PACIFIC JOURNAL OF MATHEMATICS
Vol. 67, No. 2, 1976

THE ALTITUDE FORMULA AND DVR's

L. J. RATLIFF, JR.

The main theorem in this paper characterizes a local domain (R, M) which satisfies the altitude formula in terms of certain DVR's (discrete valuation rings) in the quotient field F of R. Specifically, R satisfies the altitude formula if and only if every DVRPL (V, N) over R in F (that is, (V, N) is a DVR with quotient field F such that $R \subseteqq V$, $N \cap R = M$, and V is integral over a locality over R) is of the first kind (that is, trd $(V/N)/(R/M) =$ altitude $R - 1$).

Such a characterization is of interest and importance, since it is related to the Chain Conjecture 2.18.1. Thus, by 2.19.1, the Chain Conjecture holds if and only if every DVRPL over (I, P) in F is of the first kind, where $I = R'_{M'}$ with M' a maximal ideal in the integral closure R' of a local domain R.

The theorem is also related to the following well known result [1, Proposition 4.4] or [14, Proposition 5.1]: If (R, M) is a regular local ring and (V, N) is a valuation ring in the quotient field F of R such that $R \subseteqq V$, $N \cap R = M$, and trd $(V/N)/(R/M) =$ altitude $R - 1$, then V is a DVR and the sequence of quadratic transformations of R along V is finite. In 2.22.1 it is shown that the same conclusion holds when R is an analytically unramified local domain which satisfies the altitude formula. Moreover, 2.11 implies that a form of the converse also holds, namely: If (W, Q) is a valuation ring in F such that $R \subseteqq W$, $Q \cap R = M$ and the sequence of quadratic transformations of R along W is finite, then W is a DVR, W is a locality over R, and trd$(V/N)/(R/M) =$ altitude $R - 1$ 2.22.2. A number of other corollaries to 2.11 are given in 2.12–2.22.

In § 3 we consider a closely related subject. Namely, we prove some results concerning a locality (S, P) over a local domain (R, M) such that $P \cap R = M$ and S contains elements with property (T) 3.1. For such S, S satisfies the altitude formula 3.2.1. Also, every DVRPL V^* over S in E is such that $V^* \cap F$ is a DVRPL over R of the first kind, where E and F are the quotient fields of S and R, respectively 3.2.2. Further, either R satisfies the altitude formula or, for each DVRPL V over R in F which is of the second kind, $PV[S] = V[S]$ 3.2.3. Moreover, for each maximal ideal P' in the integral closure S' of S, $S'_{P'} \cap F$ is a DVRPL over R of the first kind 3.3. It is then shown that if R is analytically unramified and E is separably generated over R, then, for each maximal ideal P' in S', $S'_{P'}$ is analytically irreducible 3.4. Two further related corollaries are also given,and then the paper is closed with an application of 3.3.

2. The altitude formula and DVR's. All rings in this paper are assumed to be commutative with an identity element. The undefined terminology is, in general, the same as that in [5].

The main result in this section is a characterization of a local domain R which satisfies the altitude formula in terms of certain DVR's in the quotient field of R 2.11. After proving 2.11, a number of corollaries are given.

To prove 2.11, some preliminary information is needed. We begin with the following definition.

DEFINITION 2.1. Let A be an integral domain and let S be a quasi-local domain.

2.1.1. S is said to be a *locality* over A in case S is a quotient ring of a finitely generated integral domain over A (hence S is Noetherian).

2.1.2. S is said to be a *pseudo-locality* over A in case there exists a finitely generated integral domain B over A such that S is a quotient ring of an integral extension domain of B.

The proof of the following remark is straightforward, so will be omitted.

REMARK 2.2. Let A and S be as in 2.1. Then the following statements hold:

2.2.1. If S is a pseudo-locality over A and T is a pseudo-locality over S, then T is a pseudo-locality over A.

2.2.2. Let S be a pseudo-locality over A, say $S = B'_Q$, where B' is integral over a finitely generated integral domain B over A and Q is a prime ideal in B'. If there exist only finitely many prime ideals Q' in B' such that $Q' \cap B = Q \cap B$, then S is integral over a locality over A. (This holds, for example, if B is Noetherian and B' is contained in a finitely generated extension field of the quotient field of B.)

DEFINITION 2.3. A valuation ring V in a field F is said to be a *DVR* (discrete valuation ring) in case V is a discrete Archimedean valuation ring (equivalently, a regular local ring of altitude one).

REMARK 2.4. If R is a local domain with quotient field F and a valuation ring $V \neq F$ in F is a pseudolocality over R, then V is a DVR. Namely, V is a quotient ring of the integral closure I of a Noetherian domain, and I is a Krull domain [5, (33.10)].

