PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 1, 1977

METRIC COMPONENTS OF CONTINUOUS IMAGES OF ORDERED COMPACTA

JOSEPH N. SIMONE

In this paper the metric component of each point of a Hausdorff space is defined. Several properties of the metric components of continuous images of ordered compacta are then established.

A *compactum* is a compact Hausdorff space and a *continuum* is a connected compactum. Any Hausdorff space which can be obtained as a continuous image of an ordered compactum will be called an IOK. Let X be a Hausdorff space. Define the relation $\sim$ on X by $x \sim y$ if and only if there exists a metric continuum in X containing x and y. For each x in X, let $M_x = \{y \in X \mid x \sim y\}$. M_x is called the *metric component* of x. In this paper we will study the properties of metric components of connected IOK's. Our first theorem follows immediately from the above definitions.

THEOREM 1. *For each Hausdorff space X, $\sim$ is an equivalence relation on X and M_x is connected for each x in X.*

In general, the metric components of connected IOK's do not have to be compact. This can be seen by considering the "long line." However, under the hypotheses of the next theorem, we obtain the desired result.

THEOREM 2. *If X is a first countable* IOK, *then M_x is a continuum for each x in X.*

Proof. Let $x \in X$. We will show that M_x is closed in X. Let y be a limit point of M_x and let $\{U_n \mid n \in N\}$ where N denotes the set of natural numbers, be a countable base at y. For each n, let $x_n \in U_n \cap M_x$ and let K_n be a metric subcontinuum of X containing x and x_n. Let $K = \mathrm{Cl} \bigcup_n K_n$. Clearly, K is a continuum and $x, y \in K$. Since each K_n is separable, it follows immediately that K is separable. However, K is a closed subset of X, and therefore K is a separable connected IOK. It follows that K is metrizable [8]. Thus, $y \in M_x$ and hence M_x is closed.

A space X is *paraseparable* (*Suslinian*) if each collection of disjoint nonempty open sets (nondegenerate continua) in X is countable. Every Suslinian continuum is paraseparable [7] and every paraseparable IOK is first countable [5]. Thus, every Suslinian connected IOK is first countable. If S is a net whose domain is the directed set D, then we will use the notation $\{S_\alpha, \alpha \in D\}$ for S. When dealing with sequences, N will always denote the set of natural numbers. If X is a space and $A \subseteqq B \subseteqq X$, then we will use the notation ∂B to denote the boundary of B in X and the notation $\partial_B A$ to denote the boundary of A in the subspace B.

The following theorem is due to A. J. Ward ([9] and [4]).

THEOREM 3. *If X is an* IOK *and $\{F_n, n \in N\}$ is a sequence of disjoint closed subsets of X, then* $\limsup F_n$ *is separable.*

We give a proof for the case when X is paraseparable, which is all that we require.

Proof. Suppose that X is a paraseparable IOK. Then the boundary of every open subset of X is separable ([5] and [3]). Now, each F_n is closed, and therefore $\partial(X - F_n)$ is separable. Since each F_n is closed, $\partial F_n = \partial(X - F_n)$, and therefore ∂F_n is separable for every n. Hence $\bigcup_n \partial F_n$ is separable and therefore $\mathrm{Cl} \bigcup_n \partial F_n$ is separable. Let $F = \mathrm{Cl} \bigcup_n F_n$. Since $\{F_n, n \in N\}$ is a sequence of disjoint closed sets, we have that

$$\mathrm{Cl} \bigcup_n F_n = \bigcup_n F_n \cup \limsup F_n \; .$$

Now, F is a closed subset of X and hence F is a IOK. Thus, F is paraseparable, and therefore each F-open set is an F_σ in F [5]. Let

$$M = \bigcup_n F_n - \limsup F_n \; .$$

Now, $F - M = \limsup F_n$, and hence M is F-open. Thus, M is an F_σ in F, and therefore $\partial_F M$ is separable [3]. Furthermore, since F is closed, $M \subseteqq F$, and M is F-open, it follows that $\partial_F M = \bar{M} \cap (F - M)$.

