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CHOQUET SIMPLEXES AND σ-CONVEX FACES

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The purpose of this paper is to present a simple characterization of the split-faces in a Choquet simplex K, i.e., those faces F such that K is a direct convex sum of F and its complementary face. It is shown that a face F is a split-face if and only if it is σ-convex, i.e., closed under infinite convex combinations. This is proved by means of a measure-theoretic characterization of the σ-convex faces of K. As a consequence, it is shown that the lattice of σ-convex faces of a Choquet simplex forms a complete Boolean algebra.

It is well-known that infinite convex combinations are always available in a compact convex subset K of a locally convex, Hausdorff, linear topological space: given points $x_1, x_2, \cdots$ in K and nonnegative real numbers $\alpha_1, \alpha_2, \cdots$ such that $\Sigma\, \alpha_k = 1$, the series $\Sigma\, \alpha_k x_k$ must converge to some point in K. We refer to $\Sigma\, \alpha_k x_k$ as a *σ-convex combination* of the x_k. We define *σ-convex* subsets of K and *σ-convex hulls* in K in the obvious manner. A face of K which happens to be σ-convex is called a *σ-convex face*, and we note that given any subset $X \subseteq K$, there is a smallest σ-convex face containing X, called the *σ-convex face generated by X*. We also note that if F is any face of K (σ-convex or not), and if we have a σ-convex combination $\Sigma\, \alpha_k x_k \in F$ with all $\alpha_k > 0$, then all $x_k \in F$. (This follows from the defining property of a face, since

$$\alpha_j x_j + (1-\alpha_j)\left[\sum_{k \neq j} \alpha_k x_k/(1-\alpha_j)\right] = \Sigma\, \alpha_k x_k \in F$$

for all j.)

DEFINITION. Let F be a face of a compact convex set K. Then F' is defined to be the union of those faces of K which are disjoint from F. We say that F is a *split-face* of K [**2**] provided F' is a face and for each $x \in K - (F \cup F')$ there is a unique convex combination $x = \alpha y + (1-\alpha)z$ with $y \in F$, $z \in F'$. In this event, F' is a complement for F in the lattice of faces of K.

THEOREM 1. *Let F be a face of a Choquet simplex K. Then F' is a face of K. For any $x \in K - (F \cup F')$, there is at most one convex combination $x = \alpha y + (1-\alpha)z$ with $y \in F$ and $z \in F'$.*

Proof. [**1**, Theorem 1 and Proposition 2].

COROLLARY 2. *Let F be a face of a Choquet simplex K. Then F is a split-face of K if and only if the convex hull of $F \cup F'$ is K.*

DEFINITION. For any compact Hausdorff space K, we use $M_1^+(K)$ to denote the set of all probability measures on K, equipped with the vague (weak*) topology. There is also a natural norm topology on $M_1^+(K)$, obtained from the identification of the space $M(K)$ [of all signed regular Borel measures on K] with the dual Banach space $C(K)^*$. If K is a compact convex set, then for any $\mu \in M_1^+(K)$ we use x_μ to denote the resultant (barycenter) of μ in K. If K is a Choquet simplex, then for any $x \in K$ we use μ_x to denote the unique maximal measure in $M_1^+(K)$ whose resultant is x.

The existence of σ-convex combinations is more straightforward in $M_1^+(K)$ than in general compact convex sets. Given measures $\mu_1, \mu_2, \cdots$ in $M_1^+(K)$ and nonnegative real numbers $\alpha_1, \alpha_2, \cdots$ such that $\Sigma \alpha_k = 1$, we may define a Borel measure μ on K by setting $\mu(A) = \Sigma \alpha_k \mu_k(A)$ for all Borel sets $A \subseteq K$. It is clear that $\mu \in M_1^+(K)$. Observing that $\mu(f) = \Sigma \alpha_k \mu_k(f)$ for all $f \in C(K)$, we see that $\Sigma \alpha_k \mu_k$ converges to μ in the vague topology. Therefore μ coincides with the σ-convex combination $\Sigma \alpha_k \mu_k$ in $M_1^+(K)$.

PROPOSITION 3. *Let K be a compact Hausdorff space, $\mu \in M_1^+(K)$, $X \subseteq M_1^+(K)$. Then μ lies in the face generated by X if and only if there exists ν in the convex hull of X such that $\mu \leqq \alpha\nu$ for some $\alpha > 0$.*

Proof. Analogous to [**4**, Proposition 1.2].

