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INFINITE DECOMPOSITION BASES

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The concept of decomposition basis is essential to the study of the structure of mixed abelian groups. The main theorem of this paper uses invariants previously defined by the author to determine the decomposition bases of a given group. This is used to extend Ulm's theorem for mixed abelian groups to the entire class of affable groups.

1. Introduction. The word "group" will mean abelian group throughout. If X is a subset of a group A, $[X]$ denotes the subgroup generated by X. A_t (A_p) represents the torsion part (p-component) of the group A, where p is a prime. Z will denote the integers, while Q represents the rationals. Standard terminology (refer to Fuchs **[1]** and **[2]**) will be used unless otherwise indicated.

The notation $H(a)$ refers to the height matrix of an element a, while $H_p(a)$ is the p-indicator of a. If M is a height matrix, then M_p denotes the p-row or p-indicator of M. Two height matrices M and N (p-indicators M_p and N_p) are *equivalent* if there are integers m and n such that $mM = nN$ ($mM_p = nN_p$). M and N are *compatible* if there are integers m and n such that $mM \geqq N$ and $nN \geqq M$. We say the p-height of 0 is ∞', and assume $\sigma < \infty < \infty'$ for any ordinal σ. A height matrix is *proper* if it does not contain ∞' as any entry. For any group A, the subgroup generated by the elements that are not proper is A_t.

If c is an equivalence class consisting of proper height matrices (compatibility class), p is a prime and e an equivalence class of p-indicators, the invariant $\mathrm{ST}(c, p, e, A)$ is the one defined in **[6]**. If the height matrix $M \in c$ and $M_p \in e$, we say $M \in [c, p, e]$.

If σ is an ordinal or ∞, and G is a subgroup of A, then $f^p_\sigma(A)$ and $f^p_\sigma(A, G)$ denote the Ulm invariants and relative Ulm invariants respectively. A torsion group T is called *totally projective* if each T_p is a totally projective p-group.

Let $X = \{x_i\}_I$ be a subset of a group A. X is called a *decomposition basis* if $[X]$ is the free group on X, $A/[X]$ is torsion, and, for any $a = \Sigma r_i x_i$ in $[X]$, where the r_i are integers, the p-height of a, $h_p(a) = \min\{h_p(r_i x_i)\}$ for all primes p. (All heights are computed in A.) The subset $Y = \{y_i\}_I$ of A is a *subordinate decomposition basis* to X if $y_i = n_i x_i$ for all i, where the n_i are integers. It is shown in **[6]** that if X is any decomposition basis of A, then $\mathrm{ST}(c, p, e, A)$ is the cardinality of

$$\{x \in X : H(x) \in [c, p, e]\}.$$

Sections 2 and 3 of this paper are devoted mainly to the proof of Theorem 2.2. This theorem says, in effect, that the main criterion in switching from one decomposition basis to another in a group A is the preservation of ST-invariants. Henceforth the term "rank" will refer to the torsion free rank of a group.

The major application of Theorem 2.2 is to extend Ulm's theorem for mixed groups. If A is a group and G is a subgroup, define

$$A(p, G) = \{a \in A : p^k a \in G \text{ for some } k \geqq 0\}.$$

This definition is due to Wallace **[8]**. His techniques yield the following version of the generalization of Ulm's theorem (see Hill **[3]**, Walker **[7]**).

THEOREM 1.1. *Let A and B be groups containing subgroups G and H respectively such that G, (H), is p-nice in $A(p, G)$, $(B(p, H))$, and such that $f_\sigma^p(A, G) = f_\sigma^p(B, H)$ for all primes p, and for σ an ordinal or ∞. If A/G and B/H are totally projective, and $\phi: G \to H$ is a height preserving isomorphism, then ϕ extends to an isomorphism $\psi: A \to B$.*

In §4, Ulm's theorem is extended to a class of groups including the affable groups defined in **[6]**. Some interesting consequences are also discussed.

2. Equivalence of decomposition bases. In this section we begin the proof of the theorem relating basis equivalence to the ST-invariants.