DEFINITION 2.5. Let (R, M) be a quasi-local domain, and let E be an extension field of the quotient field of R. Then, by a *DVRPL over R in E*, we mean a DVR (V, N) whose quotient field is E and which is a pseudo-locality over R such that $N \cap R = M$. If (V, N) is a DVRPL over R, then V is said to be of the *first kind* in case $\operatorname{trd}(V/N)/(R/M) = \text{altitude } R - 1$. V is said to be of the *second kind*, if V isn't of the first kind. (cf. [15, p. 95]).

The following facts concerning DVRPL's will be needed in what follows.

REMARK 2.6. Let (R, M) be a local domain, and let F be the quotient field of R. Then the following statements hold:

2.6.1. If altitude $R > 0$, then there exists a DVRPL $V \subset F$ over R which is of the first kind.

2.6.2. If (V, N) is a valuation ring in F such that $R \subseteq V$, $N \cap R = M$, and $\operatorname{trd}(V/N)/(R/M) = \text{altitude } R - 1$, then V is a DVRPL over R which is of the first kind.

2.6.3. If R is analytically unramified (that is, the M-adic completion of R contains no nonzero nilpotent elements) and E is a finite separable extension field of F, then every DVRPL over R in E is, in fact, a locality over R.

Proof. 2.6.1. If $a = \text{altitude } R = 1$, then, for each maximal ideal M' in the integral closure R' of R, $R'_{M'}$ is a DVRPL over R which is of the first kind. If $a > 1$, then let $b_1, \cdots, b_a$ be a system of parameters in R, let $y_i = b_i/b_a$ $(i = 1, \cdots, a-1)$, and let $B = R[y_1, \cdots, y_{a-1}]$. Then MB is a height one prime ideal and the MB-residue classes of the y_i are algebraically independent over R/M [9, Lemma 4.3]. Let $D = B_{MB}$, and let Q be a maximal ideal in the integral closure D' of D. Then $V = D'_Q$ is a DVRPL over R. Moreover, $\operatorname{trd}(V/QV)/(R/M) = \operatorname{trd}(B/MB)/(R/M) = a - 1$, so V is of the first kind.

2.6.2. Let (V, N) be a valuation ring in F such that $R \subseteq V$, $N \cap R = M$, and $\operatorname{trd}(V/N)/(R/M) = a - 1$ ($a = \text{altitude } R$). Let $y_1, \cdots, y_{a-1}$ in V be such that the residue classes modulo N of the y_i are a transcendence basis for V/N over R/M, and let $B = R[y_1,$

$\cdots, y_{a-1}]$. Then $MB = N \cap B$ is a prime ideal and depth $MB = a - 1$ [9, Lemma 4.2]. Let $y_i = b_i/b_a$ ($i = 1, \cdots, a - 1$), where the b_i are in M. Then, by [9, Lemma 4.3], height $MB = 1$. Therefore, since V contains the integral closure I of B_{MB}, it follows that $V = I_{N \cap I}$ (since $I_{N \cap I}$ is a DVR). Therefore V is a DVRPL over R, and V is of the first kind.

2.6.3. Assume that R is analytically unramified and that E is a finite separable extension field of F, and let (V, N) be a DVRPL over R in E. Let B be a finitely generated integral domain over R such that V is a quotient ring of an integral extension domain of B. Then, since E is finitely generated over F, it may be assumed that E is the quotient field of B. Then the integral closure B' of B is contained in V and $V = B'_{N \cap B'}$. Now B' is a finite B-algebra [2, Theorem 3], so V is a locality over R.

One more definition is needed before proving our first proposition.

DEFINITION 2.7. An integral domain A is *catenary* in case, for each pair of prime ideals $P \subset Q$ in A, all saturated chains of prime ideals between P and Q have the same length (that is, if $P = P_0 \subset P_1 \subset \cdots \subset P_n = Q$ is a chain of prime ideals in A such that height $P_i/P_{i-1} = 1$, for $i = 1, \cdots, n$, then $n =$ height Q/P).

We can now state and prove the following result.

PROPOSITION 2.8. *Let (R, M) be a local domain with quotient field F. If every* DVRPL *over R in F is of the first kind, then R is catenary.*

Proof. Assume that R isn't catenary, so there exists a prime ideal P in R such that depth $P = 1$ and height $P =$ (say)$h <$ altitude $R - 1$ [11, Remark 2.6 (i)]. Let $C = (c_1, \cdots, c_h)R \subseteq P$ such that P is a minimal prime divisor of C, and let $b \in M$ such that $C: bR$ is the P-primary component of C (so $C: b^nR = C: bR$, for all $n > 0$). Let $A = R[x_1, \cdots, x_h]$, where $x_i = c_i/b$, let $P^* = PR_b \cap A$, and let $M^* = (M, X)A$, where $X = (x_1, \cdots, x_h)A$. Then $P^* = (P, X)A$, so $A/P^* = R/P$, and so M^* is proper. Also, every element in A can be written in the form c/b^n (for all large n), where $c \in (C, b)^n$. Therefore $X \cap R = C: bR$, since $C: bR \subseteq X \cap R = \cup [C(C, b)^n: b^{n+1}R] \subseteq \cup [C: b^{n+1}R] = C: bR$. Hence $A/X = R/(C: bR)$, so height $M^*/X =$ height $M/(C: bR) = 1$ (since $C: bR$ is P-primary), so height $M^* \leqq h + 1$ [15, p. 292]. Therefore, since $P^* = (P, X)A \subset M^*$ and height $P^* =$ height $P = h$, height $M^* = h + 1$.

