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INTEGRALS OF CONTINUOUS FUNCTIONS

MARK FINKELSTEIN AND ROBERT WHITLEY

Semicontinuous and related functions are characterized as integrals of continuous functions in several variables. For example: a new result of classical type is that the non-negative lower semicontinuous functions on the real line are exactly those functions f which can be written as

$$f(s) = \int_{-\infty}^{\infty} h(s, t)dt ,$$

with h nonnegative and continuous on $R \times R$ and $h(s, \cdot)$ integrable. There is a similar representation for functions of Baire class 0 or 1 but the integral involved is the (conditional) improper Riemann integral. Generalization leads to a concept of conditional integrals in a more general setting.

We will consider a locally compact but not compact metric space S. All functions on S will be real valued, not extended real valued.

Recall that [1, 2, 5, 6]: a function f is lower semicontinuous iff $f^{-1}(a, \infty)$ is open for each a, and a function f is l.s.c. iff there is a monotone increasing sequence of continuous functions converging pointwise to f.

THEOREM 1. *A nonnegative function f on S is lower semicontinuous iff there is a nonnegative function h, continuous on $S \times R$, with $h(s, \cdot)$ integrable for each s, and*

$$f(s) = \int_{-\infty}^{\infty} h(s, x)dx . \tag{1}$$

Proof. Suppose that (1) holds as described. The function

$$f_n(s) = \int_{-n}^{n} h(s, x)dx$$

is continuous. The sequence $\{f_n\}$ converges monotonically to f which is therefore l.s.c.

Conversely, suppose that f is l.s.c. and let $\{f_n\}$, with $f_1 = 0$, be a sequence of continuous functions increasing pointwise to f. By truncating each function f_n at $\pm n$ (redefine f_n to be n when $f_n(s) > n$ and redefine f_n to be $-n$ when $f_n(s) < -n$) we have $|f_n| \leqq n$. There is obviously a sequence of continuous functions h_n on R satisfying $0 \leqq h_n \leqq 1/(2n+1)2^n$ and $\int_{-\infty}^{\infty} h_n(x)dx = 1$. Consider

$$h(s, x) = \sum_{n=1}^{\infty} (f_{n+1}(s) - f_n(s))h_n(x) \ . \tag{2}$$

This nonnegative function is continuous on $S \times R$ because of the uniform convergence of the series. Integrate both sides of (2) and apply the monotone convergence theorem to obtain equation (1).

Theorem 1 directly gives a representation for an arbitrary lower semicontinuous function f; just choose g continuous with $g \leqq f$, then $f(s) = g(s) + \int h(s, x)dx$. Conversely any f which can be so written is lower semicontinuous.

There is an historical context into which Theorem 1 fits nicely: From Theorem 1 the problem which Young [4, pg. 151] solved of defining a new type of integral for $\bar{f}$ and $\underline{f}$ is exactly equivalent to defining $\int_a^b \int_{-\infty}^{\infty} h(x, y)dydx$ for a nonnegative continuous bounded function h on $[a, b] \times R$. This is an interesting relation between iterating (improper) integrals and extending the class of functions over which a single integral is defined.

A function f belongs to the Baire class zero if it is continuous, to Baire class one if it is discontinuous but is the pointwise limit of a sequence $\{f_n\}$ of continuous functions [1, 3, 6], Hausdorff, [3, § 41], has discussed a special subclass of Baire class $\leqq 1$, those functions of "type d", which are exactly those having the property that they can be written as the series

$$f(x) = \sum_{n=1}^{\infty} f_{n+1}(x) - f_n(x)$$

with the series being absolutely convergent. In this case the series $\sum (f_{n+1}(x) - f_n(x))^+$, $(a^+ = \max(a, 0))$, converges monotonically to a function g which is therefore lower semicontinuous, and the series $\sum (f_{n+1}(x) - f_n(x))^-$, $(a^- = -\min(a, 0))$, likewise converges to a l.s.c. function h. Thus $f = g - h$ is the difference of two l.s.c. functions and, conversely, any function which can be so written as the difference of two l.s.c. functions is of type d.

