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THE CONSTRUCTIVE RADON–NIKODYM THEOREM

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This paper discusses absolute continuity of integrals, and proves a version of the Radon–Nikodym Theorem and its converse, within the framework of the constructive measure theory of Bishop and Cheng.

1. Introduction. There can be few mathematicians who remain unaware of the fact that much of their subject, as commonly presented, has little or no computational significance. It will be convenient to refer to such mathematics as 'classical', in contrast to the alternative 'constructive' mathematics, in which

'*every mathematical statement ultimately expresses the fact that if we perform certain computations within the set of positive integers we shall get certain results.*' [**1**, p. 2].

In other words, the constructive philosophy (that adopted throughout this paper) insists that mathematics should be characterized by *numerical content* and *computational method.*

A simple consequence of our philosophy is the recognition of the familiar 'least upper bound principle' as an essentially nonconstructive proposition: an algorithm for computing the suprema even of sequences in $\{0, 1\}$ would provide at a stroke a method for deciding virtually all the outstanding unsolved problems of number theory [**1**, pp 6–7]. The effects of this situation appear throughout the development of constructive analysis. Thus, for example, we have no guarantee that the norm of a given bounded linear functional on a normed linear space will be computable (if it is, we call the functional *normable*); this means that we have to adopt the following as our constructive version of

THE RIESZ REPRESENTATION THEOREM. *A bounded linear functional f on a Hilbert space H is normable if and only if there exists a (unique) element ξ of H such that $f(x) = (x \mid \xi)$ for each x in H.*

The current revival of interest in the practice of constructive mathematics is due largely to Bishop, in whose fundamental book [**1**] there is developed a substantial portion of constructive functional analysis and measure theory. More recently, Bishop and Cheng have produced a much more elegant, and surely definitive, treatment of constructive measure theory [**2**]. In this paper we show how, within the framework of that theory, the classical argument of von Neumann and Stone [**5**, Ch 7] can be adapted to prove a constructive version of the Radon–Nikodym Theorem for absolutely continuous integrals. This both improves and extends Bishop's version of the Radon–Nikodym Theorem [**1**, Ch 7, Thm 2], which applies only to measures on a locally compact space.

Before proving our main theorem, we need some preliminary material on absolute continuity and certain constructions with integrals (§§2–4). Our definition of absolute continuity provides another illustration of the difference between the constructive approach and the classical: the classical definition in terms of null sets is of little computational value—it is not so much null sets as their complements that are of importance in integration theory—and so we are forced to adopt the more positive '$\epsilon - \delta$' definition, classically equivalent to the 'null sets' definition in the case of a Σ-finite integration space.

Our proof of the Radon–Nikodym Theorem in §5 is another place in which the failure of the least upper bound principle is effective: in order to apply the Riesz Representation Theorem, we are obliged to assume the normability of certain bounded linear functionals on Hilbert spaces of square-integrable functions. A second difficulty arises in connection with domains of integrable functions: because there is no constructive procedure for deciding whether or not a given point is in the domain of a given integrable function, we are unable to extend such domains as freely as can the classical mathematician. This situation is reflected in the rather complicated form of the Corollary to the Radon–Nikodym Theorem (which should be compared with the second half of the classical proof given in [**5**]).

2. Integration spaces. Throughout this paper, X will be a nonempty set with equality $=$ and apartness $\neq$, $F(X)$ the set of all real-valued functions f defined partially on X and such that $x \neq y$ whenever $f(x) \neq f(y)$, and $dmn\, f$ the domain of such a function f. We assume familiarity with the material of [**2**].

Let L be a linear subset of $F(X)$ such that $|f| \in L$ and $f \wedge 1 \in L$ whenever $f \in L$, and let I be an integral on L. We also write I for the extension of this integral to the completion $L_1(I)$ of L with respect to the seminorm $\|f\| \to I|f|$; the extended integral I is then called the *complete extension* of the original integral I on L, and L is known as an *initial integration set* for the extension I.

We adopt the following notational conventions. Where no limits of summation appear, Σ_n will always denote $\Sigma_{n=1}^{\infty}$; similar remarks apply to $\vee_n$, $\wedge_n$, $\bigcup_n$, $\bigcap_n$. If (a_n) is a sequence of nonnegative numbers, we write $\Sigma_n a_n < \infty$ to mean '$\Sigma_n a_n$ is convergent'. We write $f \leqq g$ to indicate that

$f(x) \leqq g(x)$ on a full subset of X (relative to the integral in question). If A is a complemented set with indicator (characteristic function) χ_A, we write A^1, A^0 for the subsets $\chi_A^{-1}(1)$, $\chi_A^{-1}(0)$ of X respectively, and IA for the measure $I\chi_A$ in the case that A is integrable. If $f \in F(X)$, then $(f \geqq r)$ is the complemented set

$$(\{x \in X : f(x) \geqq r\}, \quad \{x \in X : f(x) < r\});$$

we give the obvious corresponding meanings to $(f > r)$, $(f \leqq r)$, $(f < r)$. Note that if $f \in L_1(I)$, then $(f \geqq r)$ and $(f > r)$ are integrable and have the same measure for all but countably many $r > 0$ [**2**, Thm 3.6].