Now let $b_1, \cdots, b_{h+1}$ be a system of parameters in $S = A_{M^*}$, let

$y_i = b_i/b_{h+1}$, and let $B = S[y_1, \cdots, y_h]$. Then, as in the proof of 2.6.1, M^*B is a height one prime ideal and the M^*B residue classes of the y_i are algebraically independent over $S/M^*S = R/M$. Therefore, with $D = B_{M^*B}$, with D' = the integral closure of D, and with Q a maximal ideal in D', $V = D'_Q$ is a DVRPL over S of the first kind (as in the proof of 2.6.1), so

$$\operatorname{trd}(V/QV)/(R/M) = \operatorname{trd}(V/QV)/(S/M^*S) = \text{altitude } S - 1$$

$= h < a - 1$, and so V is a DVRPL over R 2.2.1 which is of the second kind. Hence, if every DVRPL over R in F is of the first kind, then R is catenary.

REMARK 2.9. The converse of 2.8 is false. That is, with the notation of 2.8, it may happen that R is catenary and there exists a DVRPL V over R in F of the second kind. For example, let R be as in [5, Example 2, pp. 203–205] in the case $m = 0$ and $r > 0$. Then R is a catenary local domain, altitude $R = r + 1 > 1$, and there exists a height one maximal ideal, say N, in the integral closure R' of R. Then $V = R'_N$ is a DVRPL over R, and $\operatorname{trd}(V/NV)/(R/M) = \operatorname{trd}(R'/N)/(R/M) = 0 <$ altitude $R - 1$, so V is of the second kind.

Before proving the main theorem in this paper, we need one more definition.

DEFINITION 2.10. An integral domain A satisfies the *altitude inequality* in case, for each finitely generated integral domain B over A and for each prime ideal P in B,

$(\sharp)$ altitude $B_P + \operatorname{trd}(B/P)/(A/(P \cap A)) \leqq$ altitude $A_{P \cap A} + \operatorname{trd} B/A$.

A satisfies the *altitude formula* in case equality always holds in $(\sharp)$.

It is known [15, Proposition 2, p. 326] that a Noetherian domain satisfies the altitude inequality.

THEOREM 2.11. *Let (R, M) be a local domain with quotient field F. Then R satisfies the altitude formula if and only if ever* DVRPL *over R in F is of the first kind.*

Proof. Assume that R satisfies the altitude formula and let $V \subset F$ be a DVRPL over R, say $V = (A_p)'$, where A is a finitely generated ring over R and p is a height one prime ideal in A 2.2.2. Then $p \cap R = M$, so, by the altitude formula, height $p + \operatorname{trd}(A/p)/(R/M) =$ height M; that is, $\operatorname{trd}(A/p)/(R/M) =$ altitude $R - 1$. Hence, since $\operatorname{trd}(V/N)/(R/M) = \operatorname{trd}(A/p)/(R/M)$, where N is the maximal ideal in V, V is a DVRPL over R of the first kind.

Conversely, assume that R doesn't satisfy the altitude formula. Then, for some prime ideal p in R such that depth $p > 1$, there exists a height one maximal ideal N' in the integral closure I of R/p [9, Theorems 2.6 and 2.19]. Therefore there exists an element $x' \in N'$ such that x' isn't in any other maximal ideal in I (since I has only finitely many maximal ideals); $x' = e'/d'$ with e' and d' in R/p. Then $(M/p, x')(R/p)[x'] = N' \cap (R/p)[x']$ is a height one maximal ideal. Let d and e be pre-images in R of d' and e', and let $A = R[x]$, where $x = e/d$. Then $pR_d \cap A =$ (say) p^* is a prime ideal such that $A/p^* \cong (R/p)[x']$, so $p^* \subset N = (M, x)A$ and height $N/p^* = 1$. We now consider two cases.

(i) If $h =$ height $N <$ height M, then let $b_1, \cdots, b_h$ be a system of parameters in $S = A_N$, and let $B = S[y_1, \cdots, y_{h-1}]$, where $y_i = b_i/b_h$. Then, as in the second paragraph of the proof of 2.8 (and since $S/NS = R/M$), there exists a DVRPL (V, N^*) over R in F such that $\operatorname{trd}(V/N^*)/(R/M) = \operatorname{trd}(B/NB)/(S/NS) = h - 1 <$ altitude $R - 1$, so V is of the second kind.

(ii) If height $N =$ height M, then $S = A_N$ isn't catenary (since height p^* + height N/p^* = height $p + 1 <$ height p + depth $p \leqq$ altitude R = height N). Therefore, by 2.8, there exists a DVRPL V over S in F of the second kind. Then V is a DVRPL over R 2.2.1 and, since $S/NS = R/M$ and height N = height M, V is of the second kind over R.

