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THE SCHUR SUBGROUP OF THE BRAUER GROUP

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Let K be a subfield of a cyclotomic extension L of the rational field Q. The Schur subgroup, $S(K)$, of the Brauer group of K, $B(K)$, consists of those algebra classes which contain an algebra which is isomorphic to a simple component of a group algebra QG for some finite group G.

In this paper we describe a set of generators for $S(K)$. We then use this theorem to determine the 2-primary part of $S(K)$ when L/K is cyclic and the fourth roots of unity are not in K.

NOTATION. In this paper K is a field contained in $Q(\varepsilon_n)$ where ε_n is a primitive nth root of unity. The completion of K at a prime P is denoted K_P. If p is the integral prime dividing P, then the residue class degree of P over p is written $f(p) = f(p, K/Q)$. The ramification index of p in $Q(\varepsilon_n)$ over K is $e(p) = e(p, Q(\varepsilon_n)/K)$.

If A is a central simple algebra over K, then $[A]$ will denote the class of A in $B(K)$. A class $[A]$ in $B(K)$ is said to have uniformly distributed invariants of values 0 or 1/2 if for each rational prime p, $[A]$ has the same Hasse invariant at each of the primes of K which divide p, and these invariants are either 0 or 1/2. The common value of the invariant of $[A]$ at the primes of K dividing p is called the p-local invariant of $[A]$ and is denoted: $\mathrm{inv}_p[A]$.

If L is an extension field of K, then the Galois group of L over K is denoted by $\mathrm{Gal}(L/K)$, and the Frobenius automorphism of a prime p unramified in L over K is written $[L/K, p]$. Let α be a factor set $\mathrm{Gal}(L/K) \times \mathrm{Gal}(L/K)$ into L. Then the crossed product algebra made with L and α is denoted by $(L/K, \alpha)$. This is a central simple K algebra having L basis $\{u_\sigma\}$ for $\sigma \in \mathrm{Gal}(L/K)$ with multiplication given by

$$\begin{aligned} u_\sigma u_\tau &= \alpha(\sigma, \tau) u_{\sigma\tau} \\ u_\sigma x &= \sigma(x) u_\sigma \qquad \text{for} \quad \sigma, \tau \in \mathrm{Gal}(L/K), \quad x \in L . \end{aligned}$$

In case $\mathrm{Gal}(L/K) = \langle\sigma\rangle$ is cyclic, we shall write (L, σ, a) for the crossed product in which

$$\begin{aligned} (u_\sigma)^i &= u_{\sigma^i} \quad 1 \leqq i < |\sigma| \\ &= a \qquad i = |\sigma| . \end{aligned}$$

If p is a rational prime which splits into an even number of primes in K over Q, then $\Omega(p)$ denotes the class of $B(K)$ with invariant 1/2 at each of the primes of K dividing p and invariant

0 elsewhere. If p_1 and p_2 are rational primes which split into an odd number of primes in K over Q, then $\Omega(p_1, p_2)$ denotes the class in $B(K)$ with invariant 1/2 at each of the primes of K dividing p_1p_2 and invariant 0 elsewhere.

Finally $|m|_2$ denotes the highest power of 2 which divides the integer m, and $t(q) = q^{f(q)} - 1$ for all rational primes q.

2. The generator theorem. In this section we give a set of generators for $S(K)$. This is a useful refinement of a result by Janusz [6].

LEMMA 1. *Let K be a field contained in $Q(\varepsilon_n)$ where n is odd. Suppose that* $\mathrm{Gal}\,(Q(\varepsilon_n)/K) = \prod_{i=1}^{r} \langle \phi_i \rangle$ *and that* $\mathrm{Gal}\,(Q(\varepsilon_{4n})/Q(\varepsilon_n)) = \langle \rho \rangle$. *If $[Q(\varepsilon_n)/K, 2] = \prod \phi_i^{g_i}$, then the 2-local index of an algebra $(Q(\varepsilon_{4n})/K, \alpha)$ is equal to 2 if and only if $\sum g_i x_i + zf(2)$ is odd where $u_\rho u_{\phi_i} = \varepsilon_4^{x_i} u_{\phi_i} u_\rho$ and $u_\rho^2 = \varepsilon_4^{2z}$.*

Proof. Set $\eta = [Q(\varepsilon_n)/K, 2]$ and suppose that η has order s. Then $u_\eta u_\rho = \varepsilon_4^\lambda u_\rho u_\eta$ where

$$\lambda = \sum_{i=1}^{r} g_i x_i \,.$$

If λ is even we have

$$u_\rho(\varepsilon_4^{\lambda/2} u_\eta) = \varepsilon_4^{-\lambda/2}\varepsilon_4^\lambda u_\eta u_\rho = (\varepsilon_4^{\lambda/2} u_\eta) u_\rho \,.$$

Let π be a prime of K dividing 2, then

$$\begin{aligned} K_{\pi\otimes K}(Q(\varepsilon_{4\eta})/K, \alpha) &= \sum_{i=0}^{1}\sum_{j=0}^{s-1} Q_2(\varepsilon_{4\eta}) u_\rho^i u_\eta^j \\ &= \sum_{i=0}^{1}\sum_{j=0}^{s-1} K_\pi(\varepsilon_4) Q_2(\varepsilon_n) u_\rho^i (\varepsilon_4^{\lambda/2} u_\eta)^j \\ &= (\sum_{i=0}^{1} K_\pi(\varepsilon_4) u_\rho^i)(\sum_{j=0}^{s-1} Q_2(\varepsilon_n)(\varepsilon_4^{\lambda/2} u_\eta)^j) \\ &= (K_\pi(\varepsilon_\eta), \rho, u_\rho^2) \otimes_{K_\pi} (Q_2(\varepsilon_n), \eta, (\varepsilon_4^{\lambda/2} u_\eta)^s) \,. \end{aligned}$$

Now $(\varepsilon_4^{\lambda/2} u_\eta)^s$ is a root of unity and $Q_2(\varepsilon_n)$ is unramified over K_π, hence by [1, Chap. V, Thm. 9.14] $(Q_2(\varepsilon_n), \eta, (\varepsilon_4^{\lambda/2} u_\eta)^s)$ has index 1. Further

$$[(K_\pi(\varepsilon_4), \rho, \varepsilon_4^{2z})] = [K_\pi \otimes_{Q_2} (Q_2(\varepsilon_4), \rho, \varepsilon_4^{2z})]$$

and $(Q_2(\varepsilon_4), \rho, \varepsilon_4^{2z})$ has index 2 if and only if z is odd, since -1 is not a norm from $Q_2(\varepsilon_4)$. Thus $K_\pi \otimes_K (Q(\varepsilon_{4n})/K, \alpha)$ has index 2 if and only if $f(2)z$ is odd in the case that λ is even.

Now suppose that λ is odd. We have that

$$u_\rho((1+\varepsilon_4^\lambda)u_\eta) = (1+\varepsilon_4^{-\lambda})\varepsilon_4^\lambda u_\eta u_\rho = ((1+\varepsilon_4^\lambda)u_\eta)u_\rho .$$

Hence

$$[K_\pi \otimes_K (Q(\varepsilon_{4n})/K, \alpha)] = [(K_\pi(\varepsilon_4), \rho, u_\rho^2) \otimes_{K_\pi} (Q(\varepsilon_n), \eta, ((1+\varepsilon_4^\lambda)u_\eta)^s)]$$

by the same reasoning used above. We have already seen that $(K_\pi(\varepsilon_4), \rho, u_\rho^2)$ has index 2 if and only if $f(2)z$ is odd; we must look at $(Q_2(\varepsilon_n), \eta, ((1+\varepsilon_4^\lambda)u_\eta)^s)$.

Let V_L denote the exponential valuation in the 2-adic field L. Then

$$\begin{aligned} V_{K_\pi}((1+\varepsilon_4^\lambda)u_\eta)^s &= \frac{1}{2}V_{K_\pi(\varepsilon_4)}((1+\varepsilon_4^\lambda)u_\eta)^s \\ &= \frac{1}{2}V_{K_\pi(\varepsilon_4)}(1+\varepsilon_4^\lambda)^s + \frac{1}{2}V_{K_\pi(\varepsilon_4)}(u_\eta^s) \\ &= \frac{s}{2}V_{K_\pi(\varepsilon_4)}(1+\varepsilon_4^\lambda) \end{aligned}$$

since u_η^s is a unit in $K_\pi(\varepsilon_4)$. Further, $(1+\varepsilon_4^\lambda)$ is a prime element in $K_\pi(\varepsilon_4)$ since λ is odd. Thus $V_{K_\pi(\varepsilon_4)}(1+\varepsilon_4^\lambda) = 1$ and

$$V_{K_\pi}((1+\varepsilon_4^\lambda)u_\eta)^s = s/2 .$$

Hence, by the definition of the Hasse invariant,

$$\begin{aligned} \operatorname{inv}(Q_2(\varepsilon_n), \eta, ((1+\varepsilon_4^\lambda)u_\eta)^s) &= \frac{s/2}{s} \bmod Z \\ &= \frac{1}{2} \bmod Z . \end{aligned}$$

Therefore, if λ is odd, we have that the index of $K_\pi \otimes_K (Q(\varepsilon_{4n})/K, \alpha)$ is 2 if and only if $f(2)z$ is even.

This completes the proof of the lemma.

We will let $S(K)_p$ denote the p-primary part of $S(K)$, and $W(K, p)$ denote the roots of unity in K with p-power order.