We shall find good use for

LEMMA 1. *Let f be nonnegative and integrable, (f_n) a representation of f in L, $\phi_0 \equiv |f_1|$, and $\phi_n \equiv |\Sigma_{k=1}^n f_k| - |\Sigma_{k=1}^{n-1} f_k|$ $(n \geqq 2)$. Then $(\phi_n)_{n \geqq 1}$ is a representation of the integrable function $\Sigma_n \phi_n$, $\Sigma_n \phi_n = f$ on the full set on which $\Sigma_n |f_n| < \infty$, and $(\Sigma_{n=1}^N \phi_n)_{N \geqq 1}$ converges in measure to f.*

Proof. As

$$\sum_n I|\phi_n| = \sum_n I\phi_n \leqq \sum_n I|f_n| < \infty$$

and

$$\sum_n \phi_n = \lim_{N \to \infty} \left| \sum_{n=1}^N f_n \right| = \left| \sum_n f_n \right| = |f| = f$$

throughout the full set F on which $\Sigma_n |f_n| < \infty$, all but the last assertion of the Lemma is clear. On the other hand, given $\epsilon > 0$ and an integrable set A, we choose in turn a positive integer ν so that

$$0 < If - \sum_{n=1}^N I\phi_n < \epsilon^2 \qquad (N \geqq \nu);$$

an integer $N \geqq \nu$; and a number α with $\epsilon < \alpha < 2\epsilon$ and $C_N \equiv (f - \Sigma_{n=1}^N \phi_n \geqq \alpha)$ integrable. Then

$$\epsilon^2 > I\left(f - \sum_{n=1}^N I\phi_n\right) \geqq I\left(\chi_{C_N}\left(f - \sum_{n=1}^N \phi_n\right)\right) \geqq \alpha IC_N,$$

and therefore $IC_N < \alpha^{-1}\epsilon^2 < \epsilon$. Setting $B_N \equiv A - C_N$, we now have B_N integrable, $B_N < A$, $I(A - B_N) < \epsilon$ and $|f - \Sigma_{n=1}^N \phi_n|\chi_{B_N} < \alpha < 2\epsilon$.

The integration space (X, L, I) is said to be *finite* if the function $x \to 1$ is integrable; and *Σ-finite* if there exists a sequence $X_1 < X_2 < \cdots$ of integrable sets such that (χ_{X_n}) converges in measure to 1. By abuse of language, we also speak of X as being finite or Σ-finite. Note that our definition of Σ-finite is different to that of Bishop and Cheng [**2**, Propn 4.4]. In the Σ-finite case, for each f in L the sequence $(\chi_{X_n} f)$ converges in measure to f, whence (by the Dominated Convergence Theorem—henceforth referred to as DCT) $I(\chi_{X_n} f) \to If$ as $n \to \infty$; in particular, if A is an integrable set, then $I(X_n \wedge A) \to IA$. We therefore may repeat the proof of [**1**, Ch. 7, Thm 4] to obtain

PROPOSITION 1. *If the integration space (X, L, I) is Σ-finite and (f_n) is a sequence of measurable functions which is Cauchy in measure, then (f_n) converges in measure to some measurable function f, and a subsequence of (f_n) converges to f pointwise on a full set.*

It follows from this that if $X_1 < X_2 < \cdots$ is as sequence of integrable sets with (χ_{X_n}) converging in measure to 1, then $\bigcup_n X_n^1$ is full.

Perhaps the most important example of an integration space occurs when X is a locally compact (metric) space, L the space $C(X)$ of continuous, real-valued functions on X with compact support, and I a nontrivial positive linear functional on $C(X)$. The proof that (X, L, I) is then an integration space is given in [**2**, pp 67–74]; a considerable simplification of the most difficult part of the proof—the verification of the constructive equivalent of countable additivity of the integral—is described in [**4**]. To prove that the complete extension of such an integration space is Σ-finite, choose a in X and a sequence $(r_k) \uparrow \infty$ of positive numbers such that the closed balls $\bar{B}(a, r_k)$ are compact [**1**, Ch. 4, Thm 8]. Define functions g_n $(n \geqq 1)$ in $C(X)$ so that $0 \leqq g_n \leqq 1$, $g_n(x) = 1$ for x in $\bar{B}(a, r_n)$ and $g_n(x) = 0$ when $d(a, x) > r_{n+1}$. Then choose β with $0 < \beta < 1$ so that $X_n \equiv (g_n \geqq \beta)$ is integrable for each integer $n \geqq 1$. Then $X_1 < X_2 < \cdots$ and $\bigcup_{n \geqq 1} X_n^1 = X$. Let A be an integrable set, $\epsilon > 0$ and choose h in $C(X)$ so that $I|\chi_A - h| < \epsilon$. Let K be a compact support of h, choose ν so that $K \subset X_\nu^1$, and set $B \equiv A \wedge X_\nu$. Then $B < A$ and

$$I(A - B) = I(|\chi_A - h|\chi_{-X_\nu}) \leqq I|\chi_A - h| < \epsilon.$$

Also, if $n \geqq \nu$, then $|1 - \chi_{X_n}|\chi_B = 0$. Hence $\chi_{X_n} \uparrow 1$ almost everywhere and $(X, L_1(I), I)$ is Σ-finite.