A number of corollaries to 2.11 will now be given. The first considers what can be said when R is analytically unramified.

COROLLARY 2.12. *Let (R, M) be an analytically unramified local domain, and let F be the quotient field of R. Then R satisfies the altitude formula if and only if each* DVRPL *V over R in F is a locality and is of the first kind.*

Proof. This follows immediately from 2.11, and 2.6.3.

The next corollary shows that the condition that all DVRPL's are of the first kind is inherited by localities and factor domains.

COROLLARY 2.13. *Let (R, M) be a local domain such that every DVRPL V over R in the quotient field of R is of the first kind, let P be a prime ideal in R, and let L be a locality over R/P. Then every* DVRPL *over L in the quotient field of L is of the first kind.*

Proof. By hypothesis and 2.11, R satisfies the altitude formula.

Therefore L satisfies the altitude formula [8, Corollary 3.7], so the conclusion follows from 2.11.

The next corollary shows, in particular, that the condition that all DVRPL's are of the first kind is inherited by localizations, since Noetherian rings which satisfy the altitude formula are catenary [8, Theorem 3.6].

COROLLARY 2.14. *Let A be a catenary Noetherian domain. Then, for each nonmaximal prime ideal P in A, every* DVRPL *V over A_P in the quotient field of A is of the first kind.*

Proof. Since A is catenary, A_P satisfies the altitude formula [10, Theorem 3.9]. Therefore the conclusion follows from 2.11.

The next corollary is closely related to 2.14. In the corollary, a taut semi-local domain A is considered. By definition, A is taut in case, for each prime ideal P in A, height P + depth $P \in \{1$, altitude $A\}$. Many properties of taut semi-local domains are given in [3] and in [13]. For our purposes, we need only that if P is a nonmaximal prime ideal in A, then A_P satisfies the altitude formula, by [3, Proposition 9] and [8, Theorem 3.1].

COROLLARY 2.15. *Let A be a taut semi-local domain. Then, for each nonmaximal prime ideal P in A, every* DVRPL *V over A_P in the quotient field of A is of the first kind.*

Proof. Since A is trut, A_P satisfies the altitude formula, as just noted. Therefore the conclusion follows from 2.11.

The next corollary considers $V^* \cap F$, where V^* is a DVRPL over R in a finitely generated extension field of the quotient field F of R.

COROLLARY 2.16. *Let (R, M) be a local domain which satisfies the altitude formula, and let F be the quotient field of R. Let (S, P) be a locality over R such that $P \cap R = M$, and let (V^*, N^*) be a* DVRPL *over S in its quotient field. Then $V = V^* \cap F$ is a* DVRPL *over R of the first kind.*

Proof. S satisfies the altitude formula [8, Corollary 3.7], so V^* is of the first kind over S 2.11. Also V^* is a pseudo-locality over R 2.2.1, so V^* is a pseudo-locality over $V = V^* \cap F$. Also, V is a DVR [5, 33.7]. ($V \neq F$, since $N = N^* \cap F$ is the maximal ideal in V and $N \cap R = N^* \cap S \cap R = P \cap R = M$.) Therefore, by 2.2.2, there exists a locality (L, Q) over V such that V^* is the integral closure of L (so altitude L = altitude V^* = altitude $V = 1$). Now V satisfies the altitude formula, so trd $(V^*/N^*)/(V/N)$ = trd $(L/Q)/(V/N)$ = trd L/V = trd V^*/V = trd S/R. Therefore trd $(V/N)/(R/M)$ = trd $(V^*/N^*)/(R/M)$ − trd $(V^*/N^*)/(V/N)$ = trd $(V^*/N^*)/(S/P)$ + trd $(S/P)/(R/M)$ − trd $(V^*/N^*)/(V/N)$. Now trd $(V^*/N^*)/(S/P)$ = altitude $S - 1$, and, by the altitude formula, trd $(S/P)/(R/M)$ = altitude R + trd S/R − altitude S. Therefore trd $(V/N)/(R/M)$ = (altitude $S - 1$) + (altitude R + trd S/R − altitude S) − trd S/R = altitude $R - 1$. Therefore V is a DVRPL over R of the first kind 2.6.2.

In 2.19, it will be seen that the next corollary to 2.11 gives an equivalence to the Chain Conjecture and to the Catenary Chain Conjecture (see 2.18 below).

COROLLARY 2.17. *If the integral closure R' of a local domain R is quasi-local, and if (D, Q) is a ring such that $R \subseteqq D \subseteqq R'$, then D satisfies the altitude formula if and only if every* DVRPL *over D in F is of the first kind, where F is the quotient field of R.*

Proof. Assume first that D satisfies the altitude formula and that (V, N) is a DVRPL over D in F. Then R satisfies the altitude formula [8, Theorem 3.10] and V is a DVRPL over R 2.2.1. Therefore, by 2.11, trd $(V/N)/(D/Q)$ = trd $(V/N)/(R/M)$ = altitude $R - 1$ = altitude $D - 1$, hence V is of the first kind over D.