THEOREM 1. *Let p be a rational prime. Then $S(K)_p$ is generated by algebra classes which contain an algebra of the form $(Q(\varepsilon_{nq})/K, \alpha)$ where the values of α are in $W(Q(\varepsilon_{nq}), p)$, q is either 4 or an odd prime, and q does not divide n.*

Proof. This is a refinement of Theorem 3 of [6]. In that theorem Janusz showed the following:

1. If p is odd, or $p = 2$ and 4 divides n, then $S(K)_p$ is generated by classes which contain algebras of the following types:

(a) $(Q(\varepsilon_{nq})/K, \alpha)$, the values of α in $W(Q(\varepsilon_n), p)$ and q a prime not dividing n.

(b) $(K(\varepsilon_{qr})/K, \beta)$, the values of β in $W(K, p)$ and q and r distinct primes not dividing n.

2. If $p = 2$ and n is odd, then $S(K)_p$ is generated by classes which contain an algebra of type (b), or of type (a′) $(Q(\varepsilon_{4nq})/K, \alpha)$, the values of α in $W(Q(\varepsilon_4), 2)$ and q an odd prime not dividing n.

In order to prove Theorem 1, we must look closely at algebras of types (b) and (a′).

Let $\mathrm{Gal}\,(K(\varepsilon_{qr})/K) = \langle\sigma\rangle \times \langle\tau\rangle$ where $\langle\sigma\rangle = \mathrm{Gal}\,(K(\varepsilon_q)/K)$ and $\langle\tau\rangle = \mathrm{Gal}\,(K(\varepsilon_r)/K)$. Also let ζ be a p^dth root of unity, the highest p-power root of unity in K. Consider the algebra

$$\varDelta_{qr} = (K(\varepsilon_{qr})/K, \beta) = \sum K(\varepsilon_{qr})u_\gamma \qquad (\gamma \in \langle\sigma\rangle \times \langle\tau\rangle)$$

where $u_\sigma u_\tau = \zeta^x u_\tau u_\sigma$, $u_\sigma^{q-1} = \zeta^y$, and $u_\tau^{r-1} = \zeta^z$. By [8, §1], the only restrictions on x, y, and z are $(\zeta^z)^{\sigma-1} = (\zeta^x)^{N(\tau)}$ and $(\zeta^y)^{\tau-1} = (\zeta^{-x})^{N(\sigma)}$ where $N(\phi) = 1 + \phi^2 + \cdots + \phi^{|\phi|-1}$. However both σ and τ fix ζ, so we get that p^d divides both $x(r-1)$ and $x(q-1)$.

Now $\varDelta_{qr}$ can have nonzero invariant only at the primes of K which divide q and r. This is because these are the only primes ramified in $K(\varepsilon_{qr})/K$.

Suppose that q is odd. Let $\tau^g = [K(\varepsilon_r)/K, q]$, the Frobenius automorphism of q in $K(\varepsilon_r)/K$, and set $t = q^{f(q)} - 1$. We have that

$$\left(\frac{\beta(\sigma, \tau^g)}{\beta(\tau^g, \sigma)}\right)^{(q-1)/t} u_\sigma^{q-1} = (\varepsilon_t)^{\mu\nu}$$

where $\mu = (q-1)/p^d$ and $\nu = xg + y(t/(q-1))$.

The inertia group of q in $K(\varepsilon_{qr})/K$ is$\langle\sigma\rangle$, so [7, Thm 3] implies that the q-local index of $\varDelta_{qr}$ is $\max\{p^{d-s}, 1\}$ where p^s exactly divides ν.

Now suppose that p^a exactly divides $f(q)$. Then p^a divides g since $[K(\varepsilon_r)/K, q] = [K(\varepsilon_r)/Q, q]^{f(q)}$. Moreover, if $p = 2$, $f(q)$ is even, and $q \equiv 3 \bmod 4$, then 2^{a+1} exactly divides $t/(q-1)$, otherwise p^a exactly divides $t/(q-1)$. In the case where $p = 2$, $f(q)$ is even and $q \equiv 3 \bmod 4$, we either have $2^d > 2$ so that x is even, or $2^d = 2$ so that $\varDelta_{qr}$ has q-local index 1.

Hence in all cases, $\max\{p^{d-s}, 1\}$ takes its highest possible value when p^s exactly divides $t/(q-1)$.

Now consider the algebra $(K(\varepsilon_q), \sigma, \zeta)$. Applying [7, Thm. 3] we see that the q-local index is $\max\{p^{d-c}, 1\}$ where p^c exactly divides $t/(q-1)$. Further, the local index of $(K(\varepsilon_q), \sigma, \zeta)$ at any prime unequal to q is 1. Note that $(K(\varepsilon_q), \sigma, \zeta)$ inflated to $Q(\varepsilon_{nq})/K$ has the form described in Theorem 1.

If r is even, then $K(\varepsilon_{qr}) = K(\varepsilon_q)$ so that the r-local index of $\varDelta_{qr}$

is 1. Thus, in this case, some power of $(K(\varepsilon_q), \sigma, \zeta)$ has exactly the same set of invariants as Δ_{qr}.

If r is odd, then we may replace q by r in the above argument. Hence, some power of $(K(\varepsilon_r), \tau, \zeta)$ has the same invariants at primes dividing r as Δ_{qr} does, and some power of $(K(\varepsilon_q), \sigma, \zeta)$ has the same invariants as Δ_{qr} at primes dividing q.

Thus $[\Delta_{qr}]$ is contained in the group generated by the classes described in the theorem.

Now suppose that $p = 2$ and n is odd. Let $G = \mathrm{Gal}\,(Q(\varepsilon_n)/K)$ be given by the direct product

$$G = \langle\phi_1\rangle \times \langle\phi_2\rangle \times \cdots \times \langle\phi_k\rangle$$

where $\langle\phi_i\rangle$ has order n_i. Further, set $\langle\rho\rangle = \mathrm{Gal}\,(Q(\varepsilon_{4n})/Q(\varepsilon_n))$ and $\langle\sigma\rangle = \mathrm{Gal}\,(Q(\varepsilon_{nq})/Q(\varepsilon_n))$, were q is an odd prime not dividing n. Let ζ be a primitive fourth root of unity.

Consider the algebra

$$\Delta_{2q} = (Q(\varepsilon_{4nq})/K, \alpha) = \sum Q(\varepsilon_{4nq})u_\gamma$$

where

$$u_\rho u_\sigma = \zeta^{x_0}u_\sigma u_\rho\ , \quad u_\rho u_{\phi_i} = \zeta^{x_i}u_{\phi_i}u_\rho\ ,$$

$$u_\sigma u_{\phi_i} = \zeta^{y_i}u_{\phi_i}u_\sigma\ , \quad u_{\phi_i}u_{\phi_j} = \zeta^{y_{ij}}u_{\phi_j}u_{\phi_i}\ ,$$

$$u_\rho^2 = \zeta^z\ , \quad u_\sigma^{q-1} = \zeta^{z_0}\ , \quad u_{\phi_i}^{n_i} = \zeta^{z_i}\ ,$$

for $i, j = 1, 2, \cdots, k$ and $i \neq j$. The conditions in [8, §1] imply that

z, y_i, and y_{ij} are even for $i, j = 1, 2, \cdots, k$ and $i \neq j$,
$2z_0 \equiv x_0(q-1) \bmod 4$,
$2z_i \equiv x_i n_i \bmod 4$ for $i = 1, 2, \cdots, k$.

We have that Δ_{2q} can have nonzero invariants only at those primes of K which divide 2, q, or some prime which ramifies in $Q(\varepsilon_n)/K$. Moreover, the invariants of Δ_{2q} can only be 0 or 1/2 since the only 2-power roots of unity in K are $\{\pm 1\}$.

Let

$$\Delta_q = (Q(\varepsilon_{nq})/K, \gamma) = \sum Q(\varepsilon_{nq})v_\tau$$

be the algebra such that

$$v_\sigma v_{\phi_i} = \zeta^{y_i}v_{\phi_i}v_\sigma\ , \quad v_{\phi_i}v_{\phi_j} = v_{\phi_j}v_{\phi_i}\ ,$$
$$v_\sigma^{q-1} = \zeta^{z_0^*}\ , \quad v_{\phi_i}^{n_i} = 1\ ,$$

for $i, j = 1, 2, \cdots, k$ where

$$z_0^* = \begin{cases} z_0 & \text{if } q \equiv 1 \bmod 4 \\ 0 & \text{if } q \equiv 3 \bmod 4 \text{ and } f(q) \text{ is even} \\ z_0 + x_0 r & \text{if } q \equiv 3 \bmod 4 \text{ and } f(q) \text{ is odd} \end{cases}$$

where

$$r^{-1} \equiv \frac{q^{f(q)} - 1}{q - 1} \bmod 4 .$$

Note that the y_i are all even, and that $z_0 + x_0 r$ is even when $q \equiv 3 \bmod 4$ and $f(q)$ is odd. Thus the values of γ are all $+1$ or -1, and Δ_q is in $S(K)$.

Further, let

$$\Delta_2 = (Q(\varepsilon_{4n})/K, \gamma') = \sum Q(\varepsilon_{4n}) w_\tau$$

be the algebra such that

$$w_\rho w_{\phi_i} = \zeta^{x_i} w_{\phi_i} w_\rho , \quad w_{\phi_i} w_{\phi_j} = \zeta^{y_{ij}} w_{\phi_j} w_{\phi_i} ,$$
$$w_\rho^2 = \zeta^{z^*} , \quad w_{\phi_i}^{n_i} = \zeta^{z_i}$$

for $i, j = 1, 2, \cdots, k$ and $i \neq j$ where

$$z^* = \begin{cases} z + x_0 & \text{if } q \equiv 3 \text{ or } 5 \bmod 8 \text{ and } f(2) \text{ is odd} \\ z & \text{otherwise} . \end{cases}$$

Observe that both Δ_q and Δ_2 belong to classes of the type described in the theorem.