Using these elementary facts and Theorem 1, any function f of type d can be represented as the integral $\int h(s, x)dx$, h a continuous function on $S \times R$ with $h(s, \cdot)$ integrable. Conversely, any f which can be so written as an integral can be written as $\int h(s, x)^+dx - \int h(s, x)^-dx$ and so is of type d. Note that if f is of type d then the series (2) can be integrated term by term by the dominated convergence theorem.

We now want to represent the functions of Baire class one in the form (1), but in order to get the entire class, and not just those of type d, the integral involved must be a conditionally convergent integral.

THEOREM 2. *A function f belongs to Baire class* 0 *or* 1 *iff there is a function h, continuous on $S \times R$, with $h(s, \cdot)$ improperly Riemann integrable on R, and*

$$f(s) = \int_{-\infty}^{\infty} h(s, x)dx \,. \tag{3}$$

Proof. If f can be represented by (3), then $f(s) = \lim \int_{-n}^{n} h(s, x)dx$ and f belongs to either Baire class 0 or Baire class 1.

Conversely, suppose that f belongs to Baire class 0 or 1; $f(s) = \lim f_n(s)$, the pointwise limit of a sequence of continuous functions. We can take $f_1 = 0$.

As in Theorem 1, by truncating the functions f_n we may suppose that $|f_n| \leqq n$. At this point in the proof we need to use more finesse than in Theorem 1 in choosing the functions $\{h_n\}$. Let $h_n(x) = 0$ for $|x| \geqq (2n+1)2^n$, $h_n(0) = 1/(2n+1)2^n$, and h_n linear otherwise. The important properties that these continuous functions have are: $0 \leqq h_n \leqq 1/(2n+1)2^n$, $\int_{-\infty}^{\infty} h_n(x) = 1$, *and* for any $a < 0 < b$, $\int_a^b h_n(x)dx$ converges monotonically to zero as n tends to infinity.

As in Theorem 1, define $h(s, x) = \sum_{n=1}^{\infty} (f_{n+1}(s) - f_n(s))h_n(x)$, a continuous function on $S \times R$. For $a < 0 < b$, apply the bounded convergence theorem to obtain

$$\int_a^b h(s, x)dx = \sum_{n=1}^{\infty} (f_{n+1}(s) - f_n(s)) \int_a^b h_n(x)dx \,. \tag{4}$$

Let $\varepsilon > 0$ be given. As $\sum (f_{n+1}(s) - f_n(s)) = f(s)$, there is an N (which may depend on s) with $|\sum_{n=k}^{\infty} f_{n+1}(s) - f_n(s)| \leqq \varepsilon$ for $k \geqq N$. Let $a_n = f_{n+1}(s) - f_n(s)$, $b_n = \int_a^b h_n(x)dx$, and $A_{N+k} = \sum_{j=N+1}^{N+k} a_j$. To estimate $\sum_{N+1}^{N+p} a_n b_n$ use Abel's formula for summation by parts [6, II, §24]: $\sum_{N+1}^{N+p} a_n b_n = A_{N+p} b_{N+p+1} - \sum_{N+1}^{N+p} A_n(b_{n+1} - b_n)$. Since $\{b_n\}$ converges to zero monotonically, $|\sum_{N+1}^{N+p} a_n b_n| \leqq \sup_{k \geqq N} |A_k| b_{N+1} \leqq \varepsilon$. Let $a \to -\infty$ and $b \to \infty$. Then $\int_a^b h_n(x)dx \to 1$ and thus $\sum_1^N (f_{n+1}(s) - f_n(s)) \int_a^b h_n(x)dx \to \sum_1^N f_{n+1}(s) - f_n(s)$ which is within ε of $f(s)$. It follows that

$$\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b h(s, x)dx = f(s) \,.$$

THEOREM 3. *A function f on S is of Baire class n or less iff* *there is a function g continuous on $S \times R \times \cdots \times R = S \times R^n$ with*

$$f(s) = \int_R \cdots \int_R g(s, x_1, x_2, \cdots, x_n) dx_1 \cdots dx_n \,. \tag{5}$$

Proof. It will suffice to consider the case $n = 2$.