THEOREM 4. *Let K be a compact Hausdorff space, $\mu \in M_1^+(K)$, $X \subseteq M_1^+(K)$. Then the following conditions are equivalent:*

(a) *μ lies in the σ-convex face generated by X.*
(b) *μ lies in the σ-convex hull of the face generated by X.*
(c) *There is some ν in the σ-convex hull of X for which $\mu \ll \nu$.*

Proof. (a) $\Rightarrow$ (c): Let F denote the set of those measures $\mu' \in M_1^+(K)$ which are absolutely continuous with respect to some ν' (depending on μ') in the σ-convex hull of X. We claim that F is a σ-convex face of $M_1^+(K)$.

First consider a σ-convex combination $\mu' = \Sigma \alpha_k \mu_k$ in $M_1^+(K)$ such that each $\mu_k \in F$. For each k, there is some ν_k in the σ-convex hull of X such that $\mu_k \ll \nu_k$. Then $\nu' = \Sigma \alpha_k \nu_k$ lies in the σ-convex hull of X, and we infer that $\mu' \ll \nu'$, whence $\mu' \in F$. Thus F is σ-convex.

Next consider a proper convex combination $\mu' = \alpha\mu_1 + (1-\alpha)\mu_2$ in $M_1^+(K)$ such that $\mu' \in F$. There is some ν' in the σ-convex hull of X such that $\mu' \ll \nu'$. Since $\mu_1 \leqq \alpha^{-1}\mu'$ and $\mu_2 \leqq (1-\alpha)^{-1}\mu'$, we find that $\mu_1, \mu_2 \ll \nu'$, and consequently $\mu_1, \mu_2 \in F$. Thus F is a face of $M_1^+(K)$.

Clearly $X \subseteq F$, hence F must contain the σ-convex face generated by X. Therefore $\mu \in F$.

(c) $\Rightarrow$ (b): There is a σ-convex combination $\nu = \Sigma\, \alpha_k \nu_k$ with each $\nu_k \in X$. Renumbering if necessary, we may assume that $\alpha_1 > 0$. For each k, set $\alpha'_k = \alpha_1 + \cdots + \alpha_k$ and $\nu'_k = (\alpha_1\nu_1 + \cdots + \alpha_k\nu_k)/\alpha'_k$, and note that ν'_k is a measure in $M_1^+(K)$ which lies in the convex hull of X.

For each positive integer n, take a Hahn Decomposition of the signed measure $n\alpha'_n\nu'_n - \mu$. This gives us a Borel set $K_n \subseteq K$ such that $\mu(A) \leqq n\alpha'_n\nu'_n(A)$ for all Borel sets $A \subseteq K_n$ and $n\alpha'_n\nu'_n(A) \leqq \mu(A)$ for all Borel sets $A \subseteq K - K_n$.

Since $2\alpha'_2\nu'_2(K_1 - K_2) \leqq \mu(K_1 - K_2) \leqq \alpha'_1\nu'_1(K_1 - K_2) \leqq \alpha'_2\nu'_2(K_1 - K_2)$, we see that $\mu(K_1 - K_2) = \nu'_2(K_1 - K_2) = 0$. Thus we may replace K_2 by $K_1 \cup K_2$, so that now $K_1 \subseteq K_2$. Continuing in this manner, we see that we may assume that $K_n \subseteq K_{n+1}$ for all n.

Set $J = K - (\cup K_n)$, and note that $\alpha'_n\nu'_n(J) \leqq \mu(J)/n$ for all n. For all $k \geqq n$, $\alpha'_k\nu'_k(J) \leqq \mu(J)/k \leqq \mu(J)/n$, hence $\nu(J) = \lim_{k\to\infty} \alpha'_k\nu'_k(J) \leqq \mu(J)/n$. Since this holds for all n, we obtain $\nu(J) = 0$. Since $\nu'_n \leqq \nu/\alpha'_n$ and $\mu \ll \nu$, it follows that $\nu'_n(J) = 0$ for all n and $\mu(J) = 0$. Thus we may replace each K_n by $K_n \cup J$, without affecting the properties obtained above. As a result, we now have $\cup K_n = K$.

Now set $L_1 = K_1$ and $L_n = K_n - K_{n-1}$ for all $n > 1$, so that $L_1, L_2, \cdots$ are pairwise disjoint Borel sets whose union is K. Set $I = \{n \mid \mu(L_n) > 0\}$. For $n \in I$, define $\mu_n \in M_1^+(K)$ by setting $\mu_n(A) = \mu(A \cap L_n)/\mu(L_n)$ for all Borel sets $A \subseteq K$. For such A, we have $A \cap L_n \subseteq L_n \subseteq K_n$ and so

$$\mu_n(A) = \mu(A \cap L_n)/\mu(L_n) \leqq n\alpha'_n\nu'_n(A \cap L_n)/\mu(L_n) \leqq n\alpha'_n\nu'_n(A)/\mu(L_n).$$

Consequently, $\mu_n \leqq [n\alpha'_n/\mu(L_n)]\nu'_n$, whence Proposition 3 shows that μ_n lies in the face generated by X.

We have $\Sigma_{n\in I}\, \mu(L_n) = 1$ and $\mu(A) = \Sigma_{n\in I}\, \mu(A \cap L_n) = \Sigma_{n\in I}\, \mu(L_n)\mu_n(A)$ for all Borel sets $A \subseteq K$. Therefore $\mu = \Sigma_{n\in I}\, \mu(L_n)\mu_n$ is a σ-convex combination of the μ_n, hence μ lies in the σ-convex hull of the face generated by X.

(b) $\Rightarrow$ (a) is clear.

In particular, Theorem 4 shows that a measure $\mu \in M_1^+(K)$ lies in the σ-convex face generated by a measure $\nu \in M_1^+(K)$ if and only if $\mu \ll \nu$. The corresponding statement for norm-closed faces is given in [**4**, Proposition 1.3]: μ lies in the norm-closure of the face generated by ν if and only if $\mu \ll \nu$. Thus the σ-convex face generated by ν coincides with the norm-closure of the face generated by ν. In general, the σ-convex faces in $M_1^+(K)$ coincide with the norm-closed faces, as the next theorem shows.

THEOREM 5. *Let K be a compact Hausdorff space. For any face F of $M_1^+(K)$, the following conditions are equivalent*:

(a) *F is a split-face.*

(b) *F is norm-closed.*

(c) *F is σ-convex.*

Proof. (a) $\Leftrightarrow$ (b) follows from [**3**, Corollary to Theorem 1], and also appears in [**4**, Theorem 2.4].

(b) $\Rightarrow$ (c) follows from the observation that infinite convex combinations in $M_1^+(K)$ must also converge in the norm topology.

(c) $\Rightarrow$ (b): Suppose that $\mu_1, \mu_2, \cdots \in F$ and $\mu \in M_1^+(K)$ such that $\|\mu_n - \mu\| \to 0$. It follows easily from Urysohn's Lemma and the regularity of the measures that $\mu_n(A) \to \mu(A)$ for all Borel sets $A \subseteq K$. Setting $\nu = \Sigma_{n=1}^{\infty} \mu_n/2^n$, we thus see that $\nu \in F$ and $\mu \ll \nu$. According to Theorem 4, $\mu \in F$.

DEFINITION. As in [**2**], any compact convex set K (in a locally convex, Hausdorff, linear topological space) is affinely homeomorphic to a weak*-compact convex subset of the dual space $A(K)^*$ (where $A(K)$ denotes the Banach space of all real-valued affine continuous functions on K). Because of this, K inherits a norm topology from $A(K)^*$.

PROPOSITION 6. *Let K be a Choquet simplex, and let K^* denote the set of maximal measures in $M_1^+(K)$. Then K^* is a σ-convex face of $M_1^+(K)$, and the rule $\phi(\mu) = x_\mu$ defines a continuous affine isomorphism ϕ of K^* onto K. The maps ϕ and ϕ^{-1} both preserve σ-convex combinations and norms.*

Proof. It is well-known that K^* is a face of $M_1^+(K)$, and that ϕ is a continuous affine isomorphism. The σ-convexity of K^* follows easily from Mokobodzki's characterization of maximal measures [**2**, Proposition I.4.5].

Since ϕ is continuous and affine, it must preserve σ-convex combinations, hence so does ϕ^{-1}.

According to [**4**, Lemma 2.6], ϕ^{-1} preserves norms, hence ϕ does also.

With the help of Proposition 6, Theorems 4 and 5 imply the corresponding results for arbitrary Choquet simplexes.

THEOREM 7. *Let K be a Choquet simplex, $x \in K$, $Y \subseteq K$. Then the following conditions are equivalent*:

(a) *x lies in the σ-convex face generated by Y.*

(b) *x lies in the σ-convex hull of the face generated by Y.*

(c) *There is some y in the σ-convex hull of Y such that $\mu_x \ll \mu_y$.*

COROLLARY 8. *If F is a face of a Choquet simplex K, then the σ-convex hull of F is also a face of K.*

THEOREM 9. *If F is a face of a Choquet simplex K, then the following conditions are equivalent*:

(a) *F is a split-face.*

(b) *F is norm-closed.*

(c) *F is σ-convex.*

The equivalence (a) $\Leftrightarrow$ (b) of Theorem 9 has appeared in [**3**, Corollary to Theorem 1] and [**4**, Theorem 2.8]. We note that the characterization (a) $\Leftrightarrow$ (c) has an apparent advantage, in that it depends only on the topology intrinsic to K, rather than on the (external) norm topology. While we have utilized the norm results as the fastest means of proving Theorems 5 and 9, it is also possible to prove the equivalence (a) $\Leftrightarrow$ (c) in these theorems without any use of norms.

THEOREM 10. *The lattice $\mathcal{F}$ of σ-convex faces of a Choquet simplex K forms a complete Boolean algebra. For $\{F_i\} \subseteq \mathcal{F}$, $\wedge F_i = \cap F_i$. For $F, G \in \mathcal{F}$, $F \vee G$ is the convex hull of $F \cup G$.*

Proof. Obviously $\mathcal{F}$ is a complete lattice in which arbitrary infima are given by intersections. For $F, G \in \mathcal{F}$, we see from Theorem 9 and [**2**, Corollary II.6.8] that the convex hull of $F \cup G$ is a σ-convex face of K. (This is also easy to prove directly, using [**1**, Proposition 3].) Thus the convex hull of $F \cup G$ equals $F \vee G$.

Given $F, G, H \in \mathcal{F}$, we automatically have $(F \wedge G) \vee (F \wedge H) \subseteq F \wedge (G \vee H)$. Now consider any $x \in F \wedge (G \vee H)$. Inasmuch as $G \vee H$ is the convex hull of $G \cup H$, there must be a convex combination $x = \alpha y + (1-\alpha)z$ with $y \in G$, $z \in H$. If $\alpha = 0$ or 1, then either $x \in F \wedge H$ or $x \in F \wedge G$. If $0 < \alpha < 1$, then since F is a face we obtain $y \in F \wedge G$, $z \in F \wedge H$. Thus $x \in (F \wedge G) \vee (F \wedge H)$ in any case, whence $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H)$. Therefore $\mathcal{F}$ is a distributive lattice.

Given $F \in \mathcal{F}$, we see from Theorem 9 that $F' \in \mathcal{F}$ as well (which is also easy to prove directly). Obviously F' is a complement for F in $\mathcal{F}$, whence $\mathcal{F}$ is a complemented lattice.

Therefore $\mathcal{F}$ is a complete, complemented, distributive lattice, i.e., a complete Boolean algebra.

DEFINITION. Let K be a Choquet simplex, let $\{x_i\} \subseteq K$, and for each i let F_i be the face generated by x_i in K. If F_i and F_j are disjoint for all $i \neq j$, we shall say that the points x_i are *facially independent* (*in* K).

COROLLARY 11. *Any σ-convex face F in a Choquet simplex K can be generated by facially independent points of K.*

Proof. Let $\mathcal{F}$ denote the lattice of σ-convex faces of K, and let $\mathcal{F}_0$ be the set of those faces in $\mathcal{F}$ which can be obtained as the σ-convex face generated by a single point of K. Note that every nonempty face in $\mathcal{F}$ contains a (nonempty) face from $\mathcal{F}_0$. Thus, since $\mathcal{F}$ is a complete Boolean algebra, there exists a family $\{F_i\}$ of pairwise disjoint faces in $\mathcal{F}_0$ such that $F = \vee F_i$ in $\mathcal{F}$. For each i, F_i is the σ-convex face generated by some $x_i \in K$. Then the x_i are facially independent points of K, and F is the σ-convex face generated by $\{x_i\}$.

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