Claim. The algebra Δ_{2q} is equivalent to $\Delta_2 \otimes_K \Delta_q$ in $B(K)$.

Proof of Claim. We will show that Δ_{2q} and $\Delta_2 \otimes \Delta_q$ have the same set of invariants. This is the same as showing that the local indices of these algebras are the same at q, 2, and the primes ramified in $Q(\varepsilon_n)/K$ because the invariants can be only 0 or 1/2.

First consider the q-local indices of Δ_{2q} and $\Delta_2 \otimes \Delta_q$. Let the Frobenius automorphism for q in $Q(\varepsilon_{4n})/K$ be $\eta_q = \rho^g \prod \phi_i^{g_i}$, and set $t = q^{f(q)} - 1$. Then

$$\left(\frac{\alpha(\sigma, \eta_q)}{\alpha(\eta_q, \sigma)}\right)^{(q-1)/t} u_\sigma^{q-1} = (\varepsilon_t)^{(q-1)\nu_0/4}$$

where

$$\nu_0 = g x_0 + \mu \sum g_i y_i + z_0(t/(q - 1))$$

where

$$\begin{aligned} \mu &= -1 \quad \text{if} \quad g = 1 \\ &= 1 \quad \text{if} \quad g = 0 . \end{aligned}$$

By [6, Thm. 3], the q-local index of Δ_{2q} is given by

$$\frac{q-1}{(\nu_0(q-1), q-1)} = \begin{matrix} 1 & \text{if} \quad \nu_0 \equiv 0 \bmod Z \\ 2 & \text{if} \quad \nu_0 \equiv 1/2 \bmod Z . \end{matrix}$$

Now q does not ramify in $Q(\varepsilon_{4n})/K$, so the q-local index of $\Delta_2 \otimes \Delta_q$ is equal to the q-local index of Δ_q.

The restriction of η_q to $Q(\varepsilon_n)$ is the Frobenius automorphism of q in $Q(\varepsilon_n)/K$; we will denote this by η'_q.

We have that

$$\left(\frac{\gamma(\sigma, \eta'_q)}{\gamma(\eta'_q, \sigma)}\right)^{(q-1)/t} v_\sigma^{q-1} = (\varepsilon_t)^{(q-1)\nu'_0}$$

where

$$\nu'_0 = \frac{1}{4}[\sum g_i y_i + z_0^*(t/(q-1))] .$$

Hence the q-local index of $\Delta_2 \otimes \Delta_q$ is given by

$$\frac{q-1}{(\nu'_0(q-1), q-1)} = \begin{matrix} 1 & \text{if} \quad \nu'_0 \equiv 0 \bmod Z \\ 2 & \text{if} \quad \nu'_0 \equiv 1/2 \bmod Z . \end{matrix}$$

Now if $q \equiv 1 \bmod 4$, then $g = 0$ and $z_0^* = z_0$, so $\nu_0 = \nu'_0$ and Δ_{2q} has the same q-local index as $\Delta_2 \otimes \Delta_q$. If $q \equiv 3 \bmod 4$ and $f(q)$ is even, then $g = 0$ and 4 divides $t/(q-1)$, so that $\nu' \equiv \nu'_0 \bmod Z$. Thus again Δ_{2q} and $\Delta_2 \otimes_K \Delta_q$ have the same q-local index. Finaliy suppose that $q \equiv 3 \bmod 4$ and $f(q)$ is odd. In this case $g = 1$ so that

$$g x_0 + z_0(t/(q-1)) \equiv z_0^*(t/(q-1)) \bmod 4 .$$

Hence $\nu_0 \equiv \nu'_0 \bmod Z$ and Δ_{2q} has the same q-local index as $\Delta_2 \otimes \Delta_q$.

Now let l be a prime which ramifies in $Q(\varepsilon_n)/K$. We will compare the l-local indices of Δ_{2q} and $\Delta_2 \otimes \Delta_q$. Let $\langle \omega \rangle$ be the inertia group of l in $Q(\varepsilon_n)/K$ where $\omega = \prod \phi_i^{a_i}$, and let $\eta_l = \rho^g \sigma^{g_0} \prod \phi_i^{g_i}$ be a Frobenius automorphism of l in $Q(\varepsilon_{4nq})/K$. Then $\eta'_l = \rho^g \prod \phi_i^{g_i}$ and $\eta''_l = \sigma^{g_0} \prod \phi_i^{g_i}$ are Frobenius automorphisms of l in $Q(\varepsilon_{4n})/K$ and $Q(\varepsilon_{nq})/K$ respectively. Let e be the ramification index of l in $Q(\varepsilon_n)/K$. Then we have $v_\omega^e = 1$ and $w_\omega^e = u_\omega^e$. Moreover

$$\frac{\alpha(\omega, \eta''_l)}{\alpha(\eta''_l, \omega)} = \frac{\gamma(\omega, \eta''_l)}{\gamma(\eta''_l, \omega)} \frac{\gamma'(\omega, \eta'_l)}{\gamma'(\eta'_l, \omega)} .$$

Hence, by [7, Thm. 3], we see that Δ_{2q} and $\Delta_2 \otimes \Delta_q$ have the same

l-local index.

Finally, we must compare the 2-local indices of Δ_{2q} and $\Delta_2 \otimes \Delta_q$. Let $\sigma^{g_0} \prod \phi_i^{g_i}$ be the Frobenius automorphism of 2 in $Q(\varepsilon_{nq})/K$, then Lemma 1 implies that the 2-local index of Δ_{2q} is 2 if and only if $\nu = g_0 x_0 + \sum g_i x_i + (z/2)f(2)$ is odd. Further, the 2-local index of $\Delta_2 \otimes_K \Delta_q$, which is the 2-local index of Δ_2, is 2 if and only if $\nu' = \sum x_i g_i + (z^*/2)f(2)$ is odd.

If $f(2)$ is even, then g_0 is even since

$$[Q(\varepsilon_{nq})/K, 2] = [Q(\varepsilon_{nq})/Q, 2]^{f(2)} .$$

Thus $\nu \equiv \nu' \bmod 2$ and Δ_{2q} has the same 2-local index as $\Delta_2 \otimes \Delta_q$. If $f(2)$ is odd and $q \equiv 1$ or $7 \bmod 8$, then 2 is a square modulo q, so that g must be even. Hence, once again $\nu \equiv \nu' \bmod 2$ and Δ_{2q} and $\Delta_2 \otimes \Delta_q$ have the same 2-local index. Finally suppose that $f(2)$ is odd and that $q \equiv 3$ or $5 \bmod 8$. Then g is odd and $z^* = z_0 + x_0$, so $g x_0 + (z/2)f(2)$ is equivalent to $(z^*/2)f(2)$ modulo 2. Thus again $\nu \equiv \nu' \bmod 2$.

This completes the proof of the claim and of the theorem.

3. $S(K)_2$ when $Q(\varepsilon_n)/K$ is cyclic. In this section we will completely chararacterize the classes in $S(K)_2$ by the behavior of of their invariants in the case where $\mathrm{Gal}\,(L/K)$ is cyclic. Before beginning these calculations we need to prove the following lemma.

LEMMA 2. *Suppose that $K \subset F$ are subfields of a cyclotomic field and that $[F: K]$ is not divisible by the rational prime p. If there are no p-power roots of unity in F which are not in K, then $S(F)_p = F \otimes_k S(K)_p$.*

Proof. Clearly $S(F)_p \supseteq F \otimes_k S(K)_p$. We need to show containment in the other direction.

Let L be the smallest cyclotomic field containing F, and let $G = \mathrm{Gal}\,(L/K)$ be given by

$$G = \prod_{i=1}^{t} \langle \phi_i \rangle \times \prod_{j=1}^{s} \langle \psi_j \rangle$$

where the order of each $\langle \phi_i \rangle$ is a power of p and the order of each $\langle \psi_j \rangle$, n_j, is relatively prime to p. It follows that $H = \mathrm{Gal}\,(L/K)$ is given by

$$H = \prod_{i=1}^{t} \langle \phi_i \rangle \times \prod_{j=1}^{s'} \langle \psi'_j \rangle$$

where $\prod_{j=1}^{s} \langle \psi_j \rangle$ is a subgroup of $\prod_{j=1}^{s'} \langle \psi'_j \rangle$.

By Theorem 1, $S(F)_p$ is generated by classes containing algebras of the form

$$(L(\varepsilon_q)/F, \alpha) = \sum_{\sigma} L(\varepsilon_q) U_\sigma$$

where q is either 4 or an odd prime and the values of α are p-power roots of unity.

Suppose that $U_{\psi_j}^{n_j} = \zeta^{z_j}$ where ζ is a primitive p^dth root of unity. The order of ψ_j is prime to p, so $\psi_j(\zeta) = \zeta$ unless ζ is not in F, in which case $S(F)_p = F \otimes_K S(K)_p = 1$. Set $\gamma = -z_j/n_j$ modulo p^d. Now replace U_{ψ_j} by $\zeta^\lambda U_{\psi_j}$ in $(L(\varepsilon_q)/F, \alpha)$. This gives an equivalent algebra, but now

$$(\zeta^\lambda U_{\psi_j})^{n_j} = \zeta^0 = 1 .$$

Hence we might as well have started with $z_j = 0$ for $j = 1, 2, \cdots, s$.