If f can be written as (5), then $f(s) = \lim_{m\to\infty} \int_{-m}^{m} \int_R g(s, x_1, x_2) dx_1 dx$ and f is a pointwise limit of functions in Baire class $\leqq 1$.

Conversely, suppose that f belongs to Baire class $\leqq 2$, $f(s) = \lim f_n(s)$, a pointwise limit of functions f_n in Baire class $\leqq 1$. As in Theorem 2, with $f_1 = 0$,

$$f(s) = \int_R \sum (f_{n+1}(s) - f_n(s)) h_n(y) dy \,.$$

The integrand is a uniformly convergent sequence of functions in Baire class 1 or 0 and so is itself in Baire class 1 or 0. Consequently the integrand can be written as

$$\int_R g(s, y, z) dz$$

with g continuous on $S \times R \times R$. This is by applying Theorem 2 to the integrand defined on $S \times R$, with S of the theorem replaced by $S \times R$. Hence f has the form (5).

The integral which occurs in (5) is the iterated improper Riemann integral; it is not, in general, absolutely convergent.

In [6, II, § 111] the notion of, say, a ul function is defined: such a function is the monotone non-increasing limit of a sequence of l.s.c. functions (which are, of course, themselves the monotone nondecreasing limit of sequences of continuous functions). Using the monotone convergence theorem twice, as in Theorem 1, any such f can be written in the form of (5) with the integral a Lebesgue integral. Similarly the other classes, lu, ulu, etc. can be so represented by iterated Lebesgue integrals.

What properties of Lebesgue measure and of the real line support these theorems? That is, for a measure space (T, μ), when can we find h continuous on $S \times T$ and represent $f(s)$ by $\int_T h(s, t) d\mu$ in place of $\int_{-\infty}^{\infty} h(s, x) dx$? We are not asking for minimal hypotheses, although that is an interesting problem, but for reasonably general sufficient conditions.

For Theorem 1 the answer is straightforward.

THEOREM 4. *Let T be a locally compact metric space which is*

σ-compact but not compact, and let μ be a regular measure on T with $\mu(T)$ infinite. A nonnegative function f on S is lower semicontinuous iff there is a nonnegative function h, continuous on $S \times T$, with $h(s, \cdot)$ integrable for each s, and

$$f(s) = \int_T h(s, t) d\mu(t) . \tag{6}$$

Proof. Suppose that (6) holds as described. The compact space T is the union $T = \cup K_n$ of a increasing sequence of compact sets K_n. Then

$$f_n(s) = \int_{K_n} h(s, t) d\mu(t)$$

is continuous. The sequence $\{f_n\}$ converges monotonically to f which is therefore l.s.c.

By the regularity of μ there is an open neighborhood U_n of K_n with $\mu(U_n) \leqq \mu(K_n) + 1 < \infty$. By the normality of T there is a continuous function g, $0 \leqq g \leqq 1$, with $g(K_n) = 1$ and $g(U_n^c) = 0$. For $c = \int_T g d\mu$, $\int_T (g/c) d\mu = 1$, and $0 \leqq g(t)/c \leqq 1/\mu(K_n) \to 0$ with n. Thus there are nonnegative continuous functions on T of integral 1 but taking on values as small as desired. Consequently there is a sequence $\{h_n\}$ of continuous functions on T satisfying $0 \leqq h_n \leqq 1/(2n + 1)2^n$ and $\int_T h_n d\mu = 1$. Proceed as in Theorem 1 to complete the proof.