3. Absolute continuity. Let (X, L, I) be an integration space, and J the complete extension of another integral originally defined on L. We say that J is *I-absolutely continuous* if there is an operation

$\delta: R^+ \to R^+$ (called a *modulus of I-absolute continuity* for J) such that, for each $\epsilon > 0$ and each complemented set E that is both I- and J-integrable, $IE < \delta(\epsilon)$ entails $JE < \epsilon$. (The meaning of such expressions as "I-integrable", "J-full" should require no explanation.)

LEMMA 2. *If J is I-absolutely continuous, $f \in L_1(I)$ and (f_n) is an I-representation of f in L, then the sequence $(\sum_{k=1}^n f_k)_{n \geq 1}$ of J-integrable functions is Cauchy in J-measure.*

Proof. Given $\epsilon > 0$ and a modulus δ of I-absolute continuity for J, we assume without loss of generality that $\delta(\epsilon) < \epsilon$, and choose in turn N_ϵ, m, n and α so that $\sum_{N_\epsilon+1}^\infty I|f_k| < \epsilon^2, m \geqq n > N_\epsilon, \epsilon < \alpha < 2\epsilon$ and

$$C \equiv \left(\left|\sum_{k=n+1}^m f_k\right| \geqq \alpha\right) \in L_1(I) \cap L_1(J).$$

Then

$$IC < \alpha^{-1}\epsilon^2 < \epsilon < \delta(\epsilon),$$

and so $JC < \epsilon$. With A any J-integrable set and $B \equiv A - C$, we now have $B \in L_1(J), B < A, J(A - B) < \epsilon$, and $|\sum_{k=n+1}^m f_k| \chi_B < \alpha < 2\epsilon$ on the J-full set $(B^1 \cup B^0) \cap (\bigcap_{k=n+1}^m dmn f_k)$.

PROPOSITION 2. *If J is I-absolutely continuous and X is Σ-finite with respect to J, then*

(i) *every I-full set is J-full;*

(ii) *every I-integrable function is J-measurable.*

Proof. Let $f \in L_1(I)$, and choose an I-representation (f_n) of f in L. From Lemma 2, Proposition 1 and the fact that $(\sum_{k=1}^n |f_k|)_{n \geq 1}$ is increasing, it follows that $\sum_n |f_n|$ converges pointwise on a J-full set G. Hence $\sum_n f_n$ converges to f on $G, G \subset dmn f$, and $dmn f$ is J-full. Conclusion (i) is now immediate. On the other hand, again by Lemma 2 and Proposition 1, there exist a strictly increasing sequence (n_k) of positive integers, a J-measurable function ϕ, and a J-full set G_1, such that $(\sum_{j=1}^{n_k} f_j)_{k \geq 1}$ converges pointwise to ϕ on G_1. It follows that $f = \phi$ on the J-full set $G \cap G_1$; whence f is measurable. This proves (ii).

COROLLARY. *Suppose that X is finite with respect to J, and that there exists a sequence $X_1 < X_2 < \cdots$ of sets in $L_1(I) \cap L_1(J)$ such that (χ_{X_n}) converges to 1 in I-measure and in J-measure. Then every I-measurable function is J-measurable.*

Proof. We first note that every I-integrable set is J-

integrable. Let A be a J-integrable set, f an I-measurable function, and ϵ a positive number. We choose in turn a positive integer N so that $J(1-\chi_{X_N})<\epsilon/2$; a sequence (f_n) of I-integrable functions converging to f in I-measure on X_N; and a positive integer ν such that, for each $n \geqq \nu$, there exists an I-integrable set $C_n < X_N$ with $J(X_N - C_n)<\epsilon/2$ and $|f-f_n|\chi_{C_n}<\epsilon$. (The choice of N is possible by the Monotone Convergence Theorem—henceforth referred to as *MCT*; that of ν by a simple application of the definition of absolute continuity.) Then, with $n \geqq \nu$ and $B_n \equiv A \wedge C_n$, we have $B_n \in L_1(J)$, $B_n < A$,

$$J(A-B_n) \leqq J(1-\chi_{B_n}) = J(1-\chi_{X_N}) + J(X_N - B_n) < \epsilon$$

and $|f-f_n|\chi_{B_n}<\epsilon$. Thus (f_n) converges to f in J-measure on A, and f is J-measurable.

REMARK. Suppose that X is Σ-finite with respect to I, finite with respect to J, and that J is I-absolutely continuous. Let $X_1 < X_2 < \cdots$ be a sequence of I-integrable sets such that (χ_{X_n}) converges to 1 in I-measure. Then (Proposition 2) each χ_{X_n} is J-measurable; whence, as $\chi_{X_n} \leqq 1$ throughout its J-full domain, χ_{X_n} is J-integrable, $JX_n \leqq J1$. Classically, we immediately deduce from the fact that each I-full set is J-full and *MCT* that (χ_{X_n}) converges in J-measure to 1. Constructively, we cannot use this argument, although we can say (from Propositions 1 and 2) that $\chi_{X_n} \uparrow 1$ pointwise on a J-full set.