Now suppose that $U_{\psi_j} U_\tau = \zeta^{x_j} U_\tau U_{\psi_j}$ for some τ in $\mathrm{Gal}\,(L(\varepsilon_q)/F)$, τ not in $\langle \psi_j \rangle$. Then

$$\begin{aligned} 1 = U_{\psi_j}^{n_j} = (U_\tau^{-1} U_{\psi_j} U_\tau)^{n_j} &= \prod_{i=0}^{n_j-1} \psi_j^i(\zeta^{x_j}) \\ &= \zeta^{n_j x_j} . \end{aligned}$$

However n_j is prime to p, so x_j must be 0. Thus $U_{\psi_j} U_\tau = U_\tau U_{\psi_j}$ for all $\tau \in \mathrm{Gal}\,(L(\varepsilon_q)/F)$. This is true for all ψ_j, $j = 1, 2, \cdots, s$.

Therefore

$$[(L(\varepsilon_q)/F, \alpha)] = [(E_1/F, \alpha_1) \otimes_F (E_2/F, \alpha_2)]$$

where E_1 is the field fixed by $\prod_{i=1}^t \langle \phi_i \rangle$ and E_2 is the field fixed by $\prod_{j=1}^s \langle \psi_j \rangle$. Moreover α_1 is the trivial factor set, so $[(E_1/F, \alpha_1)] = [F]$.

Further, $[(E_2/F, \alpha_2)] = [F \otimes_K (E_2/K, \alpha_2')]$ where α_2' restricted to $\prod_{i=1}^t \langle \phi_i \rangle$ equals α_2 and α_2' is trivial on $\mathrm{Gal}\,(F/K)$. This makes α_2' a factor set by the same reasoning we used to ascertain that α is equivalent to a factor set with nontrivial values only on $\prod_{i=1}^t \langle \phi_i \rangle$.

This completes the proof of the lemma.

Notice that this lemma implies that an algebra class $[A]$ in $S(F)_p$ has q_i-local index p^{a_i} for some sets of primes $q_1, \cdots, q_t$ if and only if there is an algebra class $[D]$ in $S(K)_p$ with exactly the same local indices. Hence, if we can find the possible local indices for classes in $S(F)_p$, then we have found them for classes in $S(K)_p$.

In the following theorems we assume that $[K\colon Q]$ is even. We may do this because $S(K)$ consists of all classes in $B(K)$ with uniformly distributed invariants of value 0 or 1/2 if $[K\colon Q]$ is odd. This follows from [2].

A. $S(K)_2$ when n is odd.

THEOREM 2. *Let K be a field contained in $L = Q(\varepsilon_n)$ where n*

is odd such that Gal(L/K) *is cyclic and* $[K\colon Q]$ *is even. Then the 2-primary part of* $S(K)$ *consists of those classes* $[A]$ *in* $B(K)$ *with uniformly distributed invariants of value* 0 *or* 1/2 *which satisfy the following conditions.*

(I) *For a prime* p *which divides* n, $\mathrm{inv}_p[A] = 0$ *if* $e(P)$ *is odd or if* $[L\colon K]/e(P)$ *is even.*

(II) *For any prime* q, $\mathrm{inv}_q[A] = 0$ *if* $f(q)$ *is even and a Frobenius automorphism of* q *is a square in* Gal(L/K).

(III) *Let* p *be a prime which divides* n *to which (I) does not apply. Suppose that* $f(p)$ *is odd and that* $|(p-1)/e(p)|_2 \geqq |p'-1|_2$ *for every prime* p' *which divides* n *and is unequal to* p. *Then the invariant of* $[A]$ *is* 1/2 *at an even number of primes in the set*

$$\{p\} \cup \{\text{primes } q\colon (q/p) = -1 \text{ and } (q, n) = 1\}$$

where (q/p) *is the Legendre symbol.*

Proof. Let $G = \mathrm{Gal}(L/K)$ be $\langle\phi\rangle$ and have order $m = 2^c c'$, $(2, c') = 1$.

Step 1. We need to determine the invariants of the generators of $S(K)_2$ given in Theorem 1.

(a) Let $\Delta_q = \Delta_q(x, y, z)$ be an algebra

$$\Delta_q = (L(\varepsilon_q)/K, \alpha) = \sum_\tau L(\varepsilon_q)U_\tau$$

where q is an odd prime not dividing n and the values of α are in $\{\pm 1\}$. Let $\langle\gamma\rangle = \mathrm{Gal}(L(\varepsilon_q)/L)$. Then the factor set α is determined by the integers x, y, and z where

$$U_\gamma U_\phi = (-1)^x U_\phi U_\gamma\,,$$
$$(U_\gamma)^{q-1} = (-1)^y\,,$$
$$(U_\phi)^m = (-1)^z\,.$$

The restrictions given in [8, §1] reduce to:

$$x = 0 \quad \text{if } m \text{ is odd}\,.$$

Suppose that the Frobenius automorphism of q in L/K is ϕ^g. Set $t(q) = q^{f(q)} - 1$. Then

$$\left(\frac{\alpha(\gamma, \phi^g)}{\alpha(\phi^g, \gamma)}\right)^{(q-1)/t(q)} U_\gamma^{q-1} = (\varepsilon_{t(q)})^{((q-1)/2)\nu}$$

where $\nu = xg + y(t(q)/(q-1))$. The inertia group of q in $L(\varepsilon_q)/K$ is $\langle\gamma\rangle$, so [8, Thm. 3] implies that the q-local index of $[\Delta_q]$ is given by

$$\frac{q-1}{(\nu(q-1)/2,\, q-1)} = 1 \qquad \text{if } \nu \text{ is even}$$
$$= 2 \qquad \text{if } \nu \text{ is odd}\,.$$

Now $t(q)/(q-1)$ is odd if and only if $f(q)$ is odd, so we get

(3.1) $$\operatorname{inv}_q [\Delta_q] = 1/2 \Longleftrightarrow xg + yf(q) \quad \text{is odd}\,.$$

Now suppose that p divides n. Let $\gamma^h\phi^{h'}$ be a Frobenius automorphism for p in $L(\varepsilon_q)/K$, and let $\langle\phi^a\rangle$ be the inertia group of p in $L(\varepsilon_q)/K$. Then

$$\left(\frac{\alpha(\phi^a, \gamma^h\phi^{h'})}{\alpha(\gamma^h\phi^{h'}, \phi^a)}\right)^{e(p)/t(p)} (U_\phi^a)^{e(p)} = (\varepsilon_{t(p)})^{(e(p)/2)\nu'}$$

where $\nu' = xah + \mu z(t(p)/e(p))$,

where $$\mu = 0 \quad \text{if} \quad a = 0$$
$$= 1 \quad \text{if} \quad a \neq 0\,.$$

Thus the p-local index of $[\Delta_q]$ is given by

$$\frac{e(p)}{(\nu' e(p)/2,\, e(p))} = 1 \qquad \text{if } \nu' \text{ is even}$$
$$= 2 \qquad \text{if } \nu' \text{ is odd}\,.$$

Hence

(3.2) $$\operatorname{inv}_p [\Delta_q] = 1/2 \Leftrightarrow xah + \mu z\left(\frac{t(p)}{e(p)}\right) \quad \text{is odd}\,.$$

(b) Let $\Delta_2 = \Delta_2(x, y, z)$ be the algebra

$$\Delta_2 = (L(\varepsilon_4)/K, \alpha) = \sum_\tau L(\varepsilon_4)U_\tau$$

where the values of α are in $\{\pm 1, \pm\varepsilon_4\}$. If $\langle\rho\rangle = \operatorname{Gal}(L(\varepsilon_4)/L)$, then the factor set α is determined by the integers x, y, and z where

$$U_\rho U_\phi = (\varepsilon_4)^x U_\phi U_\rho\,,$$
$$(U_\rho)^2 = (\varepsilon_4)^y\,,$$
$$(U_\phi)^m = (\varepsilon_4)^z\,.$$

The restrictions on x, y, and z are

(3.3) $$\begin{aligned} &y \text{ is even} \\ &xm + 2z \equiv 0 \bmod 4\,. \end{aligned}$$

Let $[L/K, 2] = \phi^g$. Then by Lemma 1, (3.4) $$\mathrm{inv}_2\,[\Delta_2] = 1/2 \Leftrightarrow xy + (y/2)f(2) \quad \text{is odd}\,.$$

Now let p be a prime dividing n. Let $\rho^k\phi^{k'}$ be a Frobenius automorphism of p in $L(\varepsilon_4)/K$, and let $\langle\phi^a\rangle$ be the inertia group of p in L/K. Then

$$\left(\frac{\alpha(\phi^a, \rho^k\phi^{k'})}{\alpha(\rho^k\phi^{k'}, \phi^a)}\right)^{e(p)/t(p)}(U_\phi^a)^{e(p)} = (\varepsilon_{t(p)})^{(e(p)/4)\nu''}$$

where

$$\nu'' = xak + \mu z\left(\frac{t(p)}{e(p)}\right)$$

where

$$\begin{aligned} \mu &= 0 \quad \text{if} \quad a = 0 \\ &= 1 \quad \text{if} \quad a \neq 0\,. \end{aligned}$$

Thus

(3.5) $$\mathrm{inv}_p\,[\Delta_2] = 1/2 \Leftrightarrow \frac{xak}{2} + \frac{\mu z}{2}\left(\frac{t(p)}{e(p)}\right) \quad \text{is odd}\,.$$

Finally observe that if l is a finite prime which does not divide nq, then l does not ramify in $L(\varepsilon_q)/K$ and so $\mathrm{inv}_l\,[\Delta_q] = 0$.