In a collection of height matrices, the same matrix may be included more than once. If $\mathbf{S}$ is a collection of height matrices, A is a group with decomposition basis X, and there is a bijection $\alpha: X \to \mathbf{S}$ such that $H(x)$ is equivalent to αx for all $x \in X$, we write $H(X) \sim \mathbf{S}$. The following definition, as well as Lemma 2.1, are due to Hunter **[4]**.

DEFINITION. Let $\mathbf{S}$ and $\mathbf{S}'$ be two collections of height matrices. Suppose A is a group having decomposition bases X and Y such that $H(X) \sim \mathbf{S}$ and $H(Y) \sim \mathbf{S}'$. Then $\mathbf{S}$ and $\mathbf{S}'$ are called *basis-equivalent*, denoted $\mathbf{S} \sim_b \mathbf{S}'$.

LEMMA 2.1. *Let $\mathbf{S}$ and $\mathbf{S}'$ be two collections of height matrices, and suppose $\mathbf{S} \sim_b \mathbf{S}'$,* (with respect to a group A). *If B is any group with a decomposition basis X such that $H(X) \sim \mathbf{S}$, then there is a decomposition basis Y of B such that $H(Y) \sim \mathbf{S}'$.*

Aside from ensuring that the concept of basis-equivalence is independent of a particular group, Lemma 2.1 also guarantees that basis-equivalence is an equivalence relation.

DEFINITION. Let A be a group with a decomposition basis $X = \{x_i\}_{i\in I}$. If $A = \bigoplus_{i\in I} A_i$, with $x_i \in A_i$ for all i, then X is called a *splitting decomposition basis.*

Given any proper height matrix M, there is a rank one group A with an element $x \in A$ of infinite order such that $H(x) = M$. Consequently, given any collection of height matrices $\mathbf{S}$, there is a group A with splitting decomposition basis X such that $H(X) \sim \mathbf{S}$. If $\mathbf{S}$ is a collection of height matrices, then $\mathrm{ST}(c, p, e, \mathbf{S})$ denotes the cardinality of the set of height matrices in $\mathbf{S}$ that are also in $[c, p, e]$.

We will now state the main result, and reduce the proof to the Lemma in §3, and Lemma 7 of **[6]**.

THEOREM 2.2. *Let* $\mathbf{S}$ *and* $\mathbf{S}'$ *be two collections of height matrices. Then the following three conditions are equivalent.*

(a) $\mathbf{S}$ *is basis-equivalent to* $\mathbf{S}'$.

(b) *For all* c, p *and* e, $\mathrm{ST}(c, p, e, \mathbf{S}) = \mathrm{ST}(c, p, e, \mathbf{S}')$.

(c) (i) *There is a bijection* $\phi: \mathbf{S} \to \mathbf{S}'$ *such that* M *and* $\phi(M)$ *are compatible for all* $M \in \mathbf{S}$.

(ii) *For each prime* p, *there is a bijection* $\psi_p: \mathbf{S} \to \mathbf{S}'$ *such that* $\psi_p(M)$ *is compatible with* M *and* $(\psi_p(M))_p$ *is equivalent to* M_p. *Moreover,* $\psi_p(M) = \phi(M)$ *for all but finitely many primes* p.

Proof. (a) implies (b) is trivial. The proof of (b) implies (c) is based on the following set theoretical argument. We claim we may write $\mathbf{S}$ as a disjoint union $\mathbf{S} = \mathbf{T} \cup (\bigcup_p \mathbf{T}_p)$ subject to the following two properties.

(α) If $M \in \mathbf{T}_p$, with $M \in [c, p, e]$, then $\mathrm{ST}(c, p, e, \mathbf{S})$ is infinite.

(β) If $\mathrm{ST}(c, p, e, \mathbf{S})$ is infinite, then $\mathrm{ST}(c, p, e, \mathbf{S}) = \mathrm{ST}(c, p, e, \mathbf{T}_p) = \mathrm{ST}(c, p, e, \mathbf{T})$.

A similar decomposition $\mathbf{S}' = \mathbf{T}' \cup (\bigcup_p \mathbf{T}'_p)$ is formed.