Let

$$S = \mathrm{Cl} \bigcup_n \partial F_n \cup \partial_F M \; .$$

Since S is the union of two closed separable subspaces of X, S is a separable IOK. We claim that $\limsup F_n \subseteqq S$. Suppose that $x \in \limsup F_n - \partial_F M$. Then $x \notin \bar{M}$. Let V be an open set such that $x \in V \subseteqq X - \bar{M}$. There exists an n_0 such that $V \cap F_{n_0} \neq \varnothing$. Since $V \cap (\bigcup_n F_n - \limsup F_n) = \varnothing$, $V \cap (F_{n_0} - \limsup F_n) = \varnothing$, and hence

$$V \cap F_{n_0} \subseteqq \limsup F_n \subseteqq \mathrm{Cl}(X - F_{n_0}) \; .$$

Therefore

$$V \cap (F_{n_0} \cap \mathrm{Cl}(X - F_{n_0})) = V \cap \partial F_{n_0} \neq \emptyset .$$

It follows that $V \cap \bigcup_n \partial F_n \neq \emptyset$, and hence $x \in \mathrm{Cl} \bigcup_n \partial F_n$. Therefore $\limsup F_n \subseteqq S$. Since every closed subset of a separable IOK is separable [5], it follows that $\limsup F_n$ is separable.

THEOREM 4. *If X is a Suslinian connected* IOK, *and $Y = \{M_x | x \in X\}$, then Y is an upper semi-continuous decomposition of X.*

Proof. By Theorem 2, each M_x is a continuum. Thus, Y is certainly a decomposition of X. Let $H \in Y$ and let U be an open set such that $H \subseteqq U$. Now, since X is Suslinian and the elements of Y are disjoint continua, Y has only countably many nondegenerate members. Let $\mathscr{G}$ denote the set of all elements M of Y such that $M \cap U \neq \emptyset$ and $M \not\subset U$. Since each element of $\mathscr{G}$ is nondegenerate, $\mathscr{G}$ is countable. Let $K = \mathrm{Cl} \cup \mathscr{G}$. We claim that $K \cap H = \emptyset$. Suppose $x \in K \cap H$. Since each element of $\mathscr{G}$ is disjoint from H, it follows that $\mathscr{G}$ is infinite. Let $\mathscr{G} = \{K_n | n \in N\}$. Since $\{K_n, n \in N\}$ is a sequence of disjoint closed sets,

$$\mathrm{Cl} \bigcup_n K_n = \bigcup_n K_n \cup \limsup K_n .$$

Since $H \cap K_n = \emptyset$ for each n, it follows that $x \in H \cap \limsup K_n$. Let $\{U_n | n \in N\}$ be a monotone decreasing countable base at x. Clearly, there exists a subsequence $\{K_{n_i}, i \in N\}$ of $\{K_n, n \in N\}$ such that $K_{n_i} \cap U_i \neq \emptyset$ for all i. It follows that $x \in \liminf K_{n_i}$. Thus, $\limsup K_{n_i}$ is a continuum. By Theorem 3, $\limsup K_{n_i}$ is separable and therefore $\limsup K_{n_i}$ is a metrizable continuum [8]. Now, for each K_{n_i}, $K_{n_i} \not\subset U$, so that $K_{n_i} \cap (X - U) \neq \emptyset$. Since $X - U$ is compact, there exists a y in $X - U$ such that $y \in \limsup K_{n_i}$. Thus, $\limsup K_{n_i}$ is a metric continuum containing x and y. However, this is impossible since H is the metric component of x, $H \subseteqq U$, and $y \notin U$. It follows that $K \cap H = \emptyset$. Let $V = U - K$. Then V is an open set, and, clearly, $H \subseteqq V \subseteqq U$. Let $L \in Y$ such that $L \cap V \neq \emptyset$. Then, $L \cap U \neq \emptyset$, and since $V = U - \mathrm{Cl} \bigcup_n K_n$, it follows that $L \subseteqq U$. Thus, Y is upper semi-continuous.

Whenever $\{A | A \in Y\}$ is a decomposition of X, it is to be assumed that Y is given the quotient topology derived from the topology of X, and that p denotes the natural map from X onto Y given by $p(x) = A$ where $x \in A \in Y$.

