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PARTIAL REGULARITY OF SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

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At the first instant of time when a viscous incompressible fluid flow with finite kinetic energy in three space becomes singular, the singularities in space are concentrated on a closed set whose one dimensional Hausdorff measure is finite.

§1. Introduction.

Let $v: R^3 \times R^+ \to R^3$ (where $R^+ = \{t \in R: t > 0\}$ represents time) be a weak solution to the Navier–Stokes equations of incompressible viscous fluid flow in 3 dimensional euclidean space with finite initial kinetic energy and viscosity equal to 1. Our definition of weak solution coincides with Leray's definition of "solution turbulente" [**4**, pp. 240, 241, 235]. In that paper, Leray showed that weak solutions always exist for prescribed initial conditions with finite energy. He also proved the following regularity theorem:

LERAY'S THEOREM. *There exists a finite or countable sequence J_0, J_1, $J_2, \cdots$ such that $J_q \subset R^+$, $J_0 = \{t: t > a\}$ for some a, J_q is an open interval for $q > 0$, the J_q are disjointed, the Lebesgue measure of $R^+ - \bigcup_{q \geqq 0} J_q$ is zero, v can be modified on a set of Lebesgue measure zero so that its restriction to each $R^3 \times J_q$ becomes smooth, and*

$$\sum_{q>0} (\text{length}\,(J_q))^{1/2}$$

is finite.

It is not known whether there exist v with singularities ($J_0 = R^+$ is a possibility). The purpose of this paper is to prove the following theorem on the nature of possible singularities of v. We assume that v has been modified to be smooth on each $R^3 \times J_q$.

THEOREM 1. *Let t_0 be the right endpoint of an interval J_q with $q > 0$. Then there exists a closed set $S \subset R^3$ such that v can be extended to a continuous function on*

$$(R^3 \times J_q) \cup ((R^3 - S) \times \{t_0\})$$

and the 1 dimensional Hausdorff measure of S is finite.

The definition of Hausdorff measure can be found in [**2**, p. 171]. We note in passing that Leray's theorem yields

THEOREM 2. *The* 1/2 *dimensional Hausdorff measure of* $R^+ - \bigcup_{q \geqq 0} J_q$ *is zero.*

There is a proof of Theorem 2 in [**7**]. Research on the Hausdorff dimension of singularities of fluid flow was started by Mandelbrot [**5**]. The conclusion of Theorem 1 resembles the partial regularity results in [**1**, IV. 13 (6), p. 126].

Leray's theorem has been generalized by M. Shinbrot and S. Kaniel to flows on a bounded domain [**8**]. I do not know whether Theorem 1 generalizes to that case.

NOTATION. We set $(a, b) = \{t\colon a < t < b\}$, $[a, b) = \{t\colon a \leqq t < b\}$, and so on for $(a, b]$ and $[a, b]$. If f is a function defined on a subset of $R^3 \times R$ then $f_{,i}$, $f_{,ij}$, etc. are the partial derivatives $(\partial/\partial x_i)f$, $(\partial^2/\partial x_i \partial x_j)f$, etc. where x_1, x_2, x_3 are the coordinates of R^3. The partial derivative with respect to the R variable is denoted by $f_{,t}$. We set $D^0 f = f$, $D^1 f = Df = (f_{,1}, f_{,2}, f_{,3})$, $D^2 f = (f_{,ij})$ for $i, j \in \{1, 2, 3\}$, and so forth for $D^n f$. We let $|D^n f(x, t)|$ be the euclidean norm. If, in addition, f has range R^3 then f_i is the corresponding component of f for $i = 1, 2, 3$. In that case we set $\operatorname{div}(f) = \Sigma_{i=1}^3 f_{i,i}$. The summation convention for repeated indices is used throughout, e.g. $\operatorname{div}(f) = f_{i,i}$. If f is a function defined on a subset of R^3 then $Df(x)$ and $|Df(x)|$ are the gradient and its norm.

An *absolute constant* is a finite positive constant that does not depend on any of the parameters in this paper. The symbol C will always denote an absolute constant, and the value of C may change from one line to the next (e.g. $2C \leqq C$). The symbols $C_1, C_2, C_3, \cdots$ are not treated in this way, and their value does not change in the course of the paper.

We begin to prove Theorem 1. Let $\phi\colon R^3 \times \{t\colon t < 0\} \to R^+$ be defined by

$$\phi(x, t) = (2\sqrt{\pi})^{-3}(-t)^{-3/2} \exp(|x|^2/(4t)). \tag{1.1}$$

Since ϕ is just the fundamental solution to the heat equation running backwards in time, it satisfies

$$\phi_{,ii} = -\phi_{,t} \tag{1.2}$$

and

$$\lim_{\epsilon \downarrow 0} \int_{R^3} f(y, t - \epsilon)\phi(y - x, -\epsilon)dy = f(x, t)$$

if f is continuous at (x, t) and $\int_{R^3} |f(y, s)|^2 dy$ is bounded as a function of s. We also define ψ: $R^3 \times \{t: t < 0\} \to R^+$ by

$$\psi(x, t) = -(4\pi)^{-1} \int_{R^3} \phi(y, t) |y - x|^{-1} dy. \tag{1.3}$$

This Newtonian potential of ϕ satisfies the Poisson equation

$$\psi_{,ii} = \phi. \tag{1.4}$$

We have the estimates

$$\begin{aligned} |D^n \phi(x, t)| &\leqq E_n(|x|^2 - t)^{-(n+3)/2}, \\ |D^n \psi(x, t)| &\leqq E_n(|x|^2 - t)^{-(n+1)/2} \end{aligned} \tag{1.5}$$

where E_n is an absolute constant for each n.

