b})$.*

Proof. Just note that each $\mathscr{H}$-class of V consists of a single element, and combine Lemmas 3.3 and 3.5.

REMARK. Lemmas 3.3 and 3.5 and their proofs were suggested by [16, Lemma 3.2] and its proof.

LEMMA 3.7. *Let* $(i, a, j), (w, b, z) \in S$. *Then,* $(i, a, j)\mathscr{D}(w, b, z)$ *if and only if* $a\mathscr{D}b(\varepsilon V)$.

Proof. Suppose that $a\mathscr{D}b(\varepsilon V)$. Hence, there exists $x \in V$ such that $a\mathscr{R}x$ and $x\mathscr{L}b$. Hence, $aa^{-1} = xx^{-1}$ and $b^{-1}b = x^{-1}x$. Thus, using Lemmas 3.3 and 3.5, $(i, a, j)\mathscr{R}(i, x, z)\mathscr{L}(w, b, z)$. Conversely, if $(i, a, j)\mathscr{D}(w, b, z)$, using (4), $a\mathscr{D}b(\varepsilon V)$.

LEMMA 3.8. *S is a bisimple semigroup if and only if V is a bisimple semigroup.*

Proof. Apply Lemma 3.7.

LEMMA 3.9 *For each* $y \in Y$, *let* $T_y = \{(i, y, j): i \in I_y$ and $j \in J_y\}$. *Then,* T_y *is a completely simple semigroup.*

Proof. First, we show that T_y is a simple semigroup. Let $(i, y, j), (u, y, v) \in T_y$. Hence, using 1(a), 2(a), and 2(c), $(i, y, j)(u, y, v)$ $(i, y, jB_u^*v) \in T_y$. Since J_y is a right group, there exists $x \in J_y$ such that $j^*x = v$. Hence, using 2(b) and 2(a),

$$(u, y, e_y)(i, y, j)(e_y, y, x) = (u, y, v) .$$

Next, we show T_y is completely simple. Let $e, f \in E(T_y)$ and suppose that $e \leqslant f$. Hence, using Lemma 3.2, $e = (i, y, j)$ and $f = (w, y, z)$, say, where $jB_i = zB_w = e_y$. Thus, $(w, y, z)(i, y, j) = (i, y, j)$ implies $w = i$. Hence, $(i, y, j)(i, y, z) = (i, y, z) = (i, y, j)$. Thus, $z = j$ and, hence, $(w, y, z) = (i, y, j)$.

LEMMA 3.10. *Let* $T = \bigcup (T_y: y \in Y)$. *Then, T is a semilattice Y of the completely simple semigroup* $(T_y: y \in Y)$.

Proof. Apply (4), 2(c), 1(a), and 2(a).

LEMMA 3.11. *T is the union of the maximal subgroups of S.*

Proof. If $x \in T_y$, x is contained in some subgroup of S (each completely simple semigroup is a union of its subgroups by [1, Theorem 2.52]). Thus, $x \in H_e$ for some $e \in E(S)$. Hence, $T \subseteq X$, the union of the maximal subgroups of S. If $c \in X$, $c\mathscr{H}e$ for some $e \in E(S)$. Hence, $e = (i, y, j)$ for some $y \in Y$ by Lemma 3.2. Thus, using Lemma 3.6, $c \in T_y$.

LEMMA 3.12. *T is a locally inverse semigroup.*

Proof. Let (i, a, j), (u, b, v), and $(w, c, z) \in E(S)$ such that $(i, a, j) \geqslant (u, b, v)$ and $(i, a, j) \geqslant (w, c, z)$. Hence, $(i, a, j)(u, b, v) = (u, b, v)(i, a, j) = (u, b, v)$. Using Lemma 3.2, $a, b \in Y$, and, hence, $a \geqslant b$. Since $i \circ e_b = u$, $i \circ u = i \circ (i \circ e_b) = i \circ e_b = u$ while $u \circ i = u \circ (u \circ i) = u$. Hence, $i \geqslant u$. Using (4), 2(c), 2(a) and 1(a), $jB_u^* v = vB_i^* j = v$. Hence, using the fact $jB_u \in H_b$ and $v \in J_b$, a right group, $jB_u = e_b$. Hence, using Lemma 3.2, $e_b = vB_u = (vB_i^* j)B_u = vB_i^* jB_u = vB_i^* e_b = vB_i$. Thus, $e_b^* j = v$. Let $\{\zeta_{p,q}: p, q \in Y\}$ denote the set of structure homomorphisms of $(J, *)$. Thus, $e_b{}^* j\zeta_{a,b} = v$. Hence, $j\zeta_{a,b} = v$. We have shown that $(i, a, j) \geqslant (u, b, v)$ implies $i \geqslant u$, $a \geqslant b$, and $v = j\zeta_{a,b}$. Similarly, $(i, a, j) \geqq (w, c, z)$ implies $i \geqslant w$, $a \geqslant c$, and $z = j\zeta_{a,c}$. Since $(I, \circ)$ is a locally inverse semigroup, $u \circ w = w \circ u$. Using 2(c), 2(a), and 1(a), $(u, b, v)(w, c, z) = (u \circ e_{bc}, bc, vB_w^* z)$ while $(w, c, z)(u, b, v) = (w \circ e_{bc}, bc, zB_u^* v)$. Let $\{\phi_{p,q}: p, q \in Y\}$ denote the set of structure homomorphisms of $(I, \circ)$. Hence, $w \circ u = w\phi_{c,bc} \circ u\phi_{b,bc} = w\phi_{c,bc} \circ e_{bc} \circ u\phi_{b,bc} = w \circ e_{bc}$. Similarly, $u \circ w = u \circ e_{bc}$. Hence, $u \circ e_{bc} = w \circ e_{bc}$. Furthermore,

$$\begin{aligned} vB_w^* z &= vB_w^* e_{bc}^* z = vB_w B_{e_{bc}}^* z \\ &= vB_{w \circ e_{bc}}{}^* z = vB_{u \circ e_{bc}}{}^* z \\ &= vB_u^* e_{bc}^* z = e_b^* e_{bc}^* z \\ &= e_{bc}^* z = z\zeta_{c,bc} \,. \end{aligned}$$

Similarly, $zB_u^* v = v\zeta_{b,bc}$. Hence, $vB_w^* z = j\zeta_{a,c}\zeta_{c,bc} = j\zeta_{a,bc} = j\zeta_{a,b}\zeta_{b,bc} = v\zeta_{b,bc} = zB_u^* v$. Thus, $(u, b, v)(w, c, z) = (w, c, z)(u, b, v)$.

THEOREM 3.13. *$(Y, I, J, V, B, \beta, g)$ is a standard regular semigroup.*

Proof. Utilize Lemmas 3.1, 3.4, and 3.10–3.12.

THEOREM 3.14. *$(Y, I, J, V, B, \beta, g)$ is a standard regular semigroup and, conversely, every standard regular semigroup is isomorphic to some $(Y, I, J, V, B, \beta, g)$.*

Proof. Combine Theorems 2.22 and 3.13.

REMARK 3.15. Let J and H be as in the statement of Theorem 3.14. Using the proof of [1, Theorem 1.27], Theorem 1.6, and Proposition 1.9, $J \cong \bigcup (H_y \times E(J_y): y \in Y)$ where, if $a \in H_y$, $c \in H_z$, $b \in E(J_y)$, and $d \in E(J_z)$, $(a, b)(c, d) = (ac, bd) \in H_{yz} \times E(J_{yz})$. The multiplications in H and $E(J)$ are given by the corresponding specializations of Theorem 1.6. These specializations yeild theorems of Clifford [1, Theorem 4.11] and Yamada and Kimura [17, Theorem 1] respectively.

REMARK 3.16. Let $e_y = (e_y, y, e_y)$. Hence, using (4), 2(c), 2(a), $e_y e_z = e_{yz}$. Thus, if we replace "Y has a greatest element" by "Y is directed from above" in the definition of $(Y, I, J, V, B, \beta, g)$, we obtain the semigroup of Remark 2.23. Hence, Theorem 3.14 with appropriate modifications characterizes these semigroups.

4. **Standar dregular semigroups of type ωY.** Let S be a regular semigroup such that T is a locally inverse ωY-semilattice A of completely simple semigroups $\{T_{(n,\delta)}: (n, \delta) \in A\}$. If $f_{(n,\delta)} \mathscr{D} f_{(m,\lambda)} (f_{(n,\delta)} \in E_{(n,\delta)}: f_{(m,\lambda)} \in E_{(m,\lambda)})$ if and only if $\delta = \lambda$, we term S a standard regular semigroup of type ωY. If δ_0 is the greatest element of Y, $(\circ, \delta_0)$ is the greatest element of A. We give a characterization of standard regular semigroups of type ωY(Theorem 4.2). A regular semigroup S such that T is a locally inverse ω-chain of completely simple semigroups $(T_n: n \in N)$ (no further condition) is termed a standard regular semigroup of type ω. We show that S is a simple (bisimple) standard regular semigroup of type ω if and only if S is a standard regular semigroup of type ωY with Y a finite chain (a single element) (Theorem 4.3) (Theorem 4.4). Hence, the structure of these semigroups is given by specializing Theorem 4.2.

To establish Theorem 4.2, we use a more general result on "split" extensions (Theorem 4.1).

Let S be a standard regular semigroup. In the notation of §2, let $\{h_c: c \in V\}$ denote the collection of $\mathscr{H}$-classes of S_0. For each $c \in V$, select $v_c \in h_c$. If $v_c v_d = v_{cd}$ for all $c, d \in V$, we term S a split extension of T by V.

Let Y, V, I, J, H, and $\{e_k\}$ be as in the definition of (I, J, V, B, β, g). Let $i \to B_i$ be a homomorphism of $(I, \circ)$ into $P(J)$, and let $v \to \beta_v$ be a homomorphism of V into $\mathrm{End}\,(J, {}^*)$ such that (1)(a) $jB_i \in H_{yz}$ for $j \in J_z$ and $i \in I_y(b) J_r \beta_b \subseteq H_{b^{-1}rb}$ (2)(a) $gBe_y = g\beta_y = g^* e_y$ for $g \in J$ (b) $jB_i = j$ for $j \in H_z$ and $i \in I_z$. Let (Y, I, J, V, B, β) denote $\{(i, a, j): a \in V, i \in I_{aa^{-1}}$, and $j \in J_{a^{-1}a}\}$ under the multiplication (3) $(i, a, j)(u, b, z) = (i \circ e_{ab(ab)^{-1}}, ab, jB_u \beta_b^* z)$.

THEOREM 4.1. *(Y, I, J, V, B, β) is a split extension of T by V. Conversely, every such semigroup is isomorphic to some (Y, I, J, V, B, β).*

Proof. Let $S = (Y, I, J, V, B, \beta)$. If we let $g(c, d) = e_{(cd)^{-1}cd}$ for all $c, d \in V$, it is easily verified that (1)-(4) of the definition of $(Y, I, J, V, B, \beta, g)$ are valid. Hence, S is a standard regular semigroup by Theorem 3.14. Let y_0 denote the greatest element of Y. Using 2(a) and Lemmas 3.9 and 3.2, $(e_{y_0}, y_0, e_{y_0}) \in E(T_{y_0})$. Using (4), 2(a) and the fact $(J, {}^*)$ is locally inverse, $(e_{y_0}, y_0, e_{y_0})S(e_{y_0}, y_0, e_{y_0}) = \{(e_{aa^{-1}}, a, j): a \in V, j \in H_{a^{-1}a}\}$. Thus, using Lemmas 3.6,

$$h_a = \{(e_{aa^{-1}}, a, j): j \in H_{a^{-1}a}\} .$$

Let $v_a = (e_{aa^{-1}}, a, e_{a^{-1}a})$. Hence, $v_a v_b = v_{ab}$. Thus, S is a split extension of T by B. Conversely, let S be a standard regular semigroup which is a split extension of T by V. Hence, using Lemma 2.17, $g(c, d) = e_{(cd)^{-1}cd}$ for all $c, d \in V$. Thus, using Theorem 3.14, $S = (Y, I, J, V, B, \beta, g)$ with $g(c, d) = e_{(cd)^{-1}cd}$. For brevity, let $(Y, I, J, V, B, \beta) = U$. Using 3(a), 1(a) and 1(b) of the definition of S, $c \rightarrow \beta_c$ defines a homomorphism of V into End $(J, {}^*)$, and (4) of the definition of S reduces to (3) of the definition of U. Hence, $S \cong U$.

Let A be an ωY-semilattice. Let $V = \{(n, k)_\delta: n, k \in N, \delta \in Y\}$ under the multiplication

$$(n, k)\delta(r, s)_\eta = (n + r - \min(k, r), k + s - \min(k, r))_{f(k,r)}$$

where $f(k, r) = \delta, \eta$, or $\delta\eta$ according to whether $k > r$, $r > k$, or $r = k$. Let $Y(=A)$, I, J, H, and $\{e_{(k,\delta)}\}$ be as in the definition of (Y, I, J, V, B, β). Let $(n, k)_\delta \rightarrow \beta_{(n,k)_\delta}$ be a homomorphism of V into End $(J, {}^*)$ and let $i \rightarrow B_i$ be a homomorphism of $(I, \circ)$ into $P(J)$ such that 1(a) $jB_i \in H_{(n,\delta)(m,\lambda)}$ for $j \in J_{(m,\lambda)}$ and $i \in I_{(n,\delta)}$ 1(b) $J_{(r,\eta)}\beta_{(n,k)_\delta} \subseteqq H_{(r+k-\min(r,n), f(r,n))}$. 2(a) $hB_{e_{(n,\delta)}} = h\beta_{(n,\delta)} = h^* e_{(n,\delta)}$ for $h \in J$ (b) $jB_i = j$ if $j \in H_{(n,\delta)}$ and $i \in I_{(n,\delta)}$. Let $I_\delta = \bigcup (I_{(n,\delta)}: n \in N)$ and let $J_\delta = \bigcup (J_{(n,\delta)}: n \in N)$. Let $(\omega Y, I, J, B, \beta)$ denote $\bigcup (I_\delta \times J_\delta: \delta \in Y)$ under the multiplication (3): if $i \in I_{(n,\delta)}$, $j \in J_{(k,\delta)}$, $u \in I_{(r,\eta)}$, and $z \in J_{(s,\eta)}$, $(i, j)(u, z) = (i \circ e_{(n+r-\min(k,r), f(k,r))}, jB_u\beta^*_{(r,s)_\eta}z)$.

THEOREM 4.2. *$(\omega Y, I, J, B, \beta)$ is a standard regular semigroup of type ωY, and conversely every such semigroup is isomorphic to some $(\omega Y, I, J, B, \beta)$.*

Proof. Let S be a standard regular semigroup of type ωY. Using Lemmas 2.1 and 2.2, $E(S_0) = \{e_{(n,\delta)}: n \in N, \delta \in Y\} \cong A$. Furthermore, $e_{(n,\delta)}\mathscr{D}e_{(m,\lambda)}(\varepsilon S_0)$ if and only if $\delta = \lambda$. Hence, use [15, Lemma 2.1, Theorem 2.3, and Corollaries 2.2 and 2.4] to show S is a split extension of T by V (given in the definition of $(\omega Y, I, J, B, \beta)$). Hence, $S \cong (A, I, J, V, B, \beta)$ by Theorem 4.1. Using [15, Corollary

2.4], $J_{(r,\eta)}\beta_{(n,k)_\delta} \subseteqq H_{(r+k-\min(r,n),f(r,n))}$ and $((n, k)_\delta(r, s)_\eta)((n, k)_\delta(r, s)_\eta)^{-1} = (n + r - \min(k, r), f(k, r))$. Hence, using Theorem 4.1 and [15, Corollary 2.4], $S \cong \{(i, (n, k)_\delta, j) \colon (n, k)_\delta \in V, i \in I_{(n,\delta)}, J \in J_{(k,\delta)}\}$ under the multiplication $(i, (n, k)_\delta, j)(u, (r, s)_\eta, z) = (i \circ e_{(n+r-\min(k,r),f(k,r))}, (n, k)_\delta(r, s)_\eta, jB_u\beta_{(r,s)_\eta}{}^*z)$. Hence, $(i, (n, k)_\delta, j)\zeta = (i, j)$ defines an isomorphism of (A, I, J, V, B, β) onto $(\omega, Y, I, J, B, \beta)$. Conversely, consider $S = (\omega Y, I, J, B, \beta)$. Using [15, Theorem 2.3 and Corollary 2.4], V is an inverse semigroup with semilattice of idempotents A and each $\mathscr{H}$-class of V consists of a single element. Using [15, Corollary 2.4], 1(b) of the definition of (A, I, J, V, B, β) is valid. Hence, ζ^{-1} defines an isomorphism of S onto (A, I, J, V, B, β). By the proof of Theorem 4.1, $(A, I, J, VB, \beta) = (A, I, J, V, B, \beta, g)$ with $g(c, d) = e_{(cd)^{-1}cd}$. Hence, using [15, Corollary 2.4] and Lemmas 3.2, 3.7, and 3.9-3.12, S is a standard regular semigroup of type ωY.

THEOREM 4.3. *S is a simple standard regular semigroup of type ω if and only if S is a standard regular semigroup of type ωY with Y a finite chain $0 > 1 > 2 > \cdots d - 1$ where d is a positive integer.*

Proof. Let S be a simple standard regular semigroup of type ω. Hence, S_0 is a simple semigroup. Thus, using Lemma 2.15 and [15, Lemma 7.5 and Theorem 2.3], V is the semigroup described in the definition of $(\omega Y, I, J, B, \beta)$ with Y the finite chain $0 > 1 > 2 \cdots > d - 1$ where d is a positive integer. Hence, using Theorem 3.14 (and its proof), [15, Corollary 2.4], Lemmas 3.2, 3.7, and 3.9-3.12, S is a standard regular semigroup of type ωY with Y the finite chain $0 > 1 > 2 \cdots > d - 1$ where d is a positive integer. Conversely, let S be a standard regular semigroup of type ωY with Y a finite chain. It is easily seen that S is a standard regular semigroup of type ω. We next show that any standard regular semigroup of type ωY is simple. Let $S = (\omega Y, I, J, B, \beta)$. Let $i \in I_{(n,\delta)}$, $j \in J_{(k,\delta)}$, $u \in I_{(r,\eta)}$, and $v \in J_{(s,\eta)}$. Let $q = e_{(n+1,\eta)}B_i\beta^*_{(n+1,s)_\eta}j\beta_{(k+1,s)_\eta}$. Using (1), $q \in H_{(s,\eta)}$. Hence, since $J_{(s,\eta)}$ is a right group, there exists $b \in J_{(s,\eta)}$ such that $q^*b = v$. Thus, (3), (2(a)), the fact $c \to \beta_c$ is a homomorphism of V into $\mathrm{End}(J, *)$, and (1), $(u, e_{(n+1),\eta})(i, j)(e_{(k+1,\eta)}, b)) = (u, v)$.

If A is a finite set, $|A|$ will denote the number of elements of A.

THEOREM 4.4. *S is a standard regular bisimple semigroup of type ω if and only if S is a standard regular semigroup of type ωY with $|Y| = 1$.*

Proof. Apply Theorem 4.3, Lemma 3.7, and [15, Corollary 2.4].

5. Some other classes of standard regular semigroups. A regular semigroup S is termed standard orthodox ($\mathscr{L}$-unipotent) if T is a locally inverse semilattice Y of rectangular groups (right groups) $(T_y: y \in Y)((J_y: y \in Y))$ where Y has a greatest element. An inverse semigroup S is termed standard inverse if T is a semilattice Y of groups $(H_y: y \in Y)$ where Y has a greatest element. In this section, we give structure theorems for these classes of semigroups.

Let Y, V, I, J, $\{e_k\}$, H, β, and g be as in the definition of $(Y, I, J, V, B, \beta, g)$. Furthermore, assume that 1(b), 1(c), 2(c), 3(a), and 3(b) of that definition are valid and that 2(a) $g\beta_y = g^*e_y$ for $y \in Y$ and $g \in J$. Let (Y, I, J, V, β, g) denote $\{(i, a, j): a \in V, i \in I_{aa^{-1}}, j \in J_{a^{-1}a}\}$ under the multiplication (4)

$$(i, a, j)(u, b, z) = (i \circ e_{(ab)(ab)^{-1}}, ab, g(a, b)^* j\beta_b^* z) .$$

REMARK 5.1. *(Y, I, J, V, β, g) is a standard orthodox semigroup and conversely every standard orthodox semigroup is isomorphic to some (Y, I, J, V, β, g).*

Proof. Let S be a standard orthodox semigroup. It is easily seen that S is standard regular. We apply Theorem 3.14. Let $j \in J$ and $u \in I$. By the proof of Lemma 2.11, $jB_u = ju$. Let $\{\zeta_{y,z}: yz \in Y\}$ denote the set of structure homomorphisms of T. Let $j \in J_y$ and $u \in I_z$. Hence, using the fact that T_{yz} is a rectangular group, $ju = j\zeta_{y,yz}u\zeta_{z,yz} = j\zeta_{y,yz}e_z\zeta_{z,yz} = je_z$. Hence $jB_u = j^*e_z$. Let $j \in J$, $u \in I_{bb^{-1}}$, and $z \in J_{b^{-1}b}$. Hence, using the fact $\beta_b \in \mathrm{End}\,(J, *)$ and 1(b), $jB_u\beta_b^* z = (j^*e_{bb^{-1}})\beta_b^* z = j\beta_b^* e_{b^{-1}b}^* z = j\beta_b^* z$. Thus, (4) of Theorem 3.14 reduces to (4) in the definition of (Y, I, J, V, β, g).

Conversely, we show (Y, I, J, V, β, g) is a standard orthodox semigroup. First, we apply Theorem 3.14 to show that (Y, I, J, V, β, g) is a standard regular semigroup. We define $jB_i = j^*e_y$ for $j \in J$ and $i \in I_y$ Let $u \in I_y$, $v \in I_z$, and $j \in J$. Hence $jB_uB_v = je_y^*e_z = j^*e_{yz} = jB_{u \circ v}$. Let $j, h \in J$ and $i \in I_y$. Hence, $(j^*h)B_i = (j^*h)^*e_y = j^*(h^*e_y) = j^*(hB_i)$. Thus, $i \rightarrow B_i$ is a homomorphism of (I, o) into $P(J, *)$. Let $j \in J_y$ and $i \in I_z$. Let $\{\gamma_{y,z}: y, z \in Y\}$ denote the set of structure homomorphisms of $(J, *)$. Hence, for $i \in I_z$, $jB_i = j^*e_z = j\gamma_{y,yz}^* e_z\gamma_{z,yz}$. However, $e_z\gamma_{z,yz} \leqslant e_z$, and $e_{yz} \leqslant e_z$. Hence, using the fact that $(J, *)$ is locally inverse and J_{yz} is a right group $e_z\gamma_{z,zy} = e_{yz}$. Thus, $jB_i = j\gamma_{y,yz}^* e_{yz} \in H_{yz}$. Hence, 1(a) of Theorem 3.14 is valid. By definition, $jB_{e_y} = j^*e_y$. If $j \in H_y$ and $i \in I_y$, $jB_i = j^*e_y = j$. Hence, 2(b) of Theorem 3.14 is valid. Let $j \in J$, $u \in I_{bb^{-1}}$, and $z \in J_{b^{-1}b}$. As above, $j\beta_b^* z = jB_u\beta_b^* z$. Thus, (4) of Theorem 3.14 and (4) of the definition of (Y, I, J, V, β, g) are equivalent. Hence, using Theorem 3.14, (Y, I, J, V, β, g) is a standard regular semigroup. Let $(i, y, j), (u, y, z) \in T_y$

(note, Lemma 3.9). Hence, using 2(c) and 2(a), $(i, y, j)(u, y, z) = (i, y, j^*z)$. Thus, using [1, Theorem 1.27], T_y is a rectangular group. Hence, using Lemmas 3.9–3.11, (Y, I, J, V, β, g) is a standard orthodox semigroup.

REMARK 5.2. By a result of Preston, Yamada, and Clifford, [2, Proposition 1], if T is a semilattice of rectangular groups, T is an orthodox semigroup. Conversely, every Cliffordian semigroup which is orthodox is a semilattice of rectangular groups.

Let Y, V, J, and β be as in Theorem 5.1. For each $y \in Y$, select $e_y \in E(J_y)$ such that $e_y^* e_z = e_{yz}$ for all $y, z \in Y$. Let H_y denote the maximal subgroup of J_y containing e_y. Let g be as in Theorem 5.1 and assume 1(b), 1(c), 2(a), 2(c), 3(a), and 3(b) of Theorem 5.1 are valid. Let (Y, J, V, β, g) denote $\{(a, j): a \in V, j \in J_{a^{-1}a}\}$ under the multiplication (4) $(a, j)(b, z) = (ab, g(a, b)^* j \beta_b^* z)$.

THEOREM 5.3. *(Y, J, V, β, g) is a standard $\mathscr{L}$-unipotent semigroup and, conversely, every standard $\mathscr{L}$-unipotent semigroup is isomorphic to some (Y, J, V, β, g).*

Proof. Let S be a standard $\mathscr{L}$-unipotent semigroup. Hence, using [1, Theorem 1.27], S is standard orthodox. Using Theorem 5.1, $S = (Y, I, J, V, \beta, g)$. Let $i \in I_y$. Using Lemma 3.2 and the proof of Lemma 5.1, $(i, y, e_y), (e_y, y, e_y) \in E(S)$. Hence, using Lemma 3.5 and the fact each $\mathscr{L}$-class of S contains precisely one idempotent, $i = e_y$. Thus, $I_y = \{e_y\}$ for each $y \in Y$. Hence, $(e_{aa^{-1}}, a, j)\phi = (a, j)$ defines an isomorphism of (Y, I, J, V, β, g) onto (Y, J, V, β, g). Conversely, we show that $S = (Y, J, V, \beta, g)$ is a standard $\mathscr{L}$-unipotent semigroup. Let $I_y = \{e_y\}$. Define $e_y \circ e_z = e_{yz}$ and let $I = \bigcup (I_y: y \in Y)$. Hence, $(I, \circ)$ is a standard regular semilattice Y of left zero semigroups $(I_y: y \in Y)$. Then, ϕ^{-1} is an isomorphism of S onto (Y, I, J, V, β, g). Hence, S is a standard orthodox semigroup. Using Lemmas 3.2 and 3.5, each $\mathscr{L}$-class of S contains precisely one idempotent. Hence, it easily follows that S is standard $\mathscr{L}$-unipotent.

REMARK 5.4. A semigroup S is termed $\mathscr{L}$-unipotent if each $\mathscr{L}$-class of S contains precisely one idempotent [13]. Hence, a standard regular semigroup is $\mathscr{L}$-unipotent in the sense of [13] if and only if it is standard $\mathscr{L}$-unipotent.

Let Y, V, and β be as in Theorem 5.3. Let $(H, {}^*)$ be a semilattice Y of groups $(H_y: y \in Y)$ and let e_y denote the identity of H_y and let g be as in Theorem 5.3 and assume 1(b), 1(c), 2(a), 2(c), 3(a) and 3(b) of Theorem 5.3 are valid. Let (Y, H, V, β, g) denote $\{(a, j): a \in V, j \in H_{a^{-1}a}\}$ under the multiplication (4)

$$(a, j)(b, z) = (ab, g(a, b)^* j\beta_b^* z) .$$

THEOREM 5.5. *(Y, H, V, β, g) is a standard inverse semigroup, and conversely every standard inverse semigroup is isomorphic to some (Y, H, V, β, g).*

Proof. Let S be a standard inverse semigroup. Then, S is standard $\mathscr{L}$-unipotent. Hence, using Theorem 5.3, $S \cong (Y, J, V, \beta, g)$. Using Lemma 3.2, $E(Y, J, V, \beta, g) = \{(y, j); y \in Y, j \in E(J_y)\}$. Let $j \in E(J_y)$. Hence, using 2(c) and 2(a), $(y, j)(y, e_y) = (y, e_y)$ and $(y, e_y)(y, j) = (y, j)$. Thus, $j = e_y$. Hence, $J_y = H_y$ for all $y \in Y$. Thus, $S \cong (Y, H, V, \beta, g)$. Conversely, let $S = (Y, H, V, \beta, g)$. Since $e_y^* e_z = e_{y,z}$, S is standard $\mathscr{L}$-unipotent by Theorem 5.3. Using Lemma 3.2, $E(S) = \{(y, e_y): y \in Y.\}$ Hence, $E(S)$ is a semilattice. Thus, it is easily seen that S is a standard inverse semigroup.

REMARK 5.6. A characterization of standard orthodox semigroups may be obtained by combining a theorem of M. Yamada [18, Theorem 2] with Theorem 5.5.

6. **The congruence t.** Let S be a standard regular semigroup. Let $t = \{(a, b) \in S^2: aa', bb' \in E(T_y)$ and $a'a, b'b \in E(T_z)$ for some $a' \in \mathscr{I}(a)$, $b' \in \mathscr{I}(b)$, and $y, z \in Y\}$. We introduced t in a special case in [12] and used it in subsequent papers (see [15] and [16] for example). We show t is a congruence on S, $S/t \cong V$, and $\ker t$ (the collection of t-classes of S containing idempotents) $= T$. We note that S is a regular extension of T by V in the sense of Yamada [19].

LEMMA 6.1. $\mathscr{I}((i, a, j)) \cap (I_{a^{-1}a} \times \{a^{-1}\} \times J_{aa^{-1}}) \neq \square$.