Now assume that $[L\colon K]$ is odd. Then $S(K)_2 = K \otimes_Q S(Q)$ by [5, Cor. 2]. This means that there is an algebra class $[A]$ in $S(K)_2$ with $\mathrm{inv}_q\,[A] = 1/2$ if and only if the order of the decomposition group of q in K/Q, $f(q)e(q, K/Q)$, is odd.

For each prime p which divides n, we must have that $e(p, K/Q)$ is even and $e(p)$ is odd. Thus condition (I) of the theorem applies, and is satisfied. Further, every element in $\mathrm{Gal}\,(L/K)$ is a square, so condition (II) reduces to: For any prime q, $\mathrm{inv}_q\,[A] = 0$ if $f(q)$ is even. Hence this condition is satisfied. Condition (III) is trivially satisfied since condition (I) applies to each prime p which divides n.

Suppose now that q is a prime not dividing n such that $f(q)$ is odd. Then the decomposition group of q in K/Q has odd order. Thus the algebra $K \otimes_Q (Q(\varepsilon_{q'}), \gamma, -1)$ has invariant $1/2$ at q and invariant 0 elsewhere, where $\langle\gamma\rangle = \mathrm{Gal}\,(Q(\varepsilon_{q'})/Q)$ and $q' = q$ unless q is even, in which case $q' = 4$. Note that K cannot be a real field in this case, so that the invariants of any algebra in $B(K)$ are 0 at the infinite primes of K.

We have now shown that the theorem holds if $[L\colon K]$ is odd. For the rest of the proof we shall assume that $[L\colon K]$ is even. By Lemma 2, we may assume that $[L\colon K] = 2^c$ for $c \geqq 1$.

Suppose that K is a real field. Pick a prime p such that $f(p)e(p, K/Q)$ is even. This can always be done since $[K\colon Q]$ is assumed to be even. Consider the algebra $K \otimes_Q D_p$ where $[D_p] \in S(Q)$ has invariant $1/2$ only at p and the infinite prime p_∞. Then $[K \otimes D_p]$

has invariant 1/2 just at the infinite primes of K. Hence $\Omega(p_\infty)$ is in K. This settles the case with respect to the infinite primes since $B(C) = \{1\}$ where C is the complex numbers. For the remainder of the proof, "prime" will mean "finite prime."

Step 2. Condition (I) is satisfied.

Suppose that p is a prime which divides n, and that $e(p) \neq 2^t$. Then a is even where $\langle \phi^a \rangle$ is the invertia group of p in L/K. Hence $(p-1)/e(p)$ is even because it is divisible by a if $e(p) \neq 1$. Thus (3.2) implies that $\mathrm{inv}_p\,[\Delta_q] = 0$ for all odd primes q which do not divide n. Now consider Δ_2. If $a = 0$, then (3.5) implies that $\mathrm{inv}_p\,[\Delta_2] = 0$ since $\mu = 0$. If $a \neq 0$, then $2^t \geqq 4$ so that $p \equiv 1 \bmod 4$. Hence $[Q(\varepsilon_4)/Q, p] = 1$, so in (3.5) we have that $k = 0$. Moreover, (3.3) implies that z is even, so $\mathrm{inv}_p\,[\Delta_2] = 0$.

We have shown that each of the generators of $S(K)_2$ has 0 invariant at p. Hence $\mathrm{inv}_p\,[A] = 0$ for all $[A]$ in $S(K)_2$ and condition (I) is satisfied.

Step 3. Condition (II) is satisfied.

Suppose that p is a prime dividing n such that $f(p)$ is even and condition (I) does not apply to p. Note that the identity element in $\mathrm{Gal}\,(L/K)$ is a Frobenius automorphism for p in L/K in this case, so condition (II) does apply to p.

Observe that $t(p)/e(p)$ is even, and in the case where $e(p) = 2$, $t(p)/e(p)$ is divisible by 4. This is so because $f(p)$ is even and $e(p) = 2^t$ must divide $p - 1$.

Let l be either 4 or an odd prime not dividing n, and suppose that γ^h is a Frobenius automorphism for p in $L(\varepsilon_l)/K$ where $\langle \gamma \rangle = \mathrm{Gal}\,(L(\varepsilon_l)/L)$. If l is an odd prime then h must be even since $f(p)$ is even. If $l = 4$, then $h = 0$. Further, by (3.3), z is even when $e(p) \geqq 4$. Thus (3.2) and (3.5) imply that $\mathrm{inv}_p\,[\Delta_{l'}] = 0$ where $l' = l$ if l is odd or $l' = 2$ if $l = 4$.

Hence, for p, condition (II) is satisfied on the generators of $S(K)_2$. Therefore condition (II) is satisfied for all primes which divide n.

Now suppose that q is a prime which does not divide n such that $f(q)$ is even and $[L/K, q] = \phi^g$ is a square in $\mathrm{Gal}\,(L/K)$. Then g is even so that $gx + f(q)y$, or $gx + f(q)y/2$ in the case of $q = 2$, is even for all permissible values of x and y. Thus, by (3.1) and (3.4), $\mathrm{inv}_q\,[\Delta_q] = 0$.

Classes of the type $[\Delta_q]$ are the only classes amongst the generating classes given by Theorem 1 which might possibly have nonzero invariant at primes of K dividing q. Hence $\mathrm{inv}_q [A] = 0$ for all $[A]$ in $S(K)_2$, and condition (II) is satisfied for primes which do not divide n.

Step 4. For each prime l to which conditions (I) and (II) do not apply, there is a class $[A]$ in $S(K)_2$ such that $\mathrm{inv}_l [A] = 1/2$.

First suppose that q is a prime which does not divide n. If $f(q)$ is odd, then the algebra

$$\begin{aligned} \varDelta_q^0 &= \varDelta_q(0, 2, 0) \qquad \text{if} \quad q = 2 \\ &= \varDelta_q(0, 1, 0) \qquad \text{if} \quad q \neq 2 \end{aligned}$$

has invariant 1/2 at q and invariant 0 elsewhere. Hence $\Omega(q) = [\varDelta_q^0]$ if $f(q)$ is odd.

Suppose that $f(q)$ is even and that $[L/K, q] = \phi^g$ where g is odd. By (3.1) and (3.4), the algebra

$$\begin{aligned} \varDelta_q^1 &= \varDelta_q(1, 0, 1) \qquad \text{if} \quad q = 2 \quad \text{and} \quad 2^e = 2 \\ &= \varDelta_q(1, 0, 0) \qquad \text{otherwise} \end{aligned}$$

has invariant 1/2 at q.

Now let p be a prime which divides n such that neither condition (I) nor condition (II) applies to p. Hence, $f(p)$ is odd. Pick an odd prime q not dividing n such that $[Q(\varepsilon_{4p})/Q, q] = \psi$ where $\langle \psi \rangle = \mathrm{Gal}\,(Q(\varepsilon_p)/Q)$. There exist infinitely many such q by the Tchebotarev density theorem. This choice of q insures that $q \equiv 1 \bmod 4$ and that $(q/p) = -1$. Hence, by quadratic reciprocity, $(p/q) = -1$. Thus h must be odd where γ^h is a Frobenius automorphism of p in $L(\varepsilon_q)/K$. Then by (3.2) $\mathrm{inv}_p [\varDelta_q^1] = 1/2$ where $\varDelta_q^1$ is the algebra described above. This is because a is odd if condition (I) does not apply.

Step 5. If condition (III) does not apply, then $\Omega(l)$ is in $S(K)_2$ for every prime l to which conditions (I) and (II) do not apply.

Let p be a prime which divides n such that condition (I) does not apply to p. This means that p is totally ramified in L/K. Hence p is the only prime which is ramified in L/K, and so p is the only prime dividing n to which condition (I) does not apply.

Now suppose that condition (II) does not apply to p. We saw in Step 3 that this means that $f(p)$ is odd. Further suppose that $|(p-1)/e(p)|_2 < |p' - 1|_2$ for some prime $p' \neq p$ which divides n. Pick an odd prime q_0 which does not divide n such that $[L(\varepsilon_4)/Q, q_0] = \psi\psi'$ where ψ generates $\mathrm{Gal}\,(Q(\varepsilon_p)/Q)$ and ψ' generates $\mathrm{Gal}\,(Q(\varepsilon_{p'})/Q)$. Now $f(q_0)$ is divisible by the same power of 2 as $p' - 1$ is, hence $[L/K, q_0] = \phi^g$ where g is even. Thus $\mathrm{inv}_{q_0} [\varDelta_{q_0}^1] = 0$. However our choice of q_0 insures that $q_0 \equiv 1 \bmod 4$ and that $(q_0/p) = -1$. Thus the argument at the end of Step 3 gives $\mathrm{inv}_p [\Delta^1_{q_0}] = 1/2$. Since p is the only prime dividing n at which $\Delta^1_{q_0}$ can have nonzero invariants, we have that $\Omega(p) = [\Delta^1_{q_0}]$.

Now let q be a prime which does not divide n such that condition (II) does not apply to q. We saw in Step 3 that $\Omega(q)$ is in $S(K)_2$ if $f(q)$ is odd. Further, if $f(q)$ is even, we have that $\mathrm{inv}_q [\Delta^1_q] = 1/2$. Thus, if $\mathrm{inv}_p [\Delta^1_q] = 0$, we have $\Omega(q) = [\Delta^1_q]$. If $\mathrm{inv}_p [\Delta^1_q] = 1/2$, then $\Omega(q) = [\Delta^1_q] \otimes_k \Omega(p)$.

Step 6. Condition (III) is satisfied.

Let p be a prime dividing n to which condition (I) does not apply. Further suppose that $f(p)$ is odd and that $|(p-1)/e(p)|_2 \geqq |p'-1|_2$ for every prime $p' \neq p$ which divides n. This hypothesis, and the assumption that $[K: Q]$ is even, forces $p \equiv 1 \bmod 4$. We also have that $\langle \phi \rangle$ is the inertia group of p in L/K.

Let q be a prime not dividing n such that $\mathrm{inv}_p [\Delta'_q] = 1/2$ where Δ'_q is one of the generators of $S(K)_2$ given in Theorem 1. Let $[L/K, q] = \phi^g$ and let γ^h be a Frobenius automorphism of p in $L(\varepsilon_{q'})/K$ where $\langle \gamma \rangle = \mathrm{Gal}\,(L(\varepsilon_{q'})/K)$, $q' = q$ if q is odd, and $q' = 4$ if $q = 2$.

(a) Suppose that q is odd. Then by (3.2), hx must be odd. However, h is odd if and only if $(p/q) = -1$ since $f(p)$ is odd. So, by the law of quadratic reciprocity, $(q/p) = -1$ and so $f(q)$ is divisible by the same power of 2 as $(p-1)/e(p)$ is. This implies that g is odd. Hence $\mathrm{inv}_q [\Delta'_q] = 1/2$.

(b) Suppose that $q = 2$. Then $h = 0$ since $[Q(\varepsilon_4)/Q, p] = 1$. Thus $z/2(t(p)/e(p))$ must be odd. This means that $t(p)/e(p) \equiv 2 \bmod 4$ and z is odd. By (3.3), this can only occur when x is odd and $e(p) = 2$. Thus $p \equiv 5 \bmod 8$, so that $(2/p) = -1$. This implies that $f(2)$ is even and that q is odd. Hence, by (3.4) $\mathrm{inv}_2 [\Delta'_2] = 1/2$.

Now let q be a prime not dividing n such that $(q/p) = -1$ and $\mathrm{inv}_q [\Delta''_q] = 1/2$ where Δ''_q is one of the algebras described in Theorem 1. Let $[L/K, q] = \phi^g$ and let γ^h be a Frobenius automorphism of p in $L(\varepsilon_{q'})/K$ where $\langle \gamma \rangle = \mathrm{Gal}\,(L(\varepsilon_{p'})/K)$ and $q' = q$ if q is odd or $q' = 4$ if $q = 2$.

By (3.1) and (3.4), xg is odd. If q is odd, then h is odd so that $\mathrm{inv}_p [\Delta''_q] = 1/2$. So suppose that $q = 2$. Then we must have $p \equiv 5 \bmod 8$. This implies that $t(p)/e(p) \equiv 2 \bmod 4$, and, by (3.3), that z is odd. Hence (3.5) implies that $\mathrm{inv}_p [\Delta''_2] = 1/2$.

We have now shown that

$$\mathrm{inv}_p [\Delta_q] = 1/2 \Leftrightarrow \mathrm{inv}_q [\Delta_q] = 1/2 \quad \text{and} \quad (q/p) = -1 \, .$$

Since every algebra class $[A]$ in $S(K)_2$ is generated by classes of this form, we have shown that condition (III) is satisfied.

Further, this proves that $\Omega(q)$ is in $S(K)_2$ if $(q/p) = 1$ and condition (II) does not apply to q. This is because $[\Delta_q]$ can have nonzero invariants only at p and q; we saw in Step 3 that we could arrange for nonzero invariants at q and we have just seen that we cannot get nonzero invariants at p.

This completes the proof of the theorem.

B. $S(K)_2$ when n is even.

Now suppose that $L = Q(\varepsilon_n)$ is a cyclotomic field containing ζ, a primitive 2^sth root of unity for $s \geqq 2$. Further suppose that $K \subset L$ does not contain a fourth root of unity, and that $\mathrm{Gal}(L/K) = \langle\phi\rangle$ has order $2^c c'$, $(c', 2) = 1$.

Let $\mathrm{Gal}(Q(\zeta)/Q) = \langle\rho\rangle \times \langle\psi\rangle$ where $\rho(\zeta) = \zeta^{-1}$ and $\psi(\zeta) = \zeta^5$. Then we may assume that $\phi = \rho\psi^{2^{r-2}}\tau$ where the order of $\langle\psi^{2^{r-2}}\rangle = 2^{s-r}$ divides the order of $\langle\tau\rangle$. Thus $\phi(\zeta) = \zeta^{-h}$ where $h = 5^{2^{r-2}}$. We will keep this notation for the rest of this section.

We must determine the invariants of the generators of $S(K)_2$ given in Theorem 1.

Let $\Delta_q = \Delta_q(x, y, z)$ be the algebra

$$\Delta_q = (L(\varepsilon_q)/K, \alpha) = \sum_\tau L(\varepsilon_q) U_\tau$$

where q is a prime not dividing n and the values of α are in $\langle\zeta\rangle$. Let $\langle\gamma\rangle = \mathrm{Gal}(L(\varepsilon_q)/L)$. The factor set α is determined by the integers x, y, and z where

$$\begin{aligned} U_\gamma U_\phi &= \zeta^x U_\phi U_\gamma , \\ U_\gamma^{q-1} &= \zeta^y , \\ U_\phi^{2^c c'} &= \zeta^z . \end{aligned}$$

The conditions in [8, §1] require that

$$\begin{aligned} &\text{(i)} & \zeta^z &= (\zeta^z)^\phi = \zeta^{-hz} \\ &\text{(ii)} & (\zeta^y)^{-h-1} &= (\zeta^y)^{\phi-1} \\ &&&= (\zeta^{-x})^{N(\gamma)} \\ &&&= \zeta^{-x(q-1)} \\ &\text{(iii)} & 1 = (\zeta^z)^{\gamma-1} &= (\zeta^x)^{N(\phi)} \end{aligned}$$

where $N(\tau) = 1 + \tau^2 + \cdots + \tau^{|\tau|-1}$ for a group element τ. Hence

$$\begin{aligned} &(3.6)\ \text{(a)} & &2^{s-1} \text{ divides } z , \\ &\ \text{(b)} & &y(h+1) - x(q-1) \equiv 0 \bmod 2^s , \\ &\ \text{(c)} & &2 \text{ divides } x \text{ if } c = s - r . \end{aligned}$$

Now suppose that $[L/K, q] = \phi^g$. Then

$$\left(\frac{\alpha(\gamma, \phi^g)}{\alpha(\phi^g, \gamma)}\right)^{(q-1)/t(q)} U_\gamma^{q-1} = (\varepsilon_{t(q)})^{(q-1)\nu}$$

where

$$\nu = \frac{1}{2^s}\left[x\left(\frac{1-(-h)^g}{1+h}\right) + y\left(\frac{t(q)}{q-1}\right)\right].$$

Thus the q-local index of Δ_q is given by

$$\begin{aligned} \frac{q-1}{((q-1)\nu, q-1)} &= 1 \qquad \text{if} \quad \nu \equiv 0 \bmod \boldsymbol{Z} \\ &= 2 \qquad \text{if} \quad \nu \equiv 1/2 \bmod \boldsymbol{Z}. \end{aligned}$$

Hence

$$\text{(3.7)} \qquad \operatorname{inv}_q [\Delta_q] = 1/2 \Leftrightarrow \nu \equiv 1/2 \bmod \boldsymbol{Z}.$$

Now suppose that p is an odd prime which divides n. Let $\gamma^b\phi^{b'}$ be a Frobenius automorphism of p in $L(\varepsilon_q)/K$, and let $\langle\phi^a\rangle$ be the inertia group of p in $L(\varepsilon_q)/K$.

Then

$$\left(\frac{\alpha(\phi^a, \gamma^b\phi^{b'})}{\alpha(\gamma^b\phi^{b'}, \phi^a)}\right)^{e(p)/t(p)} (U_{\phi^a})^{e(p)} = \varepsilon_{t(p)}^{e(p)\nu_p}$$

where

$$\nu_p = \frac{1}{2^s}\left[xb\left(\frac{1-h^a}{1+h}\right) + \mu z\left(\frac{p^{f(p)}-1}{e(p)}\right)\right]$$

where

$$\begin{aligned} \mu &= 0 \qquad \text{if} \quad a = 0 \\ &= 1 \qquad \text{if} \quad a \neq 0. \end{aligned}$$

Hence

$$\text{(3.8)} \qquad \operatorname{inv}_p [\Delta_q] = 1/2 \Longleftrightarrow \nu_p \equiv 1/2 \bmod \boldsymbol{Z}.$$

Finally suppose that 2 is ramified in L/K. Our assumption that the order of $\langle\psi^{2^{r-2}}\rangle$ divides the order of $\langle\tau\rangle$ implies that in this case $\operatorname{Gal}(L/K) = \langle\rho\rangle$.

Let $\eta = \gamma^b$ be a Frobenius automorphism of 2 in $L(\varepsilon_q)/K$. Let f be the order of $\langle\eta\rangle$. We have

$$\begin{aligned} U_\rho((1+\zeta^{xb})U_\eta) &= (1+\zeta^{-xb})U_\rho U_\eta \\ &= (1+\zeta^{-xb})\zeta^{xb}U_\eta U_\rho \\ &= [(1+\zeta^{xb})U_\eta]U_\rho. \end{aligned}$$

Let π be a prime of K which divides 2. Then

$$\begin{aligned} K_\pi \otimes \Delta_q &= \sum_{i=0}^{1}\sum_{j=0}^{f-1} K_\pi(\varepsilon_4)K_\pi(\varepsilon_q)U_\rho^i U_\eta^j \\ &= \sum_{i=0}^{1}\sum_{j=0}^{f-1} K_\pi(\varepsilon_4)K_\pi(\varepsilon_q)U_\rho^i[(1+\zeta^{xb})U_\eta]^j \\ &\cong \sum_{i=0}^{1} K_\pi(\varepsilon_4)U_\rho^i \otimes_{K_\pi} \sum_{j=0}^{f-1} K_\pi(\varepsilon_q)[(1+\zeta^{xb})U_\eta]^j \\ &\cong (K_\pi(\varepsilon_4), \rho, U_\rho^2) \otimes_{K_\pi} (K_\pi(\varepsilon_q), \eta, [(1+\zeta^{xb})U_\eta]^f)\,. \end{aligned}$$

Now $[(K_\pi(\varepsilon_4), \rho, U_\rho^2)] = K_\pi \otimes_{Q_2} (Q_2(\varepsilon_4), \rho, \zeta^z)$. Hence $\operatorname{inv}(K_\pi(\varepsilon_4), \rho, U_\rho^2)$ may be assumed to be 0, since otherwise $e(2, K/Q)$ would be odd which would mean that $K = Q(\varepsilon_{n/4})$. The Schur subgroup of a cyclotomic field is given in [5].

Now let V' and V be the exponential valuations of $K_\pi(\varepsilon_4)$ and K_π respectively. Since $e(K_\pi(\varepsilon_4)/K) = 2$, we have

$$\begin{aligned} V[(1+\zeta^{xb})U_\eta]^f &= \frac{1}{2}V'[(1+\zeta^{xb})U_\eta]^f \\ &= \frac{1}{2}[V'(1+\zeta^{xb})^f + V'(U_\eta^f)] \\ &= \frac{1}{2}fV'(1+\zeta^{xb})\,. \end{aligned}$$

Now $V'(1+\zeta^{xb})$ is odd if and only if xb is odd since $1+\zeta^{xb}$ is a prime element of $K_\pi(\varepsilon_4)$ when xb is odd. Thus from the definition of the Hasse invariant we get

$$\begin{aligned} \operatorname{inv}(K_\pi \otimes \Delta_q) &= 0 \qquad \text{if } xb \text{ is even} \\ &= 1/2 \qquad \text{if } xb \text{ is odd}\,. \end{aligned}$$

Thus

$$\operatorname{inv}_2[\Delta_q] = 1/2 \Longleftrightarrow \mu_0 xb \quad \text{is odd} \tag{3.9}$$

where

$$\begin{aligned} \mu_0 &= 0 \qquad \text{if 2 is unramified in } L/K \\ &= 1 \qquad \text{if 2 is ramified in } L/K\,. \end{aligned}$$

Observe that q and the primes which divide n are the only primes which might ramify in $L(\varepsilon_q)/K$. Hence, these are the only primes at which Δ_q can have nonzero invariants.

THEOREM 3. *The 2-primary part of $S(K)$ consists of all classes $[A]$ in $B(K)$ with uniformly distributed invariants of value 0 or 1/2 which satisfy the following conditions.*

(I) *For a prime p which divides n,* $\operatorname{inv}_p [A] = 0$ *if any of the following hold:*

(a) $e(p)$ *is odd;*

(b) $f(p)$ *is even;*

(c) $[L\colon K(\zeta)]/e(p)$ *is an even integer.*

(II) *For q a prime which does not divide n,* $\operatorname{inv}_q [A] = 0$ *if either*

(a) $t = s - r$ *and $f(q)$ is even, or*

(b) $t \neq s - r$, *$f(q)$ is even, and* $q^{f(q)} \equiv (-h)^g \bmod 2^{s+1}$ *where* $[L/K, q] = \phi^g$.

(III) *Let p be a prime which divides n such that condition* (I) *does not apply to p. If* $|e(p, K/Q)|_2 \geqq |e(p', K/Q)|_2$ *for every prime $p' \neq p$, then the invariant of $[A]$ is $1/2$ at an even number of primes in the set*

$$\{p\} \cup \{\text{primes } q\colon (p/q) = -1,\ (q, n) = 1\}$$

where (p/q) is the Legendre symbol.

Proof. We have assumed that $\langle \phi \rangle$ has even order. Hence, by Lemma 2, we may assume that $[L\colon K] = 2^e$.

First suppose that K is a real field. Pick an odd prime of q such that $f(q)e(q, K/Q)$ is even. There will always be such a prime since $[K\colon Q]$ must be even. Then the algebra $K \otimes_Q (Q(\varepsilon_q), \tau, -1)$ where $\langle \tau \rangle = \operatorname{Gal}(Q(\varepsilon_q)/Q)$ has invariant $1/2$ only at the infinite primes of K. Thus $\Omega(p_\infty)$ is in $S(K)_2$ when K is real.

For the rest of the proof, "prime" will mean "finite prime."

Step 1. Condition (I) is satisfied.

Let p be a prime which divides n. If $e(p) = 1$, then p is unramified in $L(\varepsilon_q)/K$ for any prime q not dividing n. Hence $\operatorname{inv}_p [A] = 0$ for all $[A]$ in $S(K)_2$. Now suppose that $e(p)$ is even.

If $p \neq 2$ and $\langle \phi^a \rangle$ is the intertia group of p in L/K, then 2^{s-r} divides a, or if $s = r$, 2 divides a. Since the power of 2 dividing a must divide $(p-1)/e(p)$, we have that $t(p)/e(p)$ is even. Further $h = 5^{2^{r-2}}$ so $(h^a - 1)/(h + 1)$ is not divisible by 2^s if and only if 2^{s-r+1} does not divide a, or if $s = r$, if and only if 4 does not divide a. However this happens if and only if $[L\colon K] = 2^{s-r}e(p)$, or if $s = r$, if and only if $[L\colon K] = 2e(p)$. Thus we have

$$\frac{h^a - 1}{h + 1} \not\equiv 0 \bmod 2^s \Longleftrightarrow [L\colon K(\zeta)]/e(p) \qquad \text{is odd}\,.$$

Let q be a prime which does not divide n and let $\gamma^b \phi^{b'}$ be a

Frobenius automorphism of p in $L(\varepsilon_q)/K$ where $\langle\gamma\rangle = \mathrm{Gal}\,(L(\varepsilon_q)/L)$. Then we may rewrite (3.8) to read

$$\text{(3.10)} \qquad \mathrm{inv}_p\,[\Delta_q] = 1/2 \Longleftrightarrow ([L\colon K(\zeta)]/e(p))xb \quad \text{is odd}$$

since 2^{s-1} divides z. Since b is even if $f(p)$ is even, (3.10) implies condition (I) for $p \neq 2$.

If γ^b is a Frobenius automorphism for 2 in $L(\varepsilon_q)/K$, then b is even if $f(2)$ is even. Thus (3.9) gives condition (I)(b). Since $\mathrm{Gal}\,(L/K) = \langle\rho\rangle$ when 2 is ramified in L/K, we see that condition (I) (c) never applies to 2.

Step 2. Condition (II) holds.

Let q be a prime not dividing n and let $[L/K, q] = \phi^g$. We consider the invariants of algebras of the form $\Delta_q = \Delta_q(x, y, z)$. We have

$$\phi^g(\zeta) = \zeta^{(-h)^g} = \zeta^{q^{f(q)}} .$$

Hence $q^{f(q)} = (-h)^g + V2^s$ for some integer V.
Further, by (3.6) (b), we have

$$y = \frac{x(q-1) + W2^s}{1+h}$$

for some integer W. Thus we may rewrite (3.7) to read

$$\text{(3.11)} \quad \mathrm{inv}_q\,[\Delta_q] = 1/2 \Longleftrightarrow \Big(\frac{W}{1+h}\Big)\Big(\frac{q^{f(q)}-1}{q-1}\Big) + \frac{xV}{h+1} \equiv 1/2 \bmod \boldsymbol{Z} .$$

Now $t(q)/(q-1)$ is even if $f(q)$ is even. Moreover x is even if $t = s - r$ and V is even if $q^{f(q)} \equiv (-h)^g \bmod 2^{s+1}$. Hence condition (II) is obtained directly from (3.11).

Step 3. For each prime l to which conditions (I) and (II) do not apply, there is a class $[A]$ in $S(K)_2$ such that $\mathrm{inv}_l\,[A] = 1/2$.

Suppose that q is a prime which does not divide n such that condition (II) does not apply to q. If $f(q)$ is odd, then the algebra

$$\Delta_q^0 = \Delta_q(0, 2^{s-1}, 0)$$

has invariant 1/2 at q since $W = (h+1)/2$ is odd.

If $f(q)$ is even, $t \neq s - r$, and $q^{f(q)} \not\equiv (-h)^g \bmod 2^{s+1}$, then consider the algebra

$$\Delta_q' = \Delta_q\Big(\frac{h+1}{2}, \frac{q-1}{2}, 0\Big) .$$

We have that $t(q)/(q-1)$ is even and that V is odd, thus (3.11) implies that $\mathrm{inv}_q[\Delta'_q] = 1/2$.