The analog of Theorem 2, and thus Theorem 3, is more interesting.

The first requirement is a satisfactory definition of a conditionally convergent integral. The type of integral obtained will be governed by the choice of a collection $\{K_\alpha\}$ of compact sets.

DEFINITION. Let T be a noncompact topological space, μ a regular measure on T, and $\{K_\alpha\}$ a given collection of compact sets with the limit $\lim K_\alpha = T$, the limit taken with the direction of set inclusion. For a continuous function g the (conditional $\{K_\alpha\}$-integral) of g is defined as the generalized limit, if it exists, $\lim \int_{K_\alpha} g d\mu$ taken over the sets $\{K_\alpha\}$ directed by set inclusion.

EXAMPLE 1. The real line with Lebesgue measure.

(a) $\{K_\alpha\} = \{[a, b]\}$. The integral is the conditional improper Riemann integral.

(b) $\{K_\alpha\} = \{[-c, c], c > 0\}$. The integral is the conditional Cauchy

Principal Value.

(c) $\{K_\alpha\}$ all compact sets. The integral is the (unconditional) Lebesgue integral.

EXAMPLE 2. The plane with Lebesgue measure.

$\{K_\alpha\} = \{[a, b] \times [c, d]\}$. The integral is the improper double Riemann integral.

EXAMPLE 3. Let T be the interval (a, b), and let α be a monotone increasing real valued function on T with $\alpha(a^+) = 0$ and $\alpha(b^-) = +\infty$. Take $d\mu = d\alpha$. With $\{K_\alpha\} = \{[c, d]: a < c < d < b\}$ the integral is the improper Riemann-Stieltjes integral $\int_a^b g d\alpha$.

EXAMPLE 4. Let T be the positive integers, μ counting measure, and $K_n = \{1, 2, \cdots, n\}$. Then the $\{K_n\}$-integral is the series $\int g d\mu = \sum g(i)$; in general a conditionally, not absolutely, converging series.

Any finite products of these examples will furnish additional examples.

The second requirement for a satisfactory generalization of Theorem 2 is the existence of functions on T which behave essentially like the functions $\{h_n\}$ in the proof of Theorem 2. Theorem 5 will show that it is sufficient to have the following condition satisfied:

(*) Let T and μ be given with $\mu(T) = \infty$, and let $\{K_\alpha\}$ be the collection of compact sets which are specified in order to define the conditional integral. The condition needed is that there be a sequence of compact sets $\{C_n\}$ increasing to T with $\mu(K_\alpha \cap C_n)/\mu(C_n)$ converging monotonically (to zero) for each sufficiently large K_α ($K_\alpha \geqq K_{\alpha_0}$).

For the examples above:

(1.a) $\{C_n\} = \{[-n, n]: n = 1, 2, \cdots\}$, K_{α_0} any interval containing zero.

(1.b) As in (1.a).

(1.c) Theorem 5 shows that there is no collection $\{C_n\}$ satisfying (*); for if there were, then, as is discussed following Theorem 1, every Baire class 1 function would be a function of Hausdorff's type d.

2. $\{C_n\} = \{[-n, n] \times [-n, n]: n = 1, 2, \cdots\}$, K_{α_0} any rectangle containing the origin. This is a general result. If each member of a finite product satisfies (*), then the product does also.

3. $\{C_n\} = \{[a + 1/n, b - 1/n]: n = 1, 2, \cdots\}$, K_{α_0} any interval containing zero.

4. $C_n = K_n$.

In general, if $\{K_n\}$ is a countable monotone family of compact sets, then $C_n = K_n$ will satisfy (*). Thus, by applying Theorem 5, one can represent each Baire class 0 or 1 function as in (7). Such a representation would not, in general, be possible using the Lebesgue intesgral, as we discussed following Theorem 1.