Conversely, to prove that D satisfies the altitude formula, it suffices to prove that R does [8, Theorem 3.8]. For this, let (V, N) be a DVRPL over R in F. Then $D \subseteqq R' \subseteqq V$, so V is a DVRPL over D. Therefore, by hypothesis, V is of the first kind over D. Hence trd $(V/N)/(R/M)$ = trd $(V/N)/(D/Q)$ = altitude $D - 1$ = altitude $R - 1$. Therefore R satisfies the altitude formula 2.11, hence D satisfies the altitude formula [8, Theorem 3.8].

DEFINITION 2.18. Let R' be the integral closure of a local domain R.

2.18.1. CHAIN CONJECTURE. (cf. [12, 1.6] and [8, Theorem 3.6].) R' satisfies the altitude formula.

2.18.2. CATENARY CHAIN CONJECTURE. (cf. [12, 4.2] and [8, Theorem 3.6].) If R is catenary, then R' satisfies the altitude formula.

REMARK 2.19. It is known [8, Remark 2.9 (ii)] that an integral domain A satisfies the altitude formula if and only if A_N does, for all maximal ideals N in A. Therefore, by 2.17, the following statements hold, where R' is the integral closure of a local domain R with quotient field F:

2.19.1. The Chain Conjecture is equivalent to: If M' is a maximal ideal in R', then every DVRPL over $R'_{M'}$ in F is of the first kind.

2.19.2 The Catenary Chain Conjecture is equivalent to: If R is catenary, then, for each maximal ideal M' in R', every DVRPL over $R'_{M'}$ in F is of the first kind.

To prove our final corollary to 2.11, we need the following definition and lemma.

DEFINITION 2.20. Let (R, M) be a local domain, let (V, N) be a valuation ring such that $R \subseteqq V$ and $N \cap R = M$, let $b \in M$ such that $bV = MV$, and let $A = R[b_1/b, \cdots, b_k/b]$, where $M = (b_1, \cdots, b_k)R$. Then $R_1 = A_{N \cap A}$ is the *first quadratic transformation of R along V*. Assume that (R_i, M_i) is the ith quadratic transformation of R along V $(i \geqq 1)$. Then the $(i+1)$-*st quadratic transformation of R along V* is defined to be the first quadratic transformation of R_i along V. The chain of local rings $R = R_0 \subseteqq R_1 \subseteqq \cdots \subseteqq R_i \subseteqq \cdots \subseteqq V$ is called the *sequence of quadratic transformations of R along V*.

In [5, p. 141] it is proved that R_1 is uniquely determined by R and V (that is, is independent of the basis for M).

LEMMA 2.21. *Let (R, M) be a local domain, and let (V, N) be a valuation ring in the quotient field of R such that $R \subseteqq V$ and $N \cap R = M$. Then the sequence of quadratic transformations of R along V is finite if and only if V is a DVRPL over R and V is a locality over R.*

Proof. Assume first that the sequence of quadratic transformations of R along V is finite. Then $V = R_i$, for some $i \geqq 0$, so, by 2.2.1, V is a locality over R. Therefore V is Noetherian, so V is a DVRPL over R.

Conversely, assume that V is a DVRPL over R and a locality over R. Then V is of the form A_P, where A is a finitely generated integral domain over R, say $A = R[x_1, \cdots, x_h]$. Then it is clearly sufficient to show that each x_j is in some R_i. For this, let $x = x_j$ and assume that $x \notin R_i$. Then $x = e/d$ with $d, e \in M_i$. Let $b \in M_i$ such that $bV = M_iV$. Then $e' = e/b$ and $d' = d/b$ are in R_{i+1} and, with v denoting the valuation of V, $0 \leqq v(e') = v(e)v(b^{-1}) < v(e)$, $0 \leqq v(d') < v(d)$, and $x = e'/d'$. Hence, since V is discrete, $x \in R_k$, for some k.

We close this section of the paper with the following generalization of [1, Proposition 4.4] and of [14, Proposition 5.1] together with the converse of these two referenced results.

COROLLARY 2.22. (cf. [1, Proposition 4.4] and [14, Proposition 5.1].) *Let (R, M) be a local domain which is analytically unramified and satisfies the altitude formula, and let (V, N) be a valuation ring in the quotient field of R such that $R \subsetneqq V$ and $N \cap R = M$.*

2.22.1. *If* $\operatorname{trd}(V/N)/(R/M) = $ *altitude* $R - 1$, *then the sequence of quadratic transformations of R along V is finite and V is a* DVR.