Two consequences of the definition of weak solution are:

$$\begin{aligned} &\int_{R^3} |v(x, t)|^2 dx \leqq C_1 \quad \text{if} \quad t \in \bigcup_{q \geqq 0} J_q \\ &\int_{R^3 \times R^+} |Dv|^2 \leqq C_1 \end{aligned} \tag{1.6}$$

for some $C_1 < \infty$, and

$$\operatorname{div}(v)(x, t) = 0 \quad \text{if} \quad t \in \bigcup_{q \geqq 0} J_q. \tag{1.7}$$

A third consequence is the following lemma:

LEMMA 1.1. *If $[t_1, t_2] \subset J_q$ then for $i \in \{1, 2, 3\}$ and $x \in R^3$ we have*

$$\begin{aligned} &v_i(x, t_2) \\ &= \int_{R^3} v_i(y, t_1) \phi(y - x, t_1 - t_2) dy \\ &+ \int_{t_1}^{t_2} \int_{R^3} v_j(y, t) v_i(y, t) \phi_{,j}(y - x, t - t_2) dy dt \\ &- \int_{t_1}^{t_2} \int_{R^3} v_j(y, t) v_k(y, t) \psi_{,ijk}(y - x, t - t_2) dy dt. \end{aligned} \tag{1.8}$$

Proof. We fix $i \in \{1,2,3\}$ and $x \in R^3$. Let $f: R^3 \times \{t: t < t_2\} \to R^3$ be given by

$$(1.9)\qquad \begin{aligned} f_j(y,t) &= \phi(y-x, t-t_2) - \psi_{,ij}(y-x, t-t_2) \quad \text{if} \quad j = i, \\ f_j(y,t) &= -\psi_{,ij}(y-x, t-t_2) \quad \text{if} \quad j \neq i. \end{aligned}$$

We were careful not to write $\psi_{,ii}$ in the first identity of (1.9) because there is no summation over the index i. Using (1.4) we obtain

$$(1.10)\qquad \begin{aligned} \operatorname{div}(f)(y,t) &= \phi_{,i}(y-x, t-t_2) - \psi_{,ijj}(y-x, t-t_2) \\ &= \phi_{,i}(y-x, t-t_2) - \phi_{,i}(y-x, t-t_2) = 0. \end{aligned}$$

Now take $0 < \epsilon < t_2 - t_1$. The definition of weak solution, (1.10), and the good behavior of f on $R^3 \times [t_1, t_2 - \epsilon]$ allow us to write (see (1.6))

$$(1.11)\qquad \begin{aligned} &\int_{R^3} v_j(y, t_2-\epsilon) f_j(y, t_2-\epsilon)\,dy \\ &\quad - \int_{R^3} v_j(y, t_1) f_j(y, t_1)\,dy \\ &= \int_{R^3 \times [t_1, t_2-\epsilon]} (v_j)(f_{j,kk} + f_{j,t}) \\ &\quad - \int_{R^3 \times [t_1, t_2-\epsilon]} v_k v_{j,k} f_j. \end{aligned}$$

Integration by parts with respect to the x_j and x_k variables, (1.6), and (1.7) yield

$$(1.12)\qquad \begin{aligned} &\int_{R^3} v_j(y, t_2-\epsilon)\psi_{,ij}(y-x, -\epsilon)\,dy = 0, \\ &\int_{R^3} v_j(y, t_1)\psi_{,ij}(y-x, t_1-t_2)\,dy = 0, \\ &\int_{t_1}^{t_2-\epsilon}\int_{R^3} v_j(y,t)(\psi_{,ijkk}(y-x, t-t_2) \\ &\qquad + \psi_{,ijt}(y-x, t-t_2))\,dy\,dt = 0, \\ &\int_{R^3 \times [t_1, t_2-\epsilon]} v_k v_{j,k} f_j \\ &\qquad = -\int_{R^3 \times [t_1, t_2-\epsilon]} v_k v_j f_{j,k}. \end{aligned}$$

Identities (1.9), (1.11), (1.12), (1.2) yield

$$\begin{aligned}
&\int_{R^3} v_i(y, t_2-\epsilon)\phi(y-x,-\epsilon)dy \\
&\quad -\int_{R^3} v_i(y,t_1)\phi(y-x,t_1-t_2)dy \\
&= \int_{t_1}^{t_2-\epsilon}\int_{R^3} v_i(y,t)(\phi_{,kk}(y-x,t-t_2) \\
&\quad + \phi_{,t}(y-x,t-t_2))dydt \\
&\quad + \int_{R^3\times[t_1,t_2-\epsilon]} v_k v_j f_{j,k} \\
&= 0 + \int_{t_1}^{t_2-\epsilon}\int_{R^3} v_k(y,t)v_i(y,t)\phi_{,k}(y-x,t-t_2)dydt \\
&\quad - \int_{t_1}^{t_2-\epsilon}\int_{R^3} v_k(y,t)v_j(y,t)\psi_{,ijk}(y-x,t-t_2)dydt.
\end{aligned} \tag{1.13}$$

Now (1.13), (1.6), and (1.2) yield the conclusion of the lemma.

For $a \in R^3$ and $0 < r < \infty$ we set

$$B(a,r) = \{x \in R^3 \colon |x-a| \leqq r\}. \tag{1.14}$$

If X is a set and $f\colon X \to R$ is a function we write

$$\sup(f, X) = \text{supremum}\{f(x)\colon x \in X\}. \tag{1.15}$$

LEMMA 1.2. *Let $f\colon B(a,r) \to R$ be a smooth function and let $B(b, r/4) \subset B(a,r)$. Then*

$$\int_{B(a,r)} |f|^2 \leqq Cr^2\left(\int_{B(a,r)} |Df|^2\right) + Cr^3 \sup(|f|^2, B(b, r/4)).$$