4. Two important constructions. If A is an I-integrable set, then the mapping $f \to I(\chi_A f)$ is an integral on L. Let I_A denote its complete extension, and $f \in L_1(I)$. Then it is easy to show that $f \in L_1(I_A)$, $I_A f = I(\chi_A f)$, and each I-full set is I_A-full. Moreover, if F is an I_A-full set, then $F \cup A^0$ is I-full: for, with (f_n) a sequence in L such that $\Sigma_n I|\chi_A f_n| < \infty$ and the set S on which $\Sigma_n |f_n| < \infty$ is contained in F, T the I-full set on which $\Sigma_n |\chi_A f_n| < \infty$, we have $T \subset S \cup A^0 \subset F \cup A^0$.

LEMMA 3. *If A is both I- and J-integrable, and J is I-absolutely continuous, then J_A is I_A-absolutely continuous.*

Proof. Let the complemented set E be both I_A- and J_A-integrable. We first show that

$$E_= \equiv (E^1 \cap A^1, \quad (E^0 \cap (A^1 \cup A^0)) \cup A^0)$$

is I-integrable, with $IE_= = I_A E$. To this end, let (f_n) be an I_A-representation of χ_E in L, S the set on which $\Sigma_n |f_n| < \infty$, and C the complemented set

$$(S \cap E^1 \cap A^1, \quad (S \cap E^0 \cap (A^1 \cup A^0)) \cup A^0).$$

As $C^1 \cup C^0 \subset E^1_= \cup E^0_=$ and $\chi_{E_=} = \chi_C$ on $C^1 \cup C^0$, it will suffice to prove that $C^1 \cup C^0$ is I-full and C is I-integrable, with $IC = I_A E$. That $C^1 \cup C^0$ is I-full follows from the identity

$$C^1 \cup C^0 = ((S \cap (E^1 \cup E^0)) \cup A^0) \cap (A^1 \cup A^0)$$

and the remark preceding this Lemma. On the other hand, the set G on which $\Sigma_n |\chi_A f_n| < \infty$ is I-full, $\Sigma_n \chi_A f_n$ is I-integrable, and

$$I\Big(\sum_n \chi_A f_n\Big) = \sum_n I(\chi_A f_n) = \sum_n I_A f_n = I_A E.$$

As $\chi_C = \Sigma_n \chi_A f_n$ on $(C^1 \cup C^0) \cap G$, it follows that C is I-integrable and $IC = I_A E$. In the same way, E is J-integrable, $JE = J_A E$. The proof is completed by a simple application of the definition of absolute continuity.

On a rather different tack, we now note that $f \to If + Jf$ is an integral on L, and so gives rise to a complete integration space $(X, L_1(K), K)$. We say that K is the *integral sum* of I and J. It is straightforward to show that $L_1(K) \subset L_1(I) \cap L_1(J)$, and hence that every K-full set is both I- and J-full. Moreover, if J is I-absolutely continuous, then so is K; while if, in addition, X is Σ-finite with respect to J, then f is K-integrable if and only if it is both I- and J-integrable (in which case $Kf = If + Jf$), and a set is K-full if and only if it is I-full (and therefore J-full). Of these assertions, the first is trivial and the last follows from the above and Proposition 2; to prove the remainder, we need only consider $f \geqq 0$ in $L_1(I) \cap L_1(J)$, and apply Lemmas 1 and 2, Proposition 1 and *MCT*. When J is I-absolutely continuous, we write $I + J$ for K.

Note that if X is finite with respect to both I and J, then it is also finite with respect to K; moreover, we then have $L_2(K) \subset L_1(K)$ (where $L_2(K) \equiv \{f\colon f^2 \in L_1(K)\}$), and Schwartz's inequality shows that for each $f = f.1$ in $L_2(K)$,

$$|Jf| \leqq J|f| \leqq K|f| \leqq K(1)^{\frac{1}{2}} \; (K|f|^2)^{\frac{1}{2}}.$$

Hence J is a bounded linear functional on the real Hilbert space $L_2(K)$ (with scalar product $(f, g) \equiv K(fg)$). When this linear functional is normable, we shall say that the integral J is *normable* with respect to I. A necessary and sufficient condition for this is that

$$\sup\{|Jf|\colon f^2 \in L_2(K), \; K|f|^2 \leqq 1\}$$

be computable.

5. The Radon–Nikodym theorem. Taken together with its corollary and converse, the following theorem is the main result of this paper.

THEOREM 1—THE RADON–NIKODYM THEOREM. *Let I, J be the complete extensions of two integrals defined on the same initial integration set L, such that J is I-absolutely continuous and X is J-finite. Suppose that there exists a sequence* $X_1 < X_2 < \cdots$ *of I-integrable sets such that* χ_{X_n} *converges in I- and J-measure to* 1, *and* J_{X_n} *is normable with respect to* I_{X_n} *for each* $n \geqq 1$. *Then there exists an essentially unique I-integrable function* f_0 *such that, for each f in* $L_1(I) \cap L_1(J)$, ff_0 *is I-integrable and* $I(ff_0) = Jf$.