There is a subtlety here: The smaller the family $\{K_\alpha\}$, the larger the class of continuous functions which are $\{K_\alpha\}$-integrable. One useful property of the improper Riemann integral is that the associated family $\{K_\alpha\}$ is large, and so while it integrates more continuous functions than the Lebesgue integral, still the functions which it integrates are not too badly behaved. Exactly how conflicting demands influence the "appropriate" choice of the family $\{K_\alpha\}$ is a deep matter, and our understanding of it shallow. We believe that when it is better understood, representation theorems in general settings, like Theorem 5, will be broadly used.

THEOREM 5. *Let T be a locally compact metric space which is σ-compact but not compact, and let μ be a regular measure on T with $\mu(T)$ infinite. Further suppose that there is a sequence $\{C_n\}$ of compact sets increasing monotonically to T with the property* (*).

Then a function f on S belongs to Baire class 0 or 1 iff there is a continuous function h on $S \times T$, with $h(s, \cdot)$ conditionally integrable (in the sense of the above definition) *with*

$$f(s) = \int_T h(s, t)d\mu(t) . \tag{7}$$

Proof. If f can be represented in the form (7), then $f_n(s) = \int_{C_n} h(s, t)d\mu(t)$ is a sequence of continuous functions converging to f. The function f is therefore in Baire class 0 or 1.

Conversely, suppose that f belongs to Baire class 0 or 1 and is thus the pointwise limit $f(s) = \lim f_n(s)$ of a sequence of continuous functions on S. As before, we may suppose that $|f_n| \leqq n$ and $f_1 = 0$. Set $g_n = (1/\mu(C_n))\chi_{C_n}$. Given $\delta > 0$, by Lusin's theorem there is a continuous function $w_n \geqq 0$ of compact support with $|g_n - w_n| \leqq 2/\mu(C_n)$, and $\int_T |w_n - g_n| d\mu \leqq \delta$. Let $a_n = \int w_n d\mu$. Then $1 - \delta \leqq a_n \leqq 1 + \delta$. By passing to a subsequence $\{h_n\}$ of $\{w_n/a_n\}$ and the corresponding subsequence of the $\{g_n\}$, which we will also call $\{g_n\}$, the following can be satisfied: h_n continuous,

$$0 \leqq h_n \leqq 1/(2n + 1)2^n , \quad \int_T h_n d\mu = 1 ,$$

and

$$\int_T |h_n - g_n| d\mu \leqq 1/(2n + 1)2^n .$$

Define $h(s, t) = \sum (f_{n+1}(s) - f_n(s))h_n(t)$, continuous on $S \times T$. Let K be a set in $\{K_\alpha\}$ large enough that the condition (*) on the (sub)-sequence $\{C_n\}$ is satisfied. Let $\Delta_n f(s) = f_{n+1}(s) - f_n(s)$. Given $\varepsilon > 0$, choose N so that $|\sum_{h=k}^{\infty} \Delta_n f(s)| \leqq \varepsilon$ for $k \geqq N$. Then

$$(8)\qquad \int_K h(s, t)d\mu(t) = \sum_1^\infty \Delta_n f(s)\int_K h_n(t)d\mu(t) + \sum_{N+1}^\infty \Delta_n f(s)\int_K (h_n(t) - g_n(t)d\mu(t) + \sum_{N+1}^\infty \Delta_n f(s)\int_K g_n(t)d\mu(t)\,.$$

As in the proof of Theorem 2, the last term of (8) is bounded by ε. The second term is bounded by $\sum_{N+1}^\infty |\Delta_n f(s)|\int|g_n - h_n| \leqq 1/2^N$. Letting K increase towards T, through members of $\{K_\alpha\}$, $\int_K h_n(t)d\mu(t) \rightarrow 1$, for $1 \leqq n \leqq N$, in the first term of (8). Hence $\lim \int_{K_\alpha} k(s, t)d\mu(t) = f(s)$.

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UNIVERSITY OF CALIFORNIA-IRVINE