2.22.2 *If the sequence of quadratic transformations of R along V is finite, then V is a locality over R and a* DVRPL *over R of the first kind.*

Proof. 2.22.1. Assume that $\operatorname{trd}(V/N)/(R/M) = \text{altitude } R - 1$. Then, by 2.6.2, V is a DVRPL over R. Therefore, by definition, V is of the form A'_p, where A' is the integral closure of a finitely generated integral domain A over R. Then, since R is analytically unramified, A' is a finite A-algebra [2, Theorem 3], so V is a locality over R. Hence, by 2.21, the sequence of quadratic transformations of R along V is finite.

2.22.2. Assume that the sequence of quadratic transformations of R along V is finite. Then, by 2.21, V is a locality over R and a DVRPL over R. Hence, by 2.11, V is of the first kind.

In particular, 2.22 holds for a regular local ring, since a regular local ring R is analytically irreducible [15, Corollary 1, p. 302] and satisfies the altitude formula [8, Theorem 3.1] (since R is unmixed).

3. Localities and elements with property (T). In this section we prove a Theorem 3.2 which is related to 2.11, and which shows that certain localities over a given local domain satisfy the altitude formula. Then some corollaries to 3.2 are proved.

To prove 3.2, we need the following definition.

DEFINITION 3.1. Let (S, P) be a locality over a local domain

(R, M) such that $P \cap R = M$, and let altitude $R = a$. Then it will be said that elements $y_1, \cdots, y_g$ $(g \geqq a - 1)$ in S have *property* (T) (relative to R) in case the P-residue classes of the y_i are algebraically independent over R/M and $\operatorname{trd} R[y_1, \cdots, y_g]/R = g - a + 1$.

For example, let $b_1, \cdots, b_a$ be a system of parameters in R, Let $y_i = b_i/b_a$ $(i = 1, \cdots, a - 1)$, let $y_a, \cdots, y_{a+k}$ be algebraically independent over R, and let $S = A_Q$, where $A = R[y_1, \cdots, y_{a+k}]$ and $Q = (M, y_a, \cdots, y_{a+i})A$ $(i \leqq k)$. Then $y_1, \cdots, y_{a-1}, y_{a+i+1}, \cdots, y_{a+k}$ have property (T).

It should be noted that if $y_1, \cdots, y_g$ in S are such that their residue classes modulo P are algebraically independent over R/M, then, by [9, Lemma 4.2], $MB = P \cap B$ is a prime ideal, where $B = R[y_1, \cdots, y_g]$. Therefore, by the altitude inequality for MB over R, it follows that $\operatorname{trd} B/R \geqq \text{height}\, MB + g - a \geqq g - a + 1$.

THEOREM 3.2. *Let (S, P) be a locality over a local domain (R, M) such that $P \cap R = M$, let E and F be the quotient fields of S and R, respectively, and assume that $y_1, \cdots, y_g$ in S have property* (T). *Then the following statements hold:*

3.2.1. *S satisfies the altitude formula and altitude S +* $\operatorname{trd}(S/P)/(R/M) =$ *altitude R +* $\operatorname{trd} S/R$.

3.2.2. *For each* DVRPL *V^* over S in E, V^* is of the first kind and $V^* \cap F$ is a* DVRPL *over R of the first kind.*

3.2.3. *Either R satisfies the altitude formula or, for each* DVRPL *V over R in F which is of the second kind, $PV[S] = V[S]$.*

Proof. 3.2.1. Let $B = R[y_1, \cdots, y_g]$. Then, by [9, Lemma 4.2], $MB = P \cap B$ is prime and depth $MB = g$. Also, by the altitude inequality for MB over R [15, Proposition 2, p. 326], height MB + $\operatorname{trd}(B/MB)/(R/M) \leqq$ height M + $\operatorname{trd} B/R$; that is, height MB = altitude R + (g − altitude R + 1) − $\operatorname{trd}(B/MB)/(R/M) = 1$, by 3.1. Therefore B_{MB} is Macaulay, so B_{MB} satisfies the altitude formula [5, 35.5]. Hence (*) altitude S + $\operatorname{trd}(S/P)/(B/MB)$ = altitude B_{MB} + $\operatorname{trd} S/B$. Also, $\operatorname{trd}(S/P)/(R/M) = \operatorname{trd}(S/P)/(B/MB) + g$, so altitude S + $\operatorname{trd}(S/P)/(R/M) = \operatorname{trd} S/B$ + altitude $B_{MB} + g =$ 3.1 $\operatorname{trd} S/B$ + ($\operatorname{trd} B/R$ + altitude R) = altitude R + $\operatorname{trd} S/R$. Finally, since S is a locality over B_{MB}, S satisfies the altitude formula [8, Corollary 3.7].

3.2.2. Let (V^*, N^*) be a DVRPL over S in E. Then V^* is of the first kind, by 3.2.1 and 2.11, so $\operatorname{trd}(V^*/N^*)/(S/P)$ = altitude $S - 1$. Let $V = V^* \cap F$ and $N = N^* \cap F$, so (V, N) is a DVR [5, 33.7]

($V \neq F$, since $N \cap R = N^* \cap S \cap R = M$). Therefore V satisfies the altitude formula. Now V^* is a pseudo-locality over R 2.2.1, so V^* is a pseudo-locality over V. Therefore, as in the proof of 2.16, $\operatorname{trd}(V^*/N^*)/(V/N) = \operatorname{trd} V^*/V = \operatorname{trd} S/R$.