Proof. Let $\mathscr{L}$ be the set of lines L passing through b. Let μ be the rotation invariant Radon measure on $\mathscr{L}$ that satisfies $\mu(\mathscr{L}) = 1$. For each $L \in \mathscr{L}$ the fundamental theorem of calculus yields

$$\begin{aligned}
&\int_{B(a,r)\cap L} |f|^2 \\
&\quad \leqq Cr^2\left(\int_{(B(a,r)-B(b,r/4))\cap L} |Df|^2\right) \\
&\qquad + C\sup(|f|^2, B(b,r/4)\cap L)r.
\end{aligned}$$

Hence

$$\begin{aligned}\int_{B(a,r)} |f|^2 &\leqq Cr^2 \int_{\mathscr{L}} \left(\int_{B(a,r)\cap L} |f|^2\right) d\mu \\ &\leqq Cr^4 \int_{\mathscr{L}} \left(\int_{(B(a,r)-B(b,r/4))\cap L} |Df|^2\right) d\mu \\ &\quad + Cr^3 \sup(|f|^2, B(b,r/4)) \\ &\leqq Cr^2 \left(\int_{B(a,r)-B(b,r/4)} |Df|^2\right) \\ &\quad + Cr^3 \sup(|f|^2, B(b,r/4)).\end{aligned}$$

2. The basic estimate. Throughout this section we fix $0 < d_0 < (\text{length}(J_q))^{1/2}$, where J_q is the interval in the hypotheses of Theorem 1, and we fix $x_0 \in R^3$. We define $u: R^3 \times [-1, 0) \to R^3$ by

$$u(x,t) = d_0 v(x_0 + d_0 x, t_0 + d_0^2 t), \tag{2.1}$$

where t_0 is the right endpoint of J_q as in Theorem 1, and observe that u satisfies the Navier–Stokes equations with viscosity 1 in the same way as v. Therefore Lemma 1.1 allows us to use the identity

$$\begin{aligned}u_i(x,t) &= \int_{R^3} u_i(y,-1)\phi'(y,-1)dy \\ &\quad + \left(\int_{R^3\times[-1,t]} u_j u_i \phi'_{,j}\right) \\ &\quad - \int_{R^3\times[-1,t]} u_j u_k \psi'_{,ijk}\end{aligned} \tag{2.2}$$

for $-1 < t < 0$, where

$$\phi'(y,s) = \phi(y-x, s-t),\ \psi'(y,s) = \psi(y-x, s-t). \tag{2.3}$$

We also set

$$\begin{aligned}A_p &= \{(y,s) \in R^3 \times R: |y| \leqq 1 - 2^{-p}, 2^{-2p} - 1 \leqq s < 0\} \\ B_p &= \{(y,s) \in R^3 \times R: 1 - 2^{1-p} \leqq |y| \leqq 1 + 2^{1-p}, -1 \leqq s \leqq 0\} \\ C_t &= \{(y,s) \in R^3 \times R: -1 \leqq s \leqq t\} \\ D &= \{(y,s) \in R^3 \times R: |y| \geqq 3/2, -1 \leqq s \leqq 0\} \\ E &= \{y \in R^3: |y| \geqq 3/2\} \\ F &= \{y \in R^3: |y| \leqq 2\}\end{aligned} \tag{2.4}$$

for $p = 1, 2, 3, \cdots$ and $-1 < t < 0$. In addition we set

$$A_0 = \varnothing, \quad B_{-2} = B_{-1} = B_0 = B_1. \tag{2.5}$$

LEMMA 2.1. *There exist absolute constants C_2, C_3 such that*

$$\begin{aligned} |u(x,t)| \leqq\ & C_3(t+1)^{-1/2} \int_{R^3} |u(y,-1)|^2 (1+|y|)^{-4} dy \\ & + C_3(t+1)^{-3/2} \int_{C_t} |u(y,s)|^2 (1+|y|)^{-4} dyds \\ & + C_3(t+1)^{-1/2} \int_F |Du(y,-1)|^2 dy \\ & + C_3(t+1)^{-3/2} \left(\int_{B_1 \cap C_t} |Du|^2 \right) \\ & + C_3 \left(\sum_{p=1}^{n+1} 2^{2p} \int_{B_p} |Du|^2 \right) \\ & + C_2 \left(\sum_{p=1}^{n+3} 2^{-p} \sup(|u|^2, A_p \cap C_t) \right) + C_2^{-1} 2^{-12} \end{aligned} \tag{2.6}$$

holds if $(x,t) \in A_{n+1} - A_n$ for $n \geqq 0$.

Proof. We fix $(x,t) \in A_{n+1} - A_n$ and define ϕ', ψ' as in (2.3). We set

$$G_p = \{(y,s) \in R^3 \times R : |y-x| \leqq 2^{1-p}, t - 2^{-2p} \leqq s \leqq t\} \tag{2.7}$$

for integers $p \geqq 2$. We have

$$G_{n+4} \subset G_{n+3} \subset A_{n+2} \cap C_t. \tag{2.8}$$

The integer m is defined by the relation

$$2^{4-2(m-1)} > t+1 \geqq 2^{4-2m}. \tag{2.9}$$

The requirement $(x,t) \in A_{n+1}$, (2.9), and $t+1<1$ yield

$$3 \leqq m \leqq n+3, G_p \subset C_t \quad \text{for} \quad p \geqq m. \tag{2.10}$$

For $p \in \{2, 3, 4, \cdots\}$ the point $x_p \in R^3$ is defined as follows: If $x \neq 0$ then $x_p = x - 3 \cdot 2^{-1-p} |x|^{-1} x$, and if $x = 0$ we choose x_p so that $|x_p| = 3 \cdot 2^{-1-p}$ holds. We then set

$$H_p = \{(y, s): |y - x_p| \leqq 2^{-1-p}, t - 2^{-2p} \leqq s \leqq t\}.$$

Then $H_p \subset G_p$ holds and (2.9), (2.10), and $|x| < 1$ yield

$$H_p \subset A_p \cap C_t \quad \text{for} \quad p \geqq m. \tag{2.11}$$

We set $C'_s = R^3 \times \{s\}$. For $s \in [t - 2^{-2p}, t]$ Lemma 1.2 yields

$$\int_{G_p \cap C'_s} |u|^2 \leqq C2^{-2p}\left(\int_{G_p \cap C'_s} |Du|^2\right) + C2^{-3p} \sup(|u|^2, H_p \cap C'_s). \tag{2.12}$$