Now let p be a prime which divides n such that condition (I) does not apply to p. Pick a prime q which does not divide n such that $[Q(\varepsilon_{4p})/Q, q] = \psi_p$, where ψ_p generates $\mathrm{Gal}(Q(\varepsilon_{4p})/Q(\varepsilon_4))$. This choice of q insures that $q \equiv 1 \bmod 4$ and that $(q/p) = -1$. Hence, by quadratic reciprocity, $(p/q) = -1$ so that b is odd where $\gamma^b\phi^{b'}$ is a Frobenius automorphism of p in $L(\varepsilon_q)/K$ and $\langle\gamma\rangle = \mathrm{Gal}(L(\varepsilon_q)/L)$. Hence, by (3.10) and (3.9) $\mathrm{inv}_p[\Delta'_q] = 1/2$.

Step 4. If condition (III) does not apply, then $\Omega(l)$ is in $S(K)_2$ for every prime l to which conditions (I) and (II) do not apply.

Let p be a prime dividing n to which condition (I) does not apply. Then p is totally ramified in $L/K(\zeta)$. Further, since the inertia group of a prime in $Q(\varepsilon_n)/K$ must be a subgroup of its inertia group in $Q(\varepsilon_n)/Q$, we have that p is the only prime which is ramified in L/K. Thus p is the only prime dividing n to which condition (I) does not apply.

Suppose that $|e(p, K/Q)|_2 < |e(p', K/Q)|_2$ for some prime $p' \neq p$ which divides n. Let $2^\lambda = |e(p, K/Q)|_2$.

(a) Assume that p' is odd.

Pick a prime q_0 not dividing n such that $[L/Q, q_0] = \psi_p\psi_{p'}$ where $\langle\psi_{p'}\rangle = \mathrm{Gal}(Q(\varepsilon_{p'})/Q)$ and $\psi_p = \psi$ if $p = 2$ or $\langle\psi_p\rangle = \mathrm{Gal}(Q(\varepsilon_p)/Q)$ if $p \neq 2$. There are infinitely many such q_0 by the Tchebotarev density theorem. Our choice of q_0 insures that $q_0 \equiv 5 \bmod 8$ if $p = 2$ or $(q_0/p) = -1$ if $p \neq 2$. Thus $(p/q_0) = -1$ since $q_0 \equiv 1 \bmod 4$ by choice. Let γ generate $\mathrm{Gal}(L(\varepsilon_{q_0})/L)$ and let $\gamma^b\phi^{b'}$ be a Frobenius automorphism for p in $L(\varepsilon_{q_0})/K$. Then b must be odd. Thus $\mathrm{inv}_p[\Delta'_{q_0}] = 1/2$ by (3.9) and (3.10). On the other hand, $f(q_0)$ is divisible by $|p'-1|_2$ since $[L/K, q_0] \in \mathrm{Gal}(L/K(\zeta))$ if $p \neq 2$ and $[L/K, q_0] = 1$ if $p = 2$. Hence $q_0^{f(q_0)}$ and h^g, where $[L/K, q_0] = \phi^g$, are both equivalent to 1 modulo 2^{s+1}. This is clear if $p = 2$; if $p \neq 2$, then $q_0 \equiv 1 \bmod 2^s$ and ϕ^g must be a square in $\mathrm{Gal}(L/K(\zeta))$ by our choice of q_0. Thus condition (II) applies to q_0, so $\mathrm{inv}_{q_0}[\Delta_{q_0}] = 0$. Hence $\Omega(p) = [\Delta'_{q_0}]$.

(b) Assume that $p' = 2$, that is that $2^{s-2} > 2^\lambda$.

Pick a prime q_1 not dividing n such that $[L(\varepsilon_{2^{s+1}})/Q, q'] = \psi_p\psi_{2'}^{2^{s-\lambda-2}}$, where ψ_p is the generator of the Sylow-2 subgroup of $\mathrm{Gal}(Q(\varepsilon_p)/Q)$ such that $\psi_p^{2^{\lambda+r-s}}(\varepsilon_p) = \phi(\varepsilon_p)$, and $\psi_{2'}$ is the automorphism sending $\varepsilon_{2^{s+1}}$ to $\varepsilon_{2^{s+1}}^5$. Now

$$[L(\varepsilon_{2^{s+1}})/K, q_1] = (\psi_p^{2^{\lambda+r-s}}\psi_{2'}^{2^{r-2}})^g$$

for some g, $2 \leqq g \leqq 2^{s-r}$. Hence $[L/K, q_1] = \phi^g$. Further,

$$\psi_{2^r}^{2^{r-2}g}(\varepsilon_{2^s+1}) = (\varepsilon_{2^s+1})^{h^g} = (\varepsilon_{2^s+1})^{q_1{}^{f(q_1)}} ,$$

so $h^g \equiv q_1^{f(q_1)} \bmod 2^{s+1}$. This implies that $\mathrm{inv}_{q_1}[\Delta'_{q_1}] = 0$ since we arranged for $f(q_1)$ to be even.

On the other hand, we picked q_1 so that $q_1 \equiv 1 \bmod 4$ and $(q_1/p) = -1$. Hence $(p/q_1) = -1$. Thus, by (3.10), $\mathrm{inv}_p[\Delta'_{q_1}] = 1/2$. Therefore $\Omega(p) = [\Delta'_{q_1}]$.

Now let q be a prime which does not divide n such that condition (II) does not apply to q. By Step 3, there is an algebra Δ_q^* such that $\mathrm{inv}_q[\Delta_q^*] = 1/2$. If $\mathrm{inv}_p[\Delta_q^*] = 0$, then $\Omega(q) = [\Delta_q^*]$. If $\mathrm{inv}_p[\Delta_q^*] = 1/2$, then $\Omega(q) = [\Delta_q^*] \otimes_K \Omega(p)$.

Step 5. Condition (III) holds.

Suppose that p is a prime dividing n to which condition (I) does not apply. Further suppose that $|e(p, K/Q)|_2 \geqq |e(p', K/Q)|_2$ for every prime $p' \neq p$ which divides n.

Let q be a prime not dividing n. Let $\langle\gamma\rangle = \mathrm{Gal}(L(\varepsilon_q)/L)$ and $\gamma^b\phi^{b'}$ be a Frobenius automorphism for p in $L(\varepsilon_q)/K$.

First suppose that $\mathrm{inv}_p[\Delta_q^*] = 1/2$ where Δ_q^* is an algebra of the form Δ_q. From (3.9) and (3.10) we see that this implies that xb is odd. Thus b is odd, which means that $(p/q) = -1$. Further, if $p \neq 2$, then our hypotheses insure that $p \equiv 1 \bmod 4$. Thus $(q/p) = -1$ if $p \neq 2$, or $q \equiv 3$ or $5 \bmod 8$ if $p = 2$. Suppose $p \neq 2$, then $|e(p, K/Q)/2^{s-r}|_2 > 2^{r-2}$, so the full 2-part of $e(p, K/Q)$ is equal to $|f(q)|_2$. Hence $q^{f(q)} \equiv 1 \bmod 2^{s+1}$ and $[L/K, q] = \phi^{2^{s-r}}$. Since $h^{2^{s-r}} \not\equiv 1 \bmod 2^{s+1}$, we have by (3.11) that $\mathrm{inv}_q[\Delta_q^*] = 1/2$. In the case where $p = 2$, $|f(q)|_2 = 2^{s-2}$ so $q^{f(q)} \not\equiv 1 \bmod 2^{s+1}$. However $[L/K, q] = 1$. Thus, by (3.11), $\mathrm{inv}_q[\Delta_q^*] = 1/2$.

Now suppose that $(p/q) = -1$ and $\mathrm{inv}_q[\Delta_q^*] = 1/2$. Since $(p/q) = -1$ we have that b is odd. Further, $(q/p) = -1$ if $p \neq 2$ or $q \equiv 3$ or $5 \bmod 8$ if $p = 2$. Hence $f(q)$ is divisible by $|e(p, K/Q)/2^{s-r}|_2$ if $p \neq 2$ or by 2^{s-r} if $p = 2$. This means that $f(q)$ is even so that xv is odd. Thus xb is odd. Hence (3.9) and (3.10) imply that $\mathrm{inv}_p[\Delta_q^*] = 1/2$.

We have just shown that

$$\mathrm{inv}_p[\Delta_q] = 1/2 \Longleftrightarrow \mathrm{inv}_q[\Delta_q] = 1/2 \quad \text{and} \quad (q/p) = -1 .$$

Since every algebra class $[A]$ in $S(K)_2$ is a product of classes of the form $[\Delta_q]$, this gives condition (III).

In addition, this shows that $\Omega(q)$ is in $S(K)_2$ where q is a prime not dividing n such that $(q/p) = 1$ and condition (II) does not apply to q. This is because there is an algebra $[\Delta_q^*]$ with $\mathrm{inv}_q[\Delta_q^*] = 1/2$ by Step 3, and we have just seen that $\mathrm{inv}_p[\Delta_q^*] = 0$.

This completes the proof of the theorem.

We have now determined the Schur subgroup of all fields K, not containing a fourth root of unity, which have a cyclic extension of the form $Q(\varepsilon_n)$. Observe that subfields of $Q(\varepsilon_{p^d})$ are included as special cases. The Schur group of these fields was first found by Yamada [8].

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