We now justify our claim that the decomposition can be accomplished. By transfinite induction we may write $\mathbf{S}$ as a disjoint union of countable subsets, $\mathbf{S} = \bigcup_\sigma \mathbf{S}_\sigma$, where each $\mathbf{S}_\sigma$ has the property that, whenever $\mathrm{ST}(c, p, e, \mathbf{S})$ is infinite, then either $\mathrm{ST}(c, p, e, \mathbf{S}_\sigma) = 0$ or $\mathrm{ST}(c, p, e, \mathbf{S}_\sigma) = \aleph_0$.

For any given σ, there are at most countably many $[c, p, e]$ with $\mathrm{ST}(c, p, e, \mathbf{S}_\sigma) = \aleph_0$. Order these $[c, p, e]$, and let $\mathbf{K}_i$ be the set of those elements of $\mathbf{S}_\sigma$ belonging to the i-th $[c, p, e]$. We now define subsets $\mathbf{T}_{ij}$ $(i \geqq 1, j \geqq i)$ of $\mathbf{S}_\sigma$ as follows.

Let $\mathbf{T}_{11}$ be a subset of $\mathbf{S}_\sigma$ such that

(1) $$\mathbf{T}_{11} \subseteq \mathbf{K}_1$$

(2) $$|\mathbf{T}_{11}| = \aleph_0.$$

Let M_{11} be any element of $\mathbf{T}_{11}$. Suppose that $\mathbf{T}_{ij}$ along with elements $M_{ij} \in \mathbf{T}_{ij}$ have been defined for all $j < n$. If

$$|\mathbf{K}_n \cap (\mathbf{S}_\sigma \backslash (\bigcup_{i<n} \mathbf{T}_{i,n-1}))| = \aleph_0, \qquad (\neq)$$

then we can define $\mathbf{T}_{nn}$ from elements in $\mathbf{S}_\sigma \backslash (\bigcup_{i<n} \mathbf{T}_{i,n-1})$, so that properties analogous to (1) and (2) hold. In this case, define $\mathbf{T}_{in} = \mathbf{T}_{i,n-1}$, $(i < n)$ and the element M_{in} $(i \leqq n)$ is chosen in $\mathbf{T}_{in}$ to be distinct from any previously selected element.

Suppose $(\neq)$ fails to hold. Then for some i, $|\mathbf{T}_{i,n-1} \cap \mathbf{K}_n| = \aleph_0$. Now $\mathbf{T}_{nn}$ will be a countable subset of $\mathbf{T}_{i,n-1}$ such that

$$\mathbf{T}_{nn} \cap \{M_{ii}, M_{i,i+1}, \cdots, M_{i,n-1}\} = \varnothing,$$
$$|\mathbf{T}_{nn}| = \aleph_0$$
$$|\mathbf{T}_{i,n-1} \backslash \mathbf{T}_{nn}| = \aleph_0.$$

Define $\mathbf{T}_{in} = \mathbf{T}_{i,n-1} \backslash \mathbf{T}_{nn}$, and let $\mathbf{T}_{jn} = \mathbf{T}_{j,n-1}$ for $j \neq i$, $j < n$. Select arbitrary elements M_{in} $(i \leqq n)$ distinct from the previously selected elements.

For each i, define $\mathbf{V}_i = \bigcap_{j \geqq i} \mathbf{T}_{ij}$. Since M_{ij} is in $\mathbf{V}_i$ for all $j \geqq i$, $|\mathbf{V}_i| = \aleph_0$. Let $\mathbf{V}_i$ be a disjoint union of $\mathbf{V}_{i1}$ and $\mathbf{V}_{i2}$, where $|\mathbf{V}_{i1}| = |\mathbf{V}_{i2}| = \aleph_0$. For a fixed prime p, let $\mathbf{V}_{p\sigma}$ be the union of those $\mathbf{V}_{i1}$ for which $\mathbf{K}_i$ is represented by some $[c, p, e]$. Let $\mathbf{T}_p = \bigcup_\sigma \mathbf{V}_{p\sigma}$, and let $\mathbf{T}$ be the set of elements not in any $\mathbf{T}_p$. Note that $\mathbf{T}$ contains all $\mathbf{V}_{i2}$. Then $\mathbf{S} = \mathbf{T} \cup (\bigcup_p \mathbf{T}_p)$ satisfies the required conditions.