Integration of (2.12) with respect to s and (2.11) yield

$$\int_{G_p} |u|^2 \leqq C2^{-2p}\left(\int_{G_p} |Du|^2\right) + C2^{-5p} \sup(|u|^2, A_p \cap C_t) \text{ if } p \geqq m. \tag{2.13}$$

Observing $G_{m+1} \subset G_m \subset B_1$, $B_1 \cup D = C_0$, $D \cap G_m = \varnothing$, we let f_1, f_2, f_3 be smooth functions from C_t into $[0,1]$ such that $f_1 + f_2 + f_3 = 1$, $f_1(y, s) = 1$ for $(y, s) \notin B_1$, $f_1(y, s) = 0$ for $(y, s) \notin D$, $f_2(y, s) = 0$ for $(y, s) \notin B_1$, $f_2(y, s) = 0$ for $(y, s) \in G_{m+1}$, $f_2(y, s) = 1$ for $(y, s) \notin D \cup G_m$, $|Df_2(y, s)| \leqq C$ for $(y, s) \in D \cap B_1$, $|Df_2(y, s)| \leqq C2^m$ for $(y, s) \in G_m - G_{m+1}$, $f_3(y, s) = 0$ for $(y, s) \notin G_m$ and $f_3(y, s) = 1$ for $(y, s) \in G_{m+1}$ (note that f_j is defined *only on* C_t). Using (1.5) and $x \in A_{n+1}$ we obtain

$$\begin{aligned} &\left|\int_{C_t} u_j u_i \phi'_{,j} f_1\right| + \left|\int_{C_t} u_j u_k \psi'_{,ijk} f_1\right| \\ &\leqq C \int_{D \cap C_t} |u(y, s)|^2 |y|^{-4} dy ds. \end{aligned} \tag{2.14}$$

We use integration by parts, (1.7), (1.5), the inequality $ab \leqq \epsilon a^2/2 + \epsilon^{-1} b^2/2$, (2.13), and (2.9) to estimate

$$\begin{aligned} &\left|\int_{C_t} u_j u_i \phi'_{,j} f_2\right| + \left|\int_{C_t} u_j u_k \psi'_{,ijk} f_2\right| \\ &\leqq \left|\int_{C_t} u_j u_{i,j} \phi' f_2\right| + \left|\int_{C_t} u_j u_i \phi' f_{2,j}\right| \\ &\quad + \left|\int_{C_t} u_j u_{k,j} \psi'_{,ik} f_2\right| + \left|\int_{C_t} u_j u_k \psi'_{,ik} f_{2,j}\right| \\ &\leqq C\left(\int_{(B_1 \cap C_t) - G_{m+1}} |u||Du|(|\phi'| + |D^2\psi'|)\right) \end{aligned}$$

$$
\begin{aligned}
&+ C\left(\int_{D\cap B_1\cap C_t} |u|^2(|\phi'| + |D^2\psi'|)\right) \\
&+ C\int_{G_m - G_{m+1}} |u|^2(|\phi'| + |D^2\psi'|)2^m \\
&\leqq C\left(\int_{B_1\cap C_t} |u||Du|2^{3m}\right) + C\left(\int_{B_1\cap C_t} |u|^2\right) + C\int_{G_m} |u|^2 2^{4m} \\
&\leqq C2^{3m}\left(\int_{B_1\cap C_t} |u|^2\right) + C2^{3m}\left(\int_{B_1\cap C_t} |Du|^2\right) \\
&+ C2^{2m}\left(\int_{G_m} |Du|^2\right) + C2^{-m}\sup(|u|^2, A_m\cap C_t) \\
&\leqq C(t+1)^{-3/2}\left(\int_{B_1\cap C_t} |u|^2\right) \\
&+ C(t+1)^{-3/2}\left(\int_{B_1\cap C_t} |Du|^2\right) \\
&+ C2^{2m}\left(\int_{G_m} |Du|^2\right) + C2^{-m}\sup(|u|^2, A_m\cap C_t).
\end{aligned} \tag{2.15}
$$

We use (2.10), (1.5), (2.13), (2.8), and (2.10) to estimate

$$
\begin{aligned}
&\left|\int_{C_t} u_j u_i \phi'_{,j} f_3\right| + \left|\int_{C_t} u_j u_k \psi'_{,ijk} f_3\right| \\
&\leqq C\int_{G_m} |u|^2(|D\phi'| + |D^3\psi'|) \\
&\leqq C\left(\sum_{p=m}^{n+3}\int_{G_p - G_{p+1}} |u|^2(|D\phi'| + |D^3\psi'|)\right) \\
&+ C\int_{G_{n+4}} |u|^2(|D\phi'| + |D^3\psi'|) \\
&\leqq C\left(\sum_{p=m}^{n+3} 2^{4p}\int_{G_p} |u|^2\right) \\
&+ C\left(\int_{G_{n+4}} |D\phi'| + |D^3\psi'|\right)\sup(|u|^2, G_{n+4}) \\
&\leqq C\left(\sum_{p=m}^{n+3} 2^{2p}\int_{G_p} |Du|^2\right) + C\left(\sum_{p=m}^{n+3} 2^{-p}\sup(|u|^2, A_p\cap C_t)\right) \\
&+ C2^{-n}\sup(|u|^2, A_{n+2}\cap C_t) \\
&\leqq C\left(\sum_{p=m}^{n+3} 2^{2p}\int_{G_p} |Du|^2\right) + C\left(\sum_{p=1}^{n+3} 2^{-p}\sup(|u|^2, A_p\cap C_t)\right).
\end{aligned} \tag{2.16}
$$