THEOREM 5. *Let X be a Suslinian connected* IOK *and let $Y =$*

$\{M_x \mid x \in X\}$. *If* K *is a subcontinuum of* Y, *then* $p^{-1}(K)$ *is a subcontinuum of* X.

Proof. Let K be a subcontinuum of Y, and let $K^* = p^{-1}(K)$. Since Y is upper semi-continuous, p is continuous and closed. Thus, K^* is a compact subset of X. Suppose that K^* is not connected. Then K^* is the union of two disjoint closed subsets A^* and B^* of X. Let $A = p(A^*)$ and $B = p(B^*)$. Then A and B are closed subsets of Y and $A \cup B = K$. Since K is connected we must have that $A \cap B \neq \emptyset$. Let $M_x \in A \cap B$. Then $M_x \in p(A^*) \cap p(B^*)$, and hence there exist an a in A^* and a b in B^* such that $p(a) = p(b) = M_x$. Thus, $M_x \subseteq K^* = A^* \cup B^*$. But $M_x \cap A^* \neq \emptyset$ and $M_x \cap B^* \neq \emptyset$, which contradicts the fact that M_x is connected. Therefore K^* is a continuum.

THEOREM 6. *If* X *is a Suslinian connected* IOK, *then* M_x *is metrizable for each* x *in* X.

Proof. Let $x \in X$. If $M_x = \{x\}$, then, clearly, M_x is metrizable. So suppose that M_x is nondegenerate. Let $\mathscr{S}$ denote the set of all collections of disjoint nondegenerate metric continua contained in M_x. Clearly, $\mathscr{S} \neq \emptyset$. By Zorn's Lemma it follows immediately that $\mathscr{S}$ has a maximal element $\mathscr{M}$. Since X is Suslinian, $\mathscr{M}$ is countable. Let $\mathscr{M} = \{M_n \mid n \in N'\}$, where N' is some subset of N.

Now, since each M_n is a metric continuum, each M_n has a countable dense subset D_n. Let $D = \bigcup_n D_n$. Then D is countable. We claim that D is dense in M_x. Suppose that $y \in M_x - D$. By definition there exists a metric continuum K such that $y \in K \subseteq M_x$ and $K \cap D \neq \emptyset$. Let U be any open set containing y and let V be an open set such that $y \in V$, $\bar{V} \subseteq U$ and $K \cap (X - V) \neq \emptyset$. Then $K \cap V$ is a proper K-open set. Let C be the component of y in $K \cap V$. Then $\bar{C} \cap \partial_K(K \cap V) \neq \emptyset$ [2], and hence $\bar{C}$ is a nondegenerate subcontinuum of M_x. Furthermore, $\bar{C} \subseteq K$ and therefore, $\bar{C}$ is metric. Since $\mathscr{M}$ is maximal, $\bar{C} \cap M_n \neq \emptyset$ for some n. However, $\bar{C} \subseteq U$, and hence $U \cap M_n \neq \emptyset$. But then $U \cap D \neq \emptyset$, so $y \in \bar{D}$. Thus, D is dense in M_x. Since M_x is a connected IOK, it follows that M_x is metrizable [8].

A continuum X is *netlike* if each pair of points in X can be separated by a finite set. The following theorem is proved in [7].

THEOREM 7. *If* X *is a paraseparable continuum containing no nondegenerate metric subcontinuum, then* X *is netlike if and only if it is an* IOK.

THEOREM 8. *If X is a Suslinian connected* IOK, *and* $Y = \{M_x \mid x \in X\}$, *then Y is a netlike continuum.*

Proof. Let $Y = \{M_x \mid x \in X\}$. Then Y is an upper semi-continuous decomposition of X and Y is a continuum. Let p be the natural map from X onto Y. Since Y is upper semi-continuous, p is continuous and closed. It follows from Theorem 5 that Y is Suslinian. Therefore Y is a paraseparable connected IOK.