Therefore, with B as in 3.2.1, $\operatorname{trd}(V/N)/(R/M) = \operatorname{trd}(V^*/N^*)/(R/M) - \operatorname{trd}(V^*/N^*)/(V/N) = \operatorname{trd}(V^*/N^*)/(S/P) + \operatorname{trd}(S/P)/(B/MB) + \operatorname{trd}(B/MB)/(R/M) - \operatorname{trd}(V^*/N^*)/(V/N) =$ (by (*)) (altitude $S - 1) +$ $(1 + \operatorname{trd} S/B -$ altitude $S) + g - \operatorname{trd} S/R =$ (since $g = \operatorname{trd} B/R +$ altitude $R - 1$) altitude $R - 1$. Therefore V is a DVRPL over R of the first kind 2.6.2.

3.2.3. Assume that R doesn't satisfy the altitude formula, let (V, N) be a DVRPL over R in F which is of the second kind 2.11, and suppose that $PV[S]$ is proper. Let Q be a maximal ideal in $V[S]$ such that $PV[S] \subseteq Q$, so $Q \cap S = P$, hence $Q \cap V = N$ (since $Q \cap R = Q \cap S \cap R = P \cap R = M$). Since $V[S]_Q$ is Noetherian (since S is a quotient ring of a finitely generated integral domain over R and V is Noetherian), let V^* be a DVRPL over $V[S]_Q$ in E 2.6.1. Now $V[S]_Q = S[V]_Q$ is a pseudo-locality over S (since V is integral over a locality over R 2.2.2), so V^* is a pseudo-locality over S 2.2.1, hence V^* is a DVRPL over S. Therefore V^* is of the first kind, by 3.2.2, so $V = V^* \cap F$ is a DVRPL over R of the first kind 3.2.2; contradiction. Therefore $PV[S] = V[S]$.

It is an open problem if the following holds: If (S, P) is a locality over a local domain (R, M) such that $P \cap R = M$ and $y_1, \cdots,$ y_g in S have property (T), then there exist $x_1, \cdots, x_{a-1}$ ($a =$ altitude R) in S which have property (T) (that is, the P-residue classes of the x_i are algebraically independent over R/M and $R[x_1, \cdots, x_{a-1}]$ is algebraic over R). However, the following result shows that something close to this holds.

PROPOSITION 3.3. *Let* (R, M), (S, P), F, *and* E *be as in* 3.2, *and assume that* $y_1, \cdots, y_g$ *in* S *have property* (T). *Then, for each maximal ideal* P' *in the integral closure* S' *of* S, $S'_{P'} \cap F$ *is a* DVRPL *over* R *of the first kind.*

Proof. Let $B = R[y_1, \cdots, y_g]$, and let $D = B_{MB}$, so, by the proof of 3.2.1, D is a local domain of altitude one, $D \subseteq S$, and $P \cap D$ is the maximal ideal in D. Therefore the integral closure D' of D is contained in S' and, for each maximal ideal P' in S', $P' \cap D' = Q'$ is a maximal ideal. Therefore, with F' the quotient field of B, $S'_{P'} \cap F' \supseteq D'_{Q'} =$ (say) W. Now W is a DVR, so is a maximal subring of F', so $S'_{P'} \cap F' = W$. Therefore $S'_{P'} \cap F = W \cap F =$

(say) V is a DVR ($V \neq F$, since $M = P'S'_{P'} \cap R = P'S'_{P'} \cap V \cap R$). Let N and Q be the maximal ideals in V and W, respectively. Then $\operatorname{trd}(V/N)/(R/M) = \operatorname{trd}(W/Q)/(R/M) - \operatorname{trd}(W/Q)/(V/N) = \operatorname{trd}(B/MB)/(R/M) - \operatorname{trd}(W/Q)/(V/N) =$ (as in the proof of 2.16) $g - \operatorname{trd} W/V = g - \operatorname{trd} B/R = 3.1\quad g - (g - a + 1) = a - 1$, where $a =$ altitude R. Therefore V is a DVRPL over R of the first kind 2.6.2.

It is known [4, Proposition 3, p. 417] that the completion of a separably generated integrally closed locality (L, Q) over a DVR is analytically irreducible (that is, the Q-adic completion of L is an integral domain). This will be used in the proof of the following corollary to 3.3.