Proof. With K the integral sum of I and J, we first suppose that $\chi_{X_n} = 1$ for all $n \geqq 1$ (so that X is finite with respect to I, J and K), and apply the Riesz Representation Theorem to obtain an essentially unique function g in $L_2(K)$ such that $Jf = K(fg)$ for each f in $L_2(K)$. We now prove that there exists a K-integrable set A with $KA = 0$ and $g \geqq 0$ on A^0. Let $(r_k)_{k \geqq 1}$ be a sequence of positive numbers decreasing to 0 such that each set $(-g \geqq r_k)$ is K-integrable. Supposing that $K(-g \geqq r_k) > 0$, we obtain the contradiction

$$J(-g \geqq r_k) = K(\chi_{(-g \geqq r_k)} g) \leqq -r_k K(-g \geqq r_k) < 0.$$

Hence $K(-g \geqq r_k) = 0$, and so [2, Proposition 2.10] the complemented set $A \equiv \vee_k(-g \geqq r_k)$ is K-integrable, $KA = 0$ and, clearly, $g \geqq 0$ on A^0.

Next, we let $f \geqq 0$ belong to $L_1(K)$, and show that $fg \in L_1(K)$ and $Jf = K(fg)$. For each $n \geqq 1$, $f \wedge n$ belongs to $L_2(K)$ (as X is K-finite); so that $(f \wedge n)g \in L_1(K)$ and $J(f \wedge n) = K((f \wedge n)g)$. It follows from *MCT* that $K((f \wedge n)g) \uparrow Jf$, whence (*MCT* and Proposition 1) $((f \wedge n)g)_{n \geqq 1}$ converges increasingly in K-measure, and pointwise on a K-full set, to a K-integrable function h with $Kh = Jf$. But $((f \wedge n)g)$ converges pointwise to fg on the K-full set $dmn f \cap dmn g$. Hence $fg = h$, fg is K-integrable, and $K(fg) = Kh = Jf$.

To complete the construction of f_0, we first note that, for $f \geqq 0$ in $L_2(K)$ and p a nonnegative integer, $fg^p \in L_1(K)$ and $J(fg^p) = K(fg^{p+1})$; so that

$$\begin{aligned} Jf = K(fg) &= I(fg) + J(fg) \\ &= I(fg) + K(fg^2) \\ &= I(fg) + I(fg^2) + J(fg^2) \\ &= \cdots \\ &= I\left(f \sum_{n=1}^{p} g^n\right) + J(fg^p). \end{aligned}$$

In particular, choosing $\epsilon > 0$ and then $r > (1 + J(1)^{-1})^{-1}$ so that $(g \geqq r)$ is K-integrable, we have

$$J(1) \geqq J(g \geqq r) = I\left(\chi_{(g \geqq r)} \sum_{n=1}^{p} g^n\right) + J(\chi_{(g \geqq r)} g^p)$$
$$\geqq I(g \geqq r) \sum_{n=1}^{p} r^n.$$

Letting $n \to \infty$, we get

$$J(1) \geqq I(g \geqq r) r/(1-r),$$
$$I(g \geqq r) \leqq (r^{-1} - 1) J(1) < \epsilon.$$

We therefore can construct a sequence of positive numbers r_k such that $r_k \uparrow 1$, $(g \geqq r_k)$ is K-integrable, and $I(g \geqq r_k) \downarrow 0$. By [**2**, Proposition 2.10] the complemented set $C \equiv \wedge_{k=1}^{\infty} (g \geqq r_k)$ is I-integrable, $IC = 0$. Thus C^0—on which it is clear that $g < 1$—is I-full.

We now show that $fg^n \downarrow 0$ I-almost everywhere for each $f \geqq 0$ in $L_2(K)$. To do so, we choose α, β, N so that $0 < \alpha$, $0 < \beta < 1$, the sets $(|f| \geqq \alpha)$, $(g \geqq \beta)$ are I-integrable; $I(|f| \geqq \alpha) < \epsilon$, $I(g \geqq \beta) < \epsilon$; and $\beta^n < \alpha^{-1}\epsilon$ for all $n \geqq N$. Then, with A as above and $B \equiv (|f| < \alpha) \wedge (g < \beta) \wedge -A$, we have B and $-B$ I-integrable, $I(-B) < 2\epsilon$, and

$$|fg^n| \chi_B \leqq \alpha\beta^n < \epsilon \qquad (n \geqq N).$$

It now follows from the Corollary to Proposition 2 and DCT that $J(fg^n) \downarrow 0$, and therefore that $(I(f \sum_{n=1}^{p} g^n))_{p \geqq 1}$ converges to Jf. By MCT and Proposition 1, $(f \sum_{n=1}^{p} g^n)_{p \geqq 1}$ converges increasingly in I-measure, and pointwise on an I-full set, to an I-integrable function ψ with $I\psi = Jf$. But $(f \sum_{n=1}^{p} g^n)_{p \geqq 1}$ converges pointwise to $f \sum_n g^n$ on the I-full set $A^0 \cap C^0 \cap dmnf$. Defining $f_0 \equiv \sum_n g^n$ on $A^0 \cap C^0$, we therefore have ff_0 I-integrable, $I(ff_0) = Jf$. Moreover, this obtains when $f \geqq 0$ belongs to L. For then $f \wedge n \in L_2(K)$ for each $n \geqq 1$, $I((f \wedge n) f_0) = J(f \wedge n) \uparrow Jf$, and so ($MCT$ and Proposition 1) $((f \wedge n) f_0)_{n \geqq 1}$ converges increasingly in I-measure, and pointwise on an I-full set, to an I-integrable function θ with $I\theta = Jf$. The desired result follows because $((f \wedge n) f_0)_{n \geqq 1}$ clearly converges pointwise to ff_0 on the I-full set $dmnf \cap dmnf_0$.