Combining (2.14), (2.15), (2.16), (2.10), $0 < t+1 < 1$, and $f_1 + f_2 + f_3 = 1$ we obtain

$$\begin{aligned} &\left| \int_{C_t} u_j u_i \phi'_{,j} \right| + \left| \int_{C_t} u_j u_k \psi'_{,ijk} \right| \\ &\leqq C(t+1)^{-3/2} \int_{C_t} |u(y,s)|^2 (1+|y|)^{-4} dy ds \\ &\quad + C(t+1)^{-3/2} \left(\int_{B_1 \cap C_t} |Du|^2 \right) \\ &\quad + C \left(\sum_{p=m}^{n+3} 2^{2p} \int_{G_p} |Du|^2 \right) \\ &\quad + C \left(\sum_{p=1}^{n+3} 2^{-p} \sup(|u|^2, A_p \cap C_t) \right). \end{aligned} \tag{2.17}$$

Since $(x,t) \notin A_n$, we know that either (I) $|x| \geqq 1 - 2^{-n}$ or (II) $t+1 \leqq 2^{-2n}$ holds. If (I) is satisfied then $G_p \subset B_{p-4}$ for $m \leqq p \leqq n+3$ (see (2.4), (2.5), (2.7), (2.10), and use $(x,t) \in A_{n+1}$) and hence (see (2.5))

$$\sum_{p=m}^{n+3} 2^{2p} \int_{G_p} |Du|^2 \leqq C \sum_{p=1}^{n+1} 2^{2p} \int_{B_p} |Du|^2 \tag{2.18}$$

if (I) holds. If, on the other hand, (II) holds then (2.9) yields $m \geqq n+2$ and hence (2.9), (2.10), and (2.7) yield

$$\sum_{p=m}^{n+3} 2^{2p} \int_{G_p} |Du|^2 \leqq C(t+1)^{-1} \int_{B_1 \cap C_t} |Du|^2 \tag{2.19}$$

if (II) holds. Hence (2.18), (2.19), and $0 < t+1 < 1$ yield

$$\sum_{p=m}^{n+3} 2^{2p} \int_{G_p} |Du|^2 \leqq C \left(\sum_{p=1}^{n+1} 2^{2p} \int_{B_p} |Du|^2 \right) + C(t+1)^{-3/2} \int_{B_1 \cap C_t} |Du|^2. \tag{2.20}$$

Let g_1, g_2 be smooth functions from R^3 into $[0,1]$ such that (see (2.4)) $g_1 + g_2 = 1$, $g_1 = 1$ outside F, $g_2 = 1$ outside E, $|Dg_1| \leqq C$, and $|Dg_2| \leqq C$. Using (1.1) (*not* (1.5)) we estimate

$$\left| \int_{R^3} u_i(y,-1) \phi'(y,-1) g_1(y) dy \right| \leqq C \int_E |u(y,-1)| |y|^{-4} dy. \tag{2.21}$$

We use the inequality

$$\int_{R^3} |f|^6 \leqq C \left(\int_{R^3} |Df|^2 \right)^3,$$

valid for smooth functions $f\colon R^3 \to R$ with compact support [**3**, p. 12], Hölder's inequality, and (1.1) to compute

$$
\begin{aligned}
&\left| \int_{R^3} u_i(y,-1)\phi'(y,-1)g_2(y)\,dy \right| \\
&\leqq \int_{R^3} |g_2(y)u(y,-1)|\,|\phi'(y,-1)|\,dy \\
&\leqq \left(\int_{R^3} |g_2(y)u(y,-1)|^6 dy\right)^{1/6} \left(\int_F |\phi'(y,-1)|^{6/5} dy\right)^{5/6} \\
&\leqq C\left(\int_{R^3} (|Dg_2(y)|\,|u(y,-1)| \right. \\
&\qquad \left. + |g_2(y)|\,|Du(y,-1)|)^2 dy\right)^{1/2} (t+1)^{-1/4} \\
&\leqq C(t+1)^{-1/4}\left(\int_F |u(y,-1)|^2 dy\right)^{1/2} \\
&\qquad + C(t+1)^{-1/4}\left(\int_F |Du(y,-1)|^2 dy\right)^{1/2}
\end{aligned}
\tag{2.22}
$$

Now we combine (2.17), (2.20), (2.21), (2.22), $g_1 + g_2 = 1$, and (2.2) to write

$$
\begin{aligned}
&|u(x,t)| \\
&\leqq C_2\left(\int_E |u(y,-1)|\,|y|^{-4} dy\right) \\
&\quad + C_2(t+1)^{-1/4}\left(\int_F |u(y,-1)|^2 dy\right)^{1/2} \\
&\quad + C_2(t+1)^{-1/4}\left(\int_F |Du(y,-1)|^2 dy\right)^{1/2} \\
&\quad + C_2(t+1)^{-3/2}\left(\int_{C_t} |u(y,s)|^2(1+|y|)^{-4} dy\,ds\right) \\
&\quad + C_2(t+1)^{-3/2}\left(\int_{B_1\cap C_t} |Du|^2\right) \\
&\quad + C_2\left(\sum_{p=1}^{n+1} 2^{2p}\int_{B_p} |Du|^2\right) \\
&\quad + C_2\left(\sum_{p=1}^{n+3} 2^{-p}\sup(|u|^2, A_p\cap C_t)\right),
\end{aligned}
\tag{2.23}
$$

where C_2 is fixed (see §1). For $\epsilon > 0$ we can use the inequality $ab \leqq \epsilon a^2/2 + \epsilon^{-1}b^2/2$ to write

$$
\begin{aligned}
&\int_E |u(y,-1)||y|^{-4}dy \\
(2.24) \qquad &= \int_E (|u(y,-1)||y|^{-2})(|y|^{-2})dy \\
&\leqq (\epsilon^{-1}/2)\left(\int_E |u(y,-1)|^2|y|^{-4}dy\right) + (\epsilon/2)\left(\int_E |y|^{-4}dy\right)
\end{aligned}
$$

and, for $w = u$ or $w = Du$,

$$
\begin{aligned}
&(t+1)^{-1/4}\left(\int_F |w(y,-1)|^2dy\right)^{1/2} \\
(2.25) \qquad &\leqq (\epsilon^{-1}/2)(t+1)^{-1/2}\left(\int_F |w(y,-1)|^2dy\right) + \epsilon/2.
\end{aligned}
$$