COROLLARY 3.4. *Let (R, M), (S, P), F, E, $y_1, \cdots, y_g$, and S' be as in* 3.3. *Assume that R is analytically unramified and that E is separable over F. Then, for each maximal ideal P' in S', $S'_{P'}$ is a locality over the* DVR *$V = S'_{P'} \cap F$, so $S'_{P'}$ is analytically irreducible.*

Proof. Since R is analytically unramified and E is separable over F, S' is a finite S-algebra [2, Theorem 2]. Therefore, for each maximal ideal P' in S', $S'_{P'}$ is a locality over R, hence $S'_{P'}$ is a locality over $V = S'_{P'} \cap F$, and V is a DVR 3.3. Therefore, by [4, Proposition 3, p. 417], $S'_{P'}$ is analytically irreducible.

Since the completion of a semi-local domain $(U; M_1, \cdots, M_h)$ is isomorphic to the direct sum of the completions of the U_{M_i} [5, 17.7], we have the following corollary to 3.4.

COROLLARY 3.5. *With the notation of* 3.4, *if S' has exactly h maximal ideals, then the completion of S' is a direct sum of h local domains.*

Proof. This follows immediately from 3.4 and [5, 17.7].

COROLLARY 3.6. *With the notation of* 3.4, *assume further that S' is quasi-local. Then S is analytically irreducible.*

Proof. By 3.4, S' is analytically irreducible. Also, S is a subspace of S' (since S' is a finite S-algebra [2, Theorem 2]), so S is analytically irreducible.

We close this paper with the following remark.

REMARK 3.7. 3.3 shows that the main result in [6] holds without the assumption of an infinite residue field. That is, in [6] it is proved that if (R, M) is a regular local domain of altitude two such that R/M is an infinite field, then, for each finite separable extension domain A over R, the integral closure of A is a finite A-algebra and A_P is analytically unramified, for all prime ideals P in A. (This result was extended to the case R is an arbitrary regular domain in [7], and to the case R is an analytically unramified local domain in [2].) The only place in [6] where the assumption that R/M is infinite was used was in the proof of the following result [6, Lemma 4.3]: If (S, P) is a separably generated locality over a regular local domain (R, M) (altitude $R = 2$) such that $P \cap R = M$ and S contains a set of elements with property (T), then there exists a finite integral extension domain S^* of S such that S^* is contained in the integral closure S' of S and $S^*_{P^*} \cap F$ is a DVR, for each maximal ideal P^* in S^* and where F is the quotient field of R. Now, the first paragraph of the proof of [6, Lemma 4.3] shows that it may be assumed that S' is quasi-local. Therefore 3.3 shows that, if P' is the maximal ideal in S', then $V = S'_{P'} \cap F = S' \cap F$ is a DVRPL over R of the first kind. Thus, since is regular, V is a locality over R [1, Proposition 4.4], say $V = A_P$, where A is finitely generated over R. Then $S^* = S[A] \subseteqq S[V] \subseteqq S'$ (since S' is quasilocal), S^* is a finite S-algebra, and $S^* \cap F = V$ is a DVR· Therefore, 3.3 gives as alternate proof of [6, Lemma 4.3] without assuming that R/M is an infinite field.

REFERENCES

1. S. Abhyankar, *Ramification Theoretic Methods in Algebraic Geometry*, Annals of Mathematics Studies No. 43, Princeton University Press, Princeton, N. J., 1959.
2. T. Kikuchi, *On the finiteness of the derived normal ring of an affine ring*, J. Math. Soc. Japan, **15** (1963), 360-365.
3. S. McAdam and L. J. Ratliff, Jr., *Semi-local taut rings*, Indiana Univ. Math. J., forthcoming.
4. M. Nagata, *A general theory of algebraic geometry over Dedekind domains (II)*, Amer. J. Math., **80** (1958), 382-420.
5. M. Nagata, *Local Rings*, Interscience Tracts 13, Interscience Publishers, New York, N. Y., 1962.
6. L. J. Ratliff, Jr., *Affine rings over rank two regular local rings*, Amer. J. Math., **83** (1961), 709-722.
7. ———, *Separably generated spots and affine rings over regular rings*, Amer. J. Math., **85** (1963), 126-130.
8. ———, *On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals (I)*, Amer. J. Math., **91** (1969), 508-528.
9. ———, *On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals (II)*, Amer. J. Math., **92** (1970), 99-144.
10. ———, *Characterizations of catenary rings*, Amer. J. Math., **93**, (1971), 1070-1108.

11. ———, *Catenary rings and the altitude formula*, Amer, J. Math., **94** (1972), 458-466.
12. ———, *Chain conjectures and H-domains*, pp. 222-238, *Lecture Notes in Mathematics*, No. 311, Conference on Commutative Algebra, Springer-Verlag, New York, N. Y., 1973.
13. ———, *Characterizations of taut semi-local rings*, Ann. Math. Pure Appl., forthcoming.
14. D. L. Shannon, *Monoidal transforms of regular local rings*, Ph. D. Dissertation, Purdue University, Lafayette, Indiana, June, 1972.
15. O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, D. van Nostrand, Co., Inc., New York, N. Y. 1960.

Received November 18, 1974. Research on this paper was supported in part by the National Science Foundation Grant NSF GP-28939-2.

UNIVERSITY OF CALIFORNIA-RIVERSIDE