By (b) and the above construction, we may define $\phi : \mathbf{S} \to \mathbf{S}'$ satisfying (i), and also requiring that the restriction to $\mathbf{T}$ (respectively $\mathbf{T}_p$) map onto $\mathbf{T}'$ ($\mathbf{T}'_p$). It follows from (b) that maps ψ_p satisfying the first sentence of (ii) can be defined. We also claim that the ψ_p can be required to satisfy the condition that if $M \notin \mathbf{T}_p$, and $M_p = (\phi(M))_p$, then $\psi_p(M) = \phi(M)$. This condition clearly allows the second sentence of (ii) to be satisfied. Let $\mathbf{K}$ be the subset of $\mathbf{S}$ consisting of all elements in $[c, p, e]$, and let $\mathbf{L}$ be the subset of $\mathbf{K}$ such that $M \notin \mathbf{T}_p$ and $M_p = (\phi(M))_p$. If $\mathbf{K}$ is finite, then ψ_p can be defined as indicated. If $\mathbf{K}$ is infinite, define ψ_p on $\mathbf{L}$ first. Let $\mathbf{K}'$ be the subset of $\mathbf{S}'$ consisting of all elements in $[c, p, e]$. By the above construction, $|\mathbf{K} \backslash \mathbf{L}| = |\mathbf{K}| = |\mathbf{K}'| = |\mathbf{K}' \backslash \phi(\mathbf{L})|$, so ψ_p can be defined to map $\mathbf{K} \backslash \mathbf{L}$ onto $\mathbf{K}' \backslash \phi(\mathbf{L})$.

We now begin the proof of (c) implies (a). Represent $\mathbf{S}$ by a decomposition basis X of a group A, where $H(X) \sim \mathbf{S}$, and $\alpha: X \to \mathbf{S}$ is the corresponding bijection. We will find a decomposition basis Y of A such that $H(Y) \sim \mathbf{S}'$. For a fixed prime p, $\alpha^{-1}\psi_p^{-1}\phi\alpha$ is a permutation of X, and so splits X into equivalence classes consisting of countably infinite cycles, two-cycles, and fixed points. (The element x in X is a fixed point exactly if $\psi_p\alpha x = \phi\alpha x$.) Using transfinite induction, X may be written as a disjoint union of countable subsets, $X = \cup X_\sigma$, such that if $x \in X_\sigma$, p is any prime and k is any integer, $(\alpha^{-1}\psi_p^{-1}\phi\alpha)^k x$ is also in X_σ. We will replace each X_σ by a Y_σ to obtain the new decomposition basis Y.

We proceed from X_σ to Y_σ by introducing a sequence of intermediate sets W_i $(i = 1, 2, \cdots)$, such that $(X \backslash X_\sigma) \cup W_i$ is a decomposition basis for every i. Let $\mathbf{S}_i$ be the collection of height matrices of W_i, and let $\alpha^{(i)}: W_i \to \mathbf{S}_i$ be a canonical bijection. For each i, we will have maps $\phi^{(i)}: \mathbf{S}_i \to \mathbf{S}'$, $\psi_p^{(i)}: \mathbf{S}_i \to \mathbf{S}'$ with properties analogous to (i) and (ii). (We will drop the superscripts when there is no danger of confusion.) An element $x \in W_i$ is called *permanent* if $\phi^{(i)}(\alpha^{(i)}x) = \psi_p^{(i)}(\alpha^{(i)}x)$ for all primes p. We will require that if $x \in W_i$ is permanent, then $x \in W_j$ and that $\phi^{(i)}\alpha^{(i)}x = \phi^{(j)}\alpha^{(j)}x$ for all $j \geqq i$. A prime p is said to be *repaired* (with respect to W_i) if $\phi^{(i)} \doