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VARIATION OF GREEN'S POTENTIAL

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Let u be the Green's potential of a nonnegative mass distribution μ on unit disk D, defined by

$$u(z)=\int_D G(z,w)\,d\mu(w), \tag{1.1}$$

where G is the Green's function on D. The following will be proved.

THEOREM 1. ***Suppose*** $0<\rho<1$ ***and***

$$\int_D (1-|w|)\log\frac{1}{1-|w|}\,d\mu(w)<\infty. \tag{1.2}$$

Then for almost all θ, $0\leqq\theta<2\pi$, u ***has finite variation on the line segment joining*** $\rho e^{i\theta}$ ***and*** $e^{i\theta}$.

THEOREM 2. ***Fix*** θ, $0\leqq\theta<2\pi$ ***and suppose***

$$\int_D \frac{1-|w|}{|e^{i\theta}-w|}\,d\mu(w)<\infty. \tag{1.3}$$

If L ***is a circular arc in*** D ***centered at*** $e^{i\theta}$, ***then for all*** a ***on*** L ***except a set of capacity zero,*** u ***has finite variation on the line segment joining*** a ***and*** $e^{i\theta}$.

It is known [**4**; 150] that if μ satisfies

$$\int_D (1-|w|)\,d\mu(w)<\infty, \tag{1.4}$$

then u is finite everywhere except a set of capacity zero in D.

Carleson [**2**; 32] has implicitly proved the following result.

THEOREM A. *Suppose* $0<\rho<1$ *and*

$$\int_D (1-|w|)^{\alpha}\,d\mu(w)<\infty \tag{1.5}$$

for some fixed α, $0\leqq\alpha<1$. *Then for all* θ, $0\leqq\theta<2\pi$, *except a set* E *of*

α-dimensional Hausdorff measure zero if $0 < \alpha < 1$, or of capacity zero if $\alpha = 0$, u has finite variation on the line segment joining $\rho e^{i\theta}$ and $e^{i\theta}$.

Some ideas involved in Theorems 1 and 2 are borrowed from Rudin **[6]** and Cargo **[1]**, who studied the variation of Blaschke products on line segments.

In §4 we shall extend the above three theorems to curves with certain differentiability properties, which need not be line segments.

For convenience, we shall use $l(z, w)$ to denote the line segment joining z, w and $V(F, \gamma)$ to denote the total variation of the function F on the curve γ whenever this is meaningful.

2. Lemmas.

LEMMA 1. *If l is a Jordan rectifiable curve in $\bar{D}$, then*

$$V(u, l) \leqq \int_D V(G(\cdot, w), l)\, d\mu(w).$$

The proof follows directly from the definition of variation and (1.1).

LEMMA 2. *If $z, w \in D$ and $|z - w| \geqq (1 - |w|)/2$ then $|(z - w)/(1 - \bar{w}z)| \geqq 1/5$.*

Proof. For fixed w, $|(z - w)/(1 - \bar{w}z)|$ attains minimum $1/(3|w| + 2)$ on the circle $|z - w| = (1 - |w|)/2$. The lemma follows from minimal modulus principle.

LEMMA 3. *Let $w \in D$, A be a subset of D and $m = \min\{(1 - |z|)/|1 - z| : z \in A\}$. If the distance from w to A is less than $(1 - |w|)/2$, then $(1 - |w|)/(|1 - w|) \geqq m/3$.*

Proof. Let $z \in A$ and $|z - w| \leqq (1 - |w|)/2$. It is easy to check $\frac{1}{2}(1 - |w|) \leqq 1 - |z| \leqq \frac{3}{2}(1 - |w|)$. Therefore

$$\frac{1 - |w|}{|1 - w|} \geqq \frac{1 - |w|}{|1 - z| + |z - w|} \geqq \frac{\frac{2}{3}(1 - |z|)}{|1 - z| + 1 - |z|} \geqq \frac{m}{3}.$$

LEMMA 4. (Cargo **[1**; 145]) *Suppose $a, w \in D$, $0 \leqq \theta < 2\pi$ and $z(r) = a + r(e^{i\theta} - a)$ for $0 \leqq r \leqq 1$. Then*

$$\int_0^1 \frac{1 - |w|^2}{|1 - \bar{w}z(r)|^2}\, dr < \frac{8}{(1 - |a|)\sin\left[\frac{\pi}{2}(1 + |a|)\right]} \frac{1 - |w|}{|e^{i\theta} - w|}.$$