Since $\int_E |y|^{-4}dy$ is finite and C_2 is fixed, we can choose $\epsilon > 0$ so that

$$
(2.26) \qquad C_2\left((\epsilon/2)\left(\int_E |y|^{-4}dy\right) + \epsilon\right) \leqq C_2^{-1}2^{-12}
$$

holds. Now (2.23), (2.24), (2.25), (2.26), and $0 < t+1 < 1$ yield (2.6).

LEMMA 2.2. *There exists an absolute constant $\epsilon > 0$ such that the following holds: If the conditions*

$$
\begin{aligned}
&(t+1)^{-1}\int_{C_t} |u(y,s)|^2(1+|y|)^{-4}dyds \leqq \epsilon, \\
(2.27) \qquad &(t+1)^{-1}\int_{B_1\cap C_t} |Du|^2 \leqq \epsilon, \\
&2^p\int_{B_p} |Du|^2 \leqq \epsilon
\end{aligned}
$$

are satisfied for all $t \in (-1,0)$ and $p \in \{1,2,3,\cdots\}$ then u can be extended continuously to the closure of A_1 in $R^3 \times R$.

Proof. We choose $\epsilon > 0$ so that

$$
(2.28) \qquad (12)\, C_3\epsilon \leqq C_2^{-1}2^{-12}
$$

holds (see Lemma 2.1). Let $f\colon \bigcup_{n=1}^{\infty} A_n \to R^+$ be a continuous function satisfying

$$C_2^{-1}2^{n-10} \leqq f(x,t) \leqq C_2^{-1}2^{n-7} \quad \text{if} \quad (x,t) \in A_{n+1} - A_n, \tag{2.29}$$

where $n \geqq 0$ (see (2.5)). We wish to show that (2.27) implies

$$|u(x,t)| \leqq f(x,t) \quad \text{for all} \quad (x,t) \in \bigcup_{n=1}^{\infty} A_n. \tag{2.30}$$

Assume, to the contrary, that (2.27) holds but (2.30) does not. Since u is continuous on $R^3 \times [-1, 0)$ (see first paragraph of §2) and the continuous function $f(x,t)$ tends to ∞ as (x,t) tends to

$$\{(x,-1): |x| \leqq 1\} \cup \{(x,t): |x| = 1, -1 \leqq t < 0\},$$

there must exist $(x,t) \in \bigcup_{n=1}^{\infty} A_n$ such that (2.31) and (2.32) hold:

$$|u(x,t)| = f(x,t) \tag{2.31}$$

$$|u(y,s)| \leqq f(y,s) \quad \text{if} \quad (y,s) \in \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad s \leqq t. \tag{2.32}$$

Taking the limit as t tends to -1 in (2.27) and using Fatou's lemma we obtain (recall (2.4))

$$\begin{aligned} &\int_{R^3} |u(y,-1)|^2(1+|y|)^{-4}dy \leqq \epsilon, \\ &\int_F |Du(y,-1)|^2 dy \leqq \epsilon. \end{aligned} \tag{2.33}$$

We define n by the condition $(x,t) \in A_{n+1} - A_n$ and use Lemma 2.1, (2.33), (2.27), (2.32), the inequality $t+1 \geqq 2^{-2(n+1)}$ (which follows from $(x,t) \in A_{n+1}$), (2.29), (2.28), and $n \geqq 0$ to write

$$\begin{aligned} &|u(x,t)| \\ &\leqq 4C_3(t+1)^{-1/2}\epsilon + C_3\left(\sum_{p=1}^{n+1} 2^p\epsilon\right) \\ &\quad + C_2\left(\sum_{p=1}^{n+3} 2^{-p}\sup(f^2, A_p \cap C_t)\right) + C_2^{-1}2^{-12} \\ &\leqq C_3 2^{n+3}\epsilon + C_3 2^{n+2}\epsilon + C_2\left(\sum_{p=1}^{n+3} 2^{-p}(C_2^{-1}2^{p-8})^2\right) + C_2^{-1}2^{-12} \\ &\leqq C_2^{-1}2^{n-12} + C_2^{-1}2^{n-12} + C_2^{-1}2^{-12} \\ &\leqq (3/4)C_2^{-1}2^{n-10} \leqq (3/4)f(x,t). \end{aligned} \tag{2.34}$$

However, (2.34) contradicts (2.31) since $|u(x,t)| = f(x,t)$ is positive. Hence (2.27) implies (2.30).

We set $A = B(0,1/4)\times[-3/16,0)$ (see (1.14)). From (2.30) and (2.29) we conclude that $|u|$ is bounded on A_2. Hence the integrability of $D\phi$ and $D^3\psi$ on A (see (1.5)), the boundedness of $D\phi$, $D^3\psi$ outside A, (1.6) and (1.1) allow us to extend the domain of definition of u to include the closure of A_1 by substitution of $t = 0$ in (2.2). The above integrability property allows us to construct infinite sequences of continuous functions ${}^m f_j$ and ${}^m g_{ijk}$ for $m = 1,2,3,\cdots$ and $i,j,k \in \{1,2,3\}$ such that the restrictions of ${}^m f_j$ and ${}^m g_{ijk}$ to A converge as $m\to\infty$ to $\phi_{,j}$ and $\psi_{,ijk}$, respectively, in the L^1 norm; and such that ${}^m f_j$, ${}^m g_{ijk}$ coincide with $\phi_{,j}$, $\psi_{,ijk}$ outside A. We use (1.1), (1.5), (1.6) to define

$$
\begin{aligned}
{}^m u_i(x,t) = &\int_{R^3} u_i(y,-1)\phi'(y,-1)dy \\
&+ \int_{R^3\times[-1,t]} (u_j u_i({}^m f'_j) - u_j u_k({}^m g'_{ijk}))
\end{aligned}
$$

for $-1 < t \leqq 0$, where ϕ' is as in (2.3), ${}^m f'_j(y,s) = {}^m f_j(y-x, s-t)$, ${}^m g'_{ijk}(y,s) = {}^m g_{ijk}(y-x, s-t)$. The statements in this paragraph and (2.2) imply that ${}^m u$ converges to u uniformly on the closure of A_1. The conclusion of the lemma follows because each ${}^m u$ is continuous.

3. The basic estimate and Hausdorff measure. As before, J_q is the interval in Theorem 1, and its right endpoint is t_0. We recall (1.14) and we define $S(a,r) = \{x \in R^3 : |x-a| = r\}$ for $a \in R^3$. The integral of f over $S(a,r)$ with respect to area measure will be denoted $\int_{S(a,r)} f(x)dx$ for simplicity.

LEMMA 3.1. *There exists an absolute constant $\delta > 0$ such that the following holds: If $x_0 \in R^3$, $0 < d < (\text{length}(J_q))^{1/2}$, and condition*

$$
\begin{aligned}
&d^{-2}\int_{t_0-d^2}^{t_0}\int_{R^3} |v(x,t)|^2(1+|x-x_0|/d)^{-4}dxdt \\
&\quad + \int_{t_0-d^2}^{t_0}\int_{B(x_0,2d)} |Dv(x,t)|^2 dxdt \leqq \delta d
\end{aligned}
\tag{3.1}
$$

is satisfied then v can be extended continuously to $(R^3\times J_q)\cup(V\times\{t_0\})$, where V is a neighborhood of x_0 in R^3.

Proof. We fix $x_0 \in R^3$ and $0 < d < \text{length}(J_q)^{1/2}$, and define functions k_1, k_2: $R \to \{t \in R : t \geqq 0\}$ by (see first paragraph of §3)

$$(3.2)\quad \begin{aligned} k_1(t) &= d^{-2}\int_{R^3} |v(x,t)|^2(1+|x-x_0|/d)^{-4}dx \\ &\quad + \int_{B(x_0,2d)} |Dv(x,t)|^2 dx \quad \text{if} \quad t \in (t_0 - d^2, t_0), \\ k_2(r) &= \int_{t_0-d^2}^{t_0}\int_{S(x_0,r)} |Dv(x,t)|^2 dx dt \quad \text{if} \quad r \in (0,2d), \\ k_1(t) &= 0 = k_2(r) \quad \text{if} \quad t \notin (t_0-d^2,t_0) \quad \text{and} \quad r \notin (0,2d). \end{aligned}$$

We let Mk_ι be the cubic Hardy–Littlewood maximal function of k_ι [**9**, *p.* 53]. That is,

$$(3.3)\quad Mk_\iota(a) = \sup\{(2b)^{-1}\int_{a-b}^{a+b} k_\iota(c)dc : 0 < b < \infty\}.$$

We let $\| \ \|_1$ denote the L^1 norm and $| \ |$ denote Lebesgue measure. The Hardy–Littlewood theorem for L^1 [**9**, (3.5) on p. 55] implies that (3.4) holds for some absolute constant C_4:

$$(3.4)\quad \begin{aligned} |A| &\leqq d^2/8 \quad \text{where} \quad A = \{t : Mk_1(t) > C_4(d^2/8)^{-1}\|k_1\|_1\}, \\ |B| &\leqq d/8 \quad \text{where} \quad B = \{r : Mk_2(r) > C_4(d/8)^{-1}\|k_2\|_1\}. \end{aligned}$$

We have $|\{e \in [d/2,d] : t_0 - e^2 \in A\}| \leqq d^{-1}|A| \leqq d/8$. This and (3.4) imply the existence of $d_0 \in [d/2,d]$ such that $t_0 - d_0^2 \notin A$ and $d_0 \notin B$. Now (3.2), (3.3), and (3.4) yield

$$(3.5)\quad \begin{aligned} &(2b)^{-1}\int_{t_0-d_0^2}^{t_0-d_0^2+b} d^{-2}\int_{R^3} |v(x,t)|^2(1+|x-x_0|/d)^{-4}dxdt \\ &\quad + (2b)^{-1}\int_{t_0-d_0^2}^{t_0-d_0^2+b}\int_{B(x_0,2d)} |Dv(x,t)|^2 dxdt \\ &\leqq 8C_4 d^{-2}\|k_1\|_1 \quad \text{for} \quad 0 < b < d_0^2, \end{aligned}$$

$$(3.6)\quad \begin{aligned} &(2b)^{-1}\int_{t_0-d^2}^{t_0}\int_{d_0-b\leqq|x-x_0|\leqq d_0+b} |Dv(x,t)|^2 dxdt \\ &\leqq 8C_4 d^{-1}\|k_2\|_1 \quad \text{for} \quad 0 < b \leqq d_0. \end{aligned}$$

Defining u by means of (2.1), using $d/2 \leqq d_0 \leqq d$, rewriting (3.5) and (3.6) in terms of u, and recalling (2.4), we obtain (3.7) and (3.8):

$$(t+1)^{-1}\int_{C_t}|u(y,s)|^2(1+|y|)^{-4}dyds + (t+1)^{-1}\int_{B_1\cap C_t}|Du(y,s)|^2dyds \leqq Cd^{-1}\|k_1\|_1 \quad \text{for} \quad -1<t<0, \tag{3.7}$$

$$2^p\int_{B_p}|Du|^2 \leqq Cd^{-1}\|k_2\|_1 \quad \text{for} \quad p=1,2,3,\cdots. \tag{3.8}$$

From (3.2) we obtain

$$\begin{aligned}\|k_2\|_1 &\leqq \|k_1\|_1 \\ &= d^{-2}\int_{t_0-d^2}^{t_0}\int_{R^3}|v(x,t)|^2(1+|x-x_0|/d)^{-4}dxdt \\ &\quad + \int_{t_0-d^2}^{t_0}\int_{B(x_0,2d)}|Dv(x,t)|^2dxdt.\end{aligned} \tag{3.9}$$

Now (3.7), (3.8), and (3.9) imply the existence of an absolute constant $\delta > 0$ such that (3.1) yields (2.27). The conclusion of the lemma follows from Lemma 2.2.

We fix the constant δ in Lemma 3.1 and set

$$Q = \{(x_0, 2d) \in R^3 \times (0, 2\,(\text{length}\,(J_q))^{1/2}) \colon (3.1) \text{ does not hold}\}. \tag{3.10}$$

LEMMA 3.2. *There exists a finite constant N that depends only on* C_1 *(see* (1.6)*) such that the following holds: If*

(3.11) $0 < d < (\text{length}\,(J_q))^{1/2}, B \subset R^3, (b, 2d) \in Q \quad \text{if} \quad b \in B,$
$\{B(b,2d)\colon b \in B\}$ *is a family of disjointed sets*

is satisfied then the number of points in B is at most N/d.

Proof. Let (3.11) hold. The disjointedness hypothesis implies that (3.12) holds for some absolute constant C_5:

$$\sum_{b\in B}(1+|x-b|/d)^{-4} \leqq C_5 \quad \text{for every} \quad x \in R^3. \tag{3.12}$$

Now (3.11), (3.10), (3.12), and (1.6) yield

δd (cardinality of B)

$$= \sum_{b \in B} \delta d$$

$$\leqq \sum_{b \in B} d^{-2} \int_{t_0 - d^2}^{t_0} \int_{R^3} |v(x,t)|^2 (1 + |x - b|/d)^{-4} dx dt$$

$$+ \sum_{b \in B} \int_{t_0 - d^2}^{t_0} \int_{B(b, 2d)} |Dv(x,t)|^2 dx dt$$

$$\leqq C_5 d^{-2} \int_{t_0 - d^2}^{t_0} \int_{R^3} |v(x,t)|^2 dx dt$$

$$+ \int_{t_0 - d^2}^{t_0} \int_{R^3} |Dv(x,t)|^2 dx dt \leqq C_5 C_1 + C_1.$$

Hence we can set $N = (C_5 C_1 + C_1)/\delta$.

The following lemma is a consequence of the Besicovich covering theorem [**2**, 2.8.14, 2.8.9].

LEMMA 3.3. *There exists an integral absolute constant K with the following property: If $0 < d < \infty$ and $A \subset R^3$ then there exist $Y_k \subset A$ for $k = 1, 2, \cdots, K$ such that* (I) *and* (II) *hold*:

(I) $A \subset \cup \left\{ B(y, 2d) \colon y \in \bigcup_{k=1}^{K} Y_k \right\}$

(II) *For each k, $\{B(y, 2d) \colon y \in Y_k\}$ is a family of disjointed sets.*

We can now finish the proof of Theorem 1. Let A be the set of points $x_0 \in R^3$ such that (3.1) fails to hold for every d satisfying $0 < d < (\text{length } (J_q))^{1/2}$. Lemma 3.1 implies that there exists an open set $U \subset R^3$ such that $A \cup U = R^3$ and v can be extended to a continuous function on

$$(R^3 \times J_q) \cup (U \times \{t_0\}).$$

We set $S = R^3 - U$. Since $S \subset A$, all tht remains to show is that the 1 dimensional Hausdorff measure of A is at most $4KN$.

It suffices to show [**2**, p. 171] that for every $0 < d < (\text{length } (J_q))^{1/2}$ there exists $Y \subset R^3$ such that

$$A \subset \cup \{B(y, 2d) \colon y \in Y\}$$

and

$$\sum_{y \in Y} \text{diameter}\,(B(y, 2d)) \leqq 4KN.$$

We apply Lemma 3.3 to find sets $Y_k \subset A$ satisfying (I) and (II). Lemma

3.2, (3.10), and the definition of A yield $\Sigma_{y\in Y_k}(4d) \leqq 4N$ for each k. Hence, setting $Y = \bigcup_{k=1}^{K} Y_k$, we obtain $\Sigma_{y\in Y}(4d) \leqq 4KN$. Theorem 1 is proved.

References

1. F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Memoirs of the American Mathematical Society 165, Providence, R. I., 1976.

2. H. Federer, *Geometric Measure Theory*, Springer–Verlag, New York, 1969.

3. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, revised English edition, Gordon & Breach, New York, 1964.

4. J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math., **63** (1934), 193–248.

5. B. Mandelbrot, *Les Objets Fractals*, Flammarion, Paris, 1975.

6. V. Scheffer, *Géométrie fractale de la turbulence. Équations de Navier–Stokes et dimension de Hausdorff*, C. R. Acad. Sci. Paris, **282** (January 12, 1976), Série A 121–122.

7. ———, *Turbulence and Hausdorff dimension*, to appear in the proceedings of the conference on turbulence held at U. of Paris at Orsay in June, 1975; Lecture Notes in Mathematics, Springer–Verlag, New York.

8. M. Shinbrot, *Lectures on Fluid Mechanics*, Gordon & Breach, New York, 1973.

9. E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.

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