Proof. Let $y = aa^{-1}$ and let $u \in I_{a^{-1}a}$. Hence, using (1), $g(a, a^{-1})^* j B_u \beta_{a^{-1}} \in H_{aa^{-1}}$ for $j \in J_{a^{-1}a}$. There exists $v \in H_{aa^{-1}}$ such that $g(a, a^{-1})^* j B_u \beta_{a^{-1}}^* v = e_y$. Hence, using (4), $(i, a, j)(u, a^{-1}, v) = (i, y, e_y)$. However, using 2(b), Lemmas 3.2, 3.3, and 3.5,

$$(i, a, j) = (i, y, e_y)(i, a, j) = (i, a, j)(u, a^{-1}, v)(i, a, j)$$

while $(u, a^{-1}, v)(i, a, j)(u, a^{-1}, v) = (u, a^{-1}, v)(i, y, e_y) = (u, a^{-1}, v)$.

LEMMA 6.2. *$(i, a, j)t(u, b, v)$ if and only if $a = b$.*

Proof. Using Lemmas 6.1 and 3.9, there exists $(i, a, j)' \in \mathscr{I}(i, a, j)$ and $(u,b,v)' \in \mathscr{I}(u,b,v)$ such that $(i,a,j)(i,a,j)' \in T_{aa^{-1}}$, $(u,b,v)(u,b,v)' \in T_{bb^{-1}}$, $(i, a, j)'(i, a, j) \in T_{a^{-1}a}$, and $(u, b, v)'(u, b, v) \in T_{b^{-1}b}$. Hence,

$(i, a, j)t(u, b, v)$ if and only if $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$.

THEOREM 6.3. *t is a congruence relation on S, $S/t \cong V$, and* $\ker t = T$.

Proof. The first two assertions are easily seen. Using Lemmas 3.2, 3.9, and 3.10, $\ker t = T$.

REMARK 6.4. Using Lemmas 3.9-3.11 and the fact $(i, a, j)\phi = a$ defines a homomor phism of S onto V, S is a regular extension of T by V in the sense of Yamada [19, page 4]. Thus, using Theorem 6.3, S is a regular extension of T by S/t.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys of the Amer. Math. Soc. **7**, Vol. 1, Providence, R. I., 1961; Vol. 2, Providence, R. I., 1967.
2. A. H. Clifford, *The structure of orthodox unions of groups*, Semigroup Forum, **3** (1972), 283-337.
3. T. E. Hall, *On regular semigroups whose idempotents form a semigroup*, Bull. Australian Math. Soc., **1** (1969), 195-208.
4. J. W. Hogan, *Bisimple semigroups with idempotents well ordered*, Semigroup Forum, **6** (1973), 298-316.
5. W. D. Munn, *On simple inverse semigroups*, Semigroup Forum, **1** (1970), 63-74.
6. ———, *Fundamental inverse semigroups*, Quarterly J. Math. Oxford, **21** (1970), 157-170.
7. R. J. Warne, *Extensions of completely 0-simple semigroups by completely 0-simple semigroups*, Proc. Amer. Math. Soc., **17** (1966), 524-526.
8. ———, *A class of bisimple inverse semigroups*, Pacific J. Math., **18** (1966), 563-577.
9. ———, *The idempotent separating congruences of a bisimple inverse semigroup with identity*, Publicationes Mathematicae, **13** (1966), 203-206.
10. ———, *Simple L-regular semigroups*, The Formosan Science, **24** (1970), 71-77.
11. ———, *A comment on "Simple L-regular semigroups,"* The Formosan Science, **24** (1970), 78.
12. ———, *E-bisimple semigroups*, J. Natural Sci. and Math. (Lahore), **11** (1971), 51-81.
13. ———, *$\mathscr{L}$-unipotent semigroups*, Nigerian J. Science, **5** (1972), 245-248.
14. ———, *On the structure of semigroups which are unions of groups*, Trans. Amer. Math. Soc., **186** (1973), 385-401.
15. ———, *ωY-$\mathscr{L}$-unipotent semigroups*, Jnānabha, 3 (1973), 99-118.
16. ———, *Generalized ω-$\mathscr{L}$-unipotent bisimple semigroups*, Pacific J. Math., **51** (1974), 631-648.
17. M. Yamada and N. Kimura, *Note on idempotent semigroups*, II, Proc, Japan Acad., **34** (1958), 110-112,
18. Miyuki Yamada, *Regular semigroups whose idempotents satisfy permutation identities*, Pacific J. Math., **21** (1967), 371-392.
19. ———, *On regular extensions of a semigroup which is a semilattice of completely simple semigroups*, Mem. Fac. Lit. and Sci. Shimane Univ. Nat. Sci., **7** (1974), 1-17.

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