LEMMA 5. *Let* $w, a \in D$, $0 \leqq \theta < 2\pi$ *and* b *be the projection of* a *on* $l(w, e^{i\theta})$. *If* $0 < m \leqq \min\{(1-|z|)/(|e^{i\theta} - z|): z \in l(a, e^{i\theta})\}$, *we have*

$$\begin{aligned}(2.1)\qquad & V(G(\cdot, w), l(a, e^{i\theta})) \\ & \leqq \left(\frac{40}{\sin\frac{\pi}{2}(1+|a|)} + \frac{12}{m}\log\left|\frac{1}{b-a}\right|\right)\frac{1}{1-|a|}\frac{1-|w|}{|e^{i\theta}-w|}.\end{aligned}$$

Proof. We assume that w is not on the line through a, $e^{i\theta}$, otherwise $b = a$ and (2.1) is trivial. Also we may assume $\theta = 0$ by a suitable rotation. Let $z(r) = a + r(1-a)$, $0 \leqq r \leqq 1$, be a representation of $l(a, 1)$ and $\operatorname{grad} G$ be the gradient of Green's function with respect to the first variable, that is,

$$(2.2)\qquad |\operatorname{grad} G(z, w)| = \frac{1-|w|^2}{|1-\bar{w}z||z-w|}$$

if $z \neq w$. Let $S_1 = \{r: |z(r) - w| \geqq (1-|w|)/2\} \cap [0, 1]$ and $S_2 = [0, 1] \backslash S_1$. Thus

$$\begin{aligned}V(G(\cdot, w), l(a, 1)) &\leqq |1-a| \int_{S_1+S_2} \frac{1-|w|^2}{|1-\bar{w}z(r)||z(r)-w|}\,dr \\ &= |1-a|(I_1 + I_2).\end{aligned}$$

From Lemma 2 and Lemma 4 we have

$$I_1 < \frac{40}{(1-|a|)\sin\left[\frac{\pi}{2}(1+|a|)\right]}\frac{1-|w|}{|1-w|}.$$

Assume S_2 is nonempty and let α, β ($\alpha < \beta$) be the two real numbers so that the points $z(\alpha) = a + \alpha(1-a)$ and $z(\beta) = a + \beta(1-a)$ have distance exactly $(1-|w|)/2$ from w. We observe that

$$\operatorname{Im}\frac{z(\alpha)-w}{1-a} = \operatorname{Im}\frac{z(\beta)-w}{1-a} = -\operatorname{Im}\frac{w-a}{1-a}$$

and $|\operatorname{Im}(w-a)/(1-a)|$ is the distance from w to $l(a, 1)$, which is $|(b-a)/(1-a)||1-w|$. Therefore

$$-\operatorname{Re}\frac{z(\alpha)-w}{1-a} = \operatorname{Re}\frac{z(\beta)-w}{1-a} > 0.$$

With the aid of Lemma 3, we estimate I_2 as follows:

$$\begin{aligned}
I_2 &\leqq 2\int_\alpha^\beta \frac{1}{|z(r)-w|}\,dr \\
&= \frac{2}{|1-a|}\left[\log\left(\left|\frac{z(r)-w}{1-a}\right| + \operatorname{Re}\frac{z(r)-w}{1-a}\right)\right]_\alpha^\beta \\
&= \frac{2}{|1-a|}\log\frac{\left|\dfrac{z(\beta)-w}{1-a}\right| + \operatorname{Re}\dfrac{z(\beta)-w}{1-a}}{\left|\dfrac{z(\beta)-w}{1-a}\right| - \operatorname{Re}\dfrac{z(\beta)-w}{1-a}} \\
&\leqq \frac{2}{|1-a|}\log\frac{4\left|\dfrac{z(\beta)-w}{1-a}\right|^2}{\left(\operatorname{Im}\dfrac{z(\beta)-w}{1-a}\right)^2} \\
&\leqq \frac{4}{|1-a|}\log\frac{(1-|w|)}{|b-a|\,|1-w|} \\
&\leqq \frac{12}{m|1-a|}\,\frac{1-|w|}{|1-w|}\log\frac{1}{|b-a|}.
\end{aligned}$$

Thus (2.1) follows from the estimations for I_1 and I_2.

3. Proofs and remarks.

Proof of Theorem 1. For a fixed $w \in D$, let T_1 be $\{z: \rho \leqq |z| < 1, |z-w| \geqq (1-|w|)/2\}$ and $T_2 = \{z: \rho \leqq |z| < 1\}\backslash T_1$. We consider

$$\int_{|z|\geqq\rho} |\operatorname{grad} G(z,w)|\,d\sigma_z = \int_{T_1}\cdot\,d\sigma_z + \int_{T_2}\cdot\,d\sigma_z = I_1 + I_2$$

where $d\sigma_z = r\,dr\,d\theta$, $z = re^{i\theta}$. From (2.2), Lemma 2 and Lemma 4 we obtain

$$I_1 \leqq C\int_0^{2\pi}\frac{1-|w|}{|e^{i\theta}-w|}\,d\theta$$

where C is a constant depending on ρ only. It is easy to verify that there exists a constant C_1 so that

$$I_1 \leqq C_1(1-|w|)\log\frac{1}{1-|w|} \quad \text{when} \quad |w| > \tfrac{1}{2}$$

and

$$I_1 \leqq C_1(1-|w|) \quad \text{when} \quad |w| \leqq \tfrac{1}{2}.$$

Noticing that $|1-\bar{w}z| \geqq 1-|w|$, we have

$$I_2 \leqq \int_{|z-w|<(1-|w|)/2} \frac{2}{|z-w|} d\sigma_z = 2\pi(1-|w|).$$

From (1.2), (1.4) and the estimations above, we see that

$$\int_0^{2\pi} V(u, l(\rho e^{i\theta}, e^{i\theta}))\, d\sigma \leqq \frac{1}{\rho} \int_D \int_{|z|\geqq\rho} |\operatorname{grad} G(z, w)|\, d\sigma_z\, d\mu(w) < \infty.$$

Theorem 1 follows.

Proof of Theorem 2. We may assume $\theta = 0$. Let α be the distance from 1 to L, L_0 be a closed subarc of L and m = minimum of the numbers in $\{1-|a|: a \in L_0\} \cup \{(1-|z|)/(|1-z|): z \in l(a, 1) \text{ and } a \in L_0\}$. It is clear that $m > 0$.

Let E be the set $\{a \in L_0: V(u, l(a, 1)) = \infty\}$. We shall show E is of capacity zero. If we assume otherwise, there exist $M > 0$ and a finite positive measure $\nu(a)$ on E such that

$$\int_E \log \frac{1}{|z-a|} d\nu(a) < M$$

for every complex z. With the aid of Lemma 5 we obtain

$$\begin{aligned}
&\int_E V(G(\cdot, w), l(a, 1))\, d\nu(a) \\
&\quad \leqq \int_E \left(\frac{40}{\sin \frac{\pi}{2}(2-m)} + \frac{12}{m} \log \frac{1}{|b-a|} \right) \frac{1}{m} \frac{1-|w|}{|1-w|}\, d\nu(a) \\
&\quad < \left[\frac{40}{\sin \frac{\pi}{2}(2-m)} \nu(E) + \frac{12M}{m} \right] \frac{1}{m} \frac{1-|w|}{|1-w|}.
\end{aligned}$$

From Lemma 1 and the assumption of Theorem 2 we see that $\int_E V(u, l(a, 1))\, d\nu(a) < \infty$ which is a contradiction. Hence E is of capacity zero. Theorem 2 is proved.

REMARK 1. The conclusion in Theorem 2 is best possible, because for any compact set E of capacity zero in D, there is a mass distribution μ on E such that the Green's potential of μ is $+\infty$ on E and finite on D\E [**4**; 152 and 271].

REMARK 2. Carleson [**2**; 26] showed that (1.5) implies (1.3) for all θ except a set of α-dim Hausdorff measure zero for $0 < \alpha < 1$ or of capacity zero for $\alpha = 0$. Frostman [**3**] showed that (1.2) implies (1.3) for almost all θ. But we note that Theorems A and 1 do not follow from Theorem 2 for any obvious reason.

REMARK 3. If B is the Blaschke product with zeros at z_n, then $-\log|B|$ is the Green's potential given by $\nu = \Sigma\, \delta_{z_n}$ where δ_{z_n} is the unit mass at z_n. If l is a line segment in $\bar{D}$ not passing through any z_n, then $\log(z - z_n)/(1 - \bar{z}_n z)$ is analytic in a neighborhood of l. Thus we have

$$\begin{aligned} (3.1) \qquad V(\arg B, l) &\leqq \sum V\left(\arg \frac{z - z_n}{1 - \bar{z}_n z}, l\right) \\ &\leqq \sum \int_l \left| \operatorname{grad} \arg \frac{z - z_n}{1 - \bar{z}_n z} \right| |dz| \\ &= \int_D \int_l |\operatorname{grad} G(z, w)|\,|dz|\, d\nu(w) \end{aligned}$$

and

$$\begin{aligned} (3.2) \qquad V(|B|, l) &\leqq \int_l |B'(z)|\,|dz| \\ &\leqq \int_D \int_l |\operatorname{grad} G(z, w)|\,|dz|\, d\nu(w). \end{aligned}$$

In the above theorems, when we dealt with variation of u on line segment l, we in fact studied an upper bound of it, which is $\int_D \int_l |\operatorname{grad} G(z, w)|\,|dz|\, d\mu(w)$. Therefore from (3.1) and (3.2) we see that whenever ν satisfies the same condition for μ in Theorem j, $j = 1, 2$ or A, B has the same variation property for u stated in Theorem j. With the aid of a theorem by F. Riesz [**5**; 401] on variation of analytic functions, we can conclude the following result of Cargo [**1**; 147]: if $\Sigma_{n=1}^{\infty} (1 - |z_n|)/|e^{i\theta} - z_n| < \infty$ then all the subproducts of B have finite variation on each line segment in D ending at $e^{i\theta}$.

4. Variation on curves. Let $0 < \eta < 1$, $0 < m < M$ and f be a one to one map from $\{\eta \leqq |z| \leqq 1\}$ to $\bar{D}\backslash\{0\}$ mapping $|z| = 1$ onto $|z| = 1$ and satisfying

$$m|\alpha-\beta| \leqq |f(\alpha)-f(\beta)| \leqq M|\alpha-\beta| \tag{4.1}$$

whenever $\eta \leqq |\alpha| \leqq 1$, $\eta \leqq |\beta| \leqq 1$. We have the following result.

THEOREM 3. *Let μ satisfy the condition in Theorem j, $j = 1,2$ or A, and assume that the line segments we considered in Theorem j are short enough to be in $\{\eta < |z| \leqq 1\}$. Then the variations of u on the images under f of the corresponding line segments are finite outside an exceptional set of the same size as in Theorem j.*

We note that when $j = A$ or 1, Theorem 3 can be considered as a generalization of a theorem in [7], in the sense of the size of the exceptional set, of variation vs. limit and of the more generalized curves.

Proof. Let S be the image of f and l be any fixed closed line segment in $\{\eta < |z| \leqq 1\}$. We may assume μ has support on S. In fact, if v is the Green's potential given by $\mu \mid D\backslash S$, then $v(z)$ is harmonic in the interior of S and tends to zero uniformly as z tends to the unit circle; thus $v(z)$ has finite variation on $f(l)$.

Let $g = f^{-1}$ and $\operatorname{grad} G$ be the gradient with respect to the first variable of G. Thus from (2.2) we have

$$\frac{|\operatorname{grad} G(f(z), w)|}{|\operatorname{grad} G(z, g(w))|} = \frac{1-|w|^2}{1-|g(w)|^2}\,\frac{|1-\overline{g(w)}z|}{|1-\bar{w}f(z)|}\,\frac{|g(w)-z|}{|w-f(z)|}. \tag{4.2}$$

We observe that

$$\begin{aligned} \left|\frac{1-\overline{g(w)}z}{1-\bar{w}f(z)}\right| &\leqq \frac{1-|g(w)|^2+|g(w)||g(w)-z|}{|1-\bar{w}f(z)|} \\ &\leqq 2\,\frac{1-|g(w)|}{1-|w|}+\frac{|g(w)-z|}{|w-f(z)|}. \end{aligned} \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we have

$$|\operatorname{grad} G(f(z), w)| \leqq M_1 |\operatorname{grad} G(z, g(w))|$$

for some constant M_1. Because $f(l)$ is Lipschitz, for fixed w, we have

$$
\begin{aligned}
(4.4) \qquad V(G(z,w), f(l)) &\leqq \int_{f(l)} |\operatorname{grad} G(z,w)|\,|dz| \\
&\leqq \int_l |\operatorname{grad} G(f(z),w)|\,M\,|dz| \\
&\leqq MM_1 \int_l |\operatorname{grad} G(z,g(w))|\,|dz|.
\end{aligned}
$$

From (4.4) and the proof of Theorem j, we see that it is enough to show

$$(4.5) \qquad \int_D (1-|g(w)|)^\alpha \, d\mu(w) < \infty,$$

$$(4.6) \qquad \int_D (1-|g(w)|)\log\frac{1}{1-|g(w)|}\, d\mu(w) < \infty,$$

$$(4.7) \qquad \int_D \frac{1-|g(w)|}{|g(e^{i\theta})-g(w)|}\, d\mu(w) < \infty,$$

for Theorems A, 1, 2 respectively. (4.5) and (4.7) are simple consequences of (4.1). We shall show (4.6). Let δ be the distance from 0 to S and assume $w \in S$. We have

$$\log\frac{1}{1-|g(w)|} - \log\frac{1}{1-|w|} < \log M,$$

thus

$$\log\frac{1}{1-|g(w)|} < \left(1+\frac{\log M}{\log\frac{1}{1-\delta}}\right)\log\frac{1}{1-|w|},$$

therefore

$$
\begin{aligned}
&(1-|g(w)|)\log\frac{1}{1-|g(w)|} \\
(4.8) \qquad &< \left(1+\frac{\log M}{\log\frac{1}{1-\delta}}\right)\frac{1}{m}(1-|w|)\log\frac{1}{1-|w|}.
\end{aligned}
$$

Since we assume μ has support on S, (1.2) and (4.8) imply (4.6). The proof of Theorem 3 is complete.

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