We claim that Y contains no nondegenerate metric subcontinuum. Suppose that K is a nondegenerate metric subcontinuum of Y. Let $H = p^{-1}(K)$. By Theorem 5, H is a nondegenerate subcontinuum of X. Now, since X is Suslinian, Y has only countably many nondegenerate members. Let $\{p(x_n) \mid n \in N'\}$, where $N' \subseteqq N$, denote the set of nondegenerate elements of Y contained in H. By Theorem 6, each M_{x_n} is a metric continuum and hence contains a countable dense set D_n. Now, K is a metric subcontinuum of Y, and hence K has a countable dense subset E. Let $F = E - \{p(x_n) \mid n \in N'\}$. Thus, if $M_x \in F$, then $M_x = \{x\}$. Let $A = p^{-1}(F)$. Then A is a countable subset of X. Let $D = A \cup \bigcup_n D_n$. D is a countable set. We claim that D is dense in H. Let U be an H-open set. If $U \cap M_{x_n} \neq \emptyset$ for some n, then $U \cap D_n \neq \emptyset$ and therefore $U \cap D \neq \emptyset$. Suppose that $U \cap M_{x_n} = \emptyset$ for each n. Thus, if $x \in U$, then $p(x) = \{x\}$ and therefore $p^{-1}(p(U)) = U$. Hence

$$p(H - U) = p(H) - p(U) = K - p(U) .$$

Now, since $H - U$ is a closed set and p in a closed map, $K - p(U)$ is closed. Thus, $p(U)$ is K-open. Hence $p(U) \cap E \neq \emptyset$. However, $x \in U$ implies that $p(x) = \{x\}$, and therefore $p(U) \cap F \neq \emptyset$. Let $\{y\} \in p(U) \cap F$. Then $y \in A$ and $y \in p^{-1}(p(U)) = U$, so that $U \cap A \neq \emptyset$. Thus, $U \cap D \neq \emptyset$, and therefore D is dense in H. Since H is a separable connected IOK, H is metrizable, and therefore $p(H) = K$ is degenerate. Hence Y contains no nondegenerate metric subcontinuum. By Theorem 7, Y is netlike.

An *hereditarily locally connected* continuum is a continuum in which each subcontinuum is locally connected. By combining Theorems 4, 6 and 8 we immediately obtain the following result.

THEOREM 9. *If X is a Suslinian connected hereditarily locally connected* IOK, *then there exists an upper semi-continuous decomposition Y of X such that the space Y is a netlike continuum and each element of Y is a Peano space.*

The author would like to thank Professor A. J. Ward for his comments on this paper, and in particular for his remarks which led to the present form of Theorem 6.

REFERENCES

1. Z. Frolík, *Concerning topological convergence of sets*, Czechoslovak Math. J., **10** (85)(1960), 168-180.
2. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley Publishing Co., Reading, 1961.
3. S. Mardešić, *Continuous images of ordered compacta and a new dimension which neglects metric subcontinua*, Trans. Amer. Math. Soc., **121** (1966), 424-433.
4. ———, *On the Hahn-Mazurkiewicz problem in non-metric spaces*, General Topology and its Relation to Modern Analysis and Algebra II (1967), 248-255.
5. S. Mardešić and P. Papić, *Continuous images of ordered compacta*, Glasnik Mat.-Fiz. Astronom., **17** (1962), 3-25.
6. B. J. Pearson, *Mapping arcs and dendritic spaces onto netlike continua*, Coll. Math., **34** (1975), 39-48.
7. J. N. Simone, *Continuous images of ordered compacta and hereditarily locally connected continua*, submitted to Fund. Math.
8. L. B. Treybig, *Concerning continua which are continuous images of compact ordered spaces*, Duke Math. J., **32** (1965), 417-422.
9. A. J. Ward, *Some properties of images of ordered compacta, with special reference to topological limits*, unpublished.
10. L. E. Ward, Jr., *The Hahn-Mazurkiewicz theorem for rim-finite continua*, General Topology and Appl., **6** (1976), 183-190.
11. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., **28** (1942).

Received April 20, 1976. This paper is part of the author's doctoral dissertation under the direction of Professor B. J. Pearson.

UNIVERSITY OF MISSOURI