To extend this to $f \geqq 0$ in $L_1(I) \cap L_1(J)$, we choose an I-representation (f_n) on f in L, and note that

$$(\phi_n) \equiv (|f_1|, \quad |f_1 + f_2| - |f_1|, \quad |f_1 + f_2 + f_3| - |f_1 + f_2|, \cdots)$$

is both an I- and a J-representation in L of the K-integrable function $\Sigma_n \phi_n$, and (Lemma 1) that $f = \Sigma_n \phi_n$ on the I-, J-, and K-full set F on which $\Sigma_n |\phi_n| < \infty$. As

$$\sum_n I|\phi_n f_0| = \sum_n J|\phi_n| < \infty,$$

we see that ff_0—equal to $\Sigma_n \phi_n f_0$ on the I-full set $F \cap dmn f_0$—is I-integrable, with

$$I(ff_0) = \sum_n I(\phi_n f_0) = \sum_n J\phi_n = J\Big(\sum_n \phi_n\Big) = Jf.$$

In particular, we note that f_0 is I-integrable, $If_0 = Jf$.

Returning now to the general case, we define the integrals $I_n \equiv I_{X_n}$, $J_n \equiv J_{X_n}$ for each $n \geqq 1$. Given $N \geqq 1$, and bearing in mind Lemma 3, we can produce an I_N-integrable function $h_N \geqq 0$ such that, if $f \in L_1(I) \cap L_1(J)$, then fh_N is I_N-integrable and $J_N f = I_N(fh_N)$. We now define

$$\psi_N(x) \equiv \begin{cases} h_N(x) & (x \in X_N^1 \cap dmn\, h_N) \\ 0 & (x \in X_N^0). \end{cases}$$

Note that $dmn\psi_N$—a superset of $((X_N^1 \cup X_N^0) \cap dmn\, h_N) \cup X_N^0$—is I-full. Let $f \in L_1(I) \cap L_1(J)$. Then there exists a sequence (ϕ_n) in L such that

$$\sum_n I|\chi_{X_n}\phi_n| = \sum_n I_N|\phi_n| < \infty,$$

$fh_N = \Sigma_n \phi_n$ on the I_N-full set Γ on which $\Sigma_n |\phi_n| < \infty$, and

$$J(\chi_{X_N} f) = J_N f = I_N(fh_N) = \sum_n I(\chi_{X_N}\phi_n).$$

Clearly, the set G on which $\Sigma_n |\chi_{X_N}\phi_n| < \infty$ is I-full, $\Sigma_n \chi_{X_N}\phi_n$ is I-integrable, and

$$I\Big(\sum_n \chi_{X_N}\phi_n\Big) = \sum_n I(\chi_{X_N}\phi_n) = J(\chi_{X_N} f).$$

Now

$$(G \cap \Gamma) \cup (G \cap X_N^0) \supset G \cap (\Gamma u X_N^0),$$

whence $(G \cap \Gamma) \cup (G \cap X_N^0)$ is I-full. As $f\psi_N = \Sigma_n \chi_{X_N}\phi_n$ on this set, $f\psi_N$ is I-integrable, and $I(f\psi_N) = J(\chi_{X_N} f)$.

In particular, as $\chi_{X_{N+1}}\psi_N = \chi_{X_N}\psi_N$ on the I-full set $(X_N^1 \cup X_N^0) \cap dmn\, \psi_N$, we see that, for any I-integrable set E,

$$\begin{aligned} I(\chi_{X_N}\chi_E\psi_{N+1}) &= J(\chi_{X_{N+1}}\chi_{X_N}\chi_E) \\ &= I(\chi_{X_{N+1}}\chi_E\psi_N) \\ &= I(\chi_{X_N}\chi_E\psi_N). \end{aligned}$$

Hence $\chi_{X_N}\psi_{N+1} = \chi_{X_N}\psi_N$ on an I-full set F_N. Thus there is a unique function f_0 defined on the I-full set

$$H \equiv \left(\bigcap_{N=1}^{\infty} F_N \right) \cap \left(\bigcup_{n=1}^{\infty} X_n^1 \right)$$

and such that, for each $N \geqq 1$ and each x in $H \cap X_N^1$, $f_0(x) = \psi_N(x)$. In fact, (ψ_n) converges in I-measure to f_0: for, given $\epsilon > 0$ and an I-integrable set E, choosing ν so that $IE - I(X_\nu \wedge E) < \epsilon$ for $n \geqq \nu$, and defining $B \equiv X_\nu \wedge E$, we have $B \in L_1(I)$, $B < E$, $I(E - B) < \epsilon$ and

$$|f_0 - \psi_n|\chi_B \leqq |h - \psi_n|\chi_{X_\nu} = 0 \qquad (n \geqq \nu).$$

Thus f_0 is I-measurable. Note also that $f_0 \geqq 0$, and that

$$\psi_1 \leqq \psi_2 \leqq \cdots \leqq \psi_n = \chi_{X_n} f_0 \leqq \chi_{X_{n+1}} f_0 = \psi_{n+1} \leqq \cdots$$

It follows that $I(\chi_{X_n} f_0) = J\chi_{X_n}$, and therefore (from *MCT* and Proposition 1) that f_0 is I-integrable. The identity $I(f\psi_n) = J(\chi_{X_n} f)$, *MCT*, Proposition 1 and the foregoing combine to show that, when $f \geqq 0$ belongs to $L_1(I) \cap L_1(J)$, then ff_0 is I-integrable and $I(ff_0) = Jf$. The extension to the case of general f in $L_1(I) \cap L_1(J)$ is trivial.

We omit the straightforward proof that f_0 is essentially unique, in the sense that, if f_0' is another nonnegative function in $L_1(I)$ such that $ff_0' \in L_1(I)$ and $I(ff_0') = Jf$ whenever f belongs to $L_1(I) \cap L_1(J)$, then $f_0 = f_0'$ on an I-full set.

COROLLARY. *For each $r > 0$ and each f with J-full domain, define*

$$f_r(x) \equiv \begin{cases} f(x) & \text{if } x \in dmn\, f \text{ and } f_0(x) \geqq r \\ 0 & \text{if } f_0(x) < r. \end{cases}$$

Let $\Gamma \equiv \{r > 0 \colon (f_0 \geqq r) \in L_1(I)\}$. Then f is J-integrable if and only if $f_r f_0$ is I-integrable for each r in Γ and $(f_r f_0)_{r \in \Gamma}$ converges in I-measure to an

I-integrable function g as $r \to 0$; in which case $Ig = Jf$, $g = ff_0$ on a J-full set, and $g = 0$ whenever $f_0 = 0$.

Proof. We first show that, for each J-full set F and each r in Γ,

$$F_= \equiv (\{x \in X : f_0(x) \geqq r\} \cap F) \cup \{x \in X : f_0(x) < r\}$$

is I-full. We suppose without loss of generality that $F = E^1 \cup E^0$ for some J-null set E. Let (ϕ_n) be a J-representation of χ_E in L, and write $\chi_r \equiv \chi_{(f_0 \geqq r)}$ for each $r > 0$. Then, as $\chi_r \leqq r^{-1} f_0$ on the I-full set

$$(\{x \in X : f_0(x) \geqq r\} \cup \{x \in X : f_0(x) < r\}) \cap dmn\, f_0,$$

we have

$$\sum_n I|\chi_r \phi_n| \leqq r^{-1} \sum_n I(|\phi_n| f_0) = r^{-1} \sum_n J|\phi_n| < \infty.$$

Thus the set H on which $\Sigma_n \chi_r |\phi_n| < \infty$ is I-full. But, as $x \in H$ if and only if either $\chi_r(x) = 0$ or $\Sigma_n |\phi_n(x)| < \infty$, $H = F_=$.

Now let E be an I-integrable set, and $\epsilon > 0$. Choosing in turn ρ, r in Γ with $\rho IE < \epsilon$ and $r \leqq \rho$, we see that $\chi_E(1 - \chi_r) \in L_2(K)$; whence

$$\begin{aligned} J(E \wedge (f_0 < r)) &= J(\chi_E(1 - \chi_r)) = I(\chi_E(1 - \chi_r) f_0) \\ &\leqq rI(\chi_E(1 - \chi_r)) \leqq rIE \leqq IE \\ &< \epsilon. \end{aligned}$$

In particular, if we choose N so that $J(1 - \chi_{X_N}) < \epsilon$, and set $E \equiv X_N$, $B \equiv (f_0 \geqq \rho)$, we have $B \in L_1(J)$,

$$\begin{aligned} J(-B) &= J(1 - \chi_{X_N}) + J(X_N - B) \\ &< \epsilon + J(X_N \wedge (f_0 < \rho)) \\ &< 2\epsilon. \end{aligned}$$

On the other hand, for each $r \leqq \rho$ in Γ and each x in the J-full set $(B^1 \cup B^0) \cap dmn\, f$, we clearly have $f_r(x) = f(x)$. Thus we have shown that $(f_r)_{r \in \Gamma}$ converges in J-measure to f as $r \to 0$.

Supposing that $r \in \Gamma$ and that f_r is J-integrable, with J-representation (ϕ_n), we see that $\Sigma_n \chi_r \phi_n = \chi_r f_r = f_r$ on the J-full set F on which $\Sigma_n |\phi_n| < \infty$, and therefore that $\Sigma_n \chi_r \phi_n f_0 = f_r f_0$ on the I-full $F_=$. But

$$\sum_n I|\chi_r \phi_n f_0| = \sum_n J|\chi_r \phi_n| \leqq \sum_n J|\phi_n| < \infty;$$

so that $\Sigma_n \chi_r \phi_n f_0$ is I-integrable, $\Sigma_n \chi_r \phi_n$ is J-integrable, and

$$I\Big(\sum_n \chi_r \phi_n f_0\Big) = \sum_n I(\chi_r \phi_n f_0)$$
$$= \sum_n J(\chi_r \phi_n)$$
$$= J\Big(\sum_n \chi_r \phi_n\Big).$$

Hence $f_r f_0$ is I-integrable, $I(f_r f_0) = Jf_r$. On the other hand, as

$$f_r = (\max(f_0, r)^{-1})\chi_r f_r f_0 \leqq r^{-1} f_r f_0$$

on the I-full set

$$(\{x \in X: f_0(x) \geqq r\} \cap dmn f) \cup \{x \in X: f_0(x) < r\},$$

if $f_r f_0$ is I-integrable, then f_r is I-integrable, and so J-integrable. Thus, $f_r f_0$ is I-integrable if and only if $f_r \in L_1(I) \cap L_1(J)$, in which case $Jf_r = I(f_r f_0)$.

To complete the proof, we suppose without loss of generality that $f \geqq 0$. If f is J-integrable, then (as $f_r = \chi_r f$ on a J-full set) f_r is J-integrable for each r in Γ; whence $f_r f_0$ is I-integrable, $I(f_r f_0) = Jf_r$, and (MCT) $I(f_r f_0) \uparrow Jf$ as $r \downarrow 0$ through Γ. Applying MCT once again, we see that $(f_r f_0)_{r \in \Gamma}$ converges in I-measure to an I-integrable function g as $r \to 0$, and that $Ig = Jf$. Conversely, suppose that $f_r f_0 \in L_1(I)$ for each r in Γ, and that $(f_r f_0)_{r \in \Gamma}$ converges in I-measure to an I-integrable function g as $r \to 0$ through Γ. Then, for each r in Γ, $f_r \in L_1(J)$, $Jf_r = I(f_r f_0)$ and so (MCT) $Jf_r \uparrow Ig$ as $r \downarrow 0$. It follows from this, MCT and Proposition 1 that, as $r \downarrow 0$, $(f_r)_{r \in \Gamma}$ converges increasingly in J-measure, and pointwise on a J-full set, to a J-integrable function ψ with $J\psi = Ig$. It is clear from the foregoing and Proposition 1 that $\psi = f$ on a J-full set; so that f is J-integrable, $Jf = J\psi = Ig$.

REMARKS.

1. Let $f \geqq 0$ be defined on a J-full set, and suppose that there exists I-integrable $g \geqq 0$ such that $g = ff_0$ on a J-full set F and $g = 0$ on $\{x \in X: f_0(x) = 0\} \cap dmn\, g$. Given r in Γ, we have $f_r f_0 = \chi_r g \leqq g$; whence $f_r f_0$ is I-integrable, f_r is J-integrable, $(f_r)_{r \in \Gamma}$ converges in J-measure to f as $r \to 0$, and f is J-measurable (cf. proof of the above Corollary).

Note that f is classically J-integrable: for we have $Jf_r \uparrow$ as $r \downarrow 0$ through Γ, and $Jf_r = I(f_r f_0) \leqq Ig$. Unfortunately, we have been unable to produce a constructive proof of the J-integrability of f except in the trivial case where f is bounded, and that in which χ_r converges in I-measure as $r \to 0$ through Γ. In the latter case, we see that χ_r must converge to the indicator χ_0 of the complemented set $(f_0 > 0)$; so that $dmn\,\chi_0$ is I-full. Choosing an I-integrable set E, positive ϵ, $R > 0$ so that $I(g \geqq R) < \epsilon$, and $\rho > 0$ such that, whenever $r \in \Gamma$ and $r \leqq \rho$, there exists an I-integrable set $C_r < E$ with $I(E - C_r) < \epsilon$, $|\chi_0 - \chi_r|\chi_{C_r} < R^{-1}\epsilon$, we set $B_r \equiv C_r \wedge (g < R)$ to obtain $B_r \in L_1(I)$, $B_r < E$, $I(E - B_r) < 2\epsilon$, $|(\chi_0 - \chi_r)g|\chi_{B_r} < \epsilon$. Thus

$$f_r f_0 = \chi_r g \uparrow \chi_0 g = g$$

in I-measure as $r \downarrow 0$ through Γ. The above Corollary now ensures that f is J-integrable, $Jf = Ig$.

In general, it is easily seen that if f is J-integrable, then $Jf = Ig$.

2. A simple argument, which we omit, shows that, up to equality on I-full sets, f_0 is the unique nonnegative I-integrable function with the property stated in the above Corollary.

We still have to show that the conditions of the Radon–Nikodym Theorem do obtain in a non-trivial context. That this is so is the substance of the following converse of Theorem 1.

THEOREM 2. *Let (X, L, I) be an integration space and f_0 a non-negative element of $L_1(I)$ such that $If_0 > 0$, and $ff_0 \in L_1(I)$ for each f in L. Then $f \to I(ff_0)$ is an integral on L whose complete extension J is I-absolutely continuous. Moreover, if there exists a sequence $X_1 < X_2 < \cdots$ of I-integrable sets such that $\chi_{X_n} \uparrow 1$ in I-measure, then each X_n is J-integrable, $\chi_{X_N} \uparrow 1$ in J-measure, X is J-finite, and J_{X_n} is normable with respect to I_{X_N} for each $n \geqq 1$.*

The proof follows the lines of the well-known classical analogue (with obvious modifications where the classical argument succumbs to the lure of nonconstructivity), and is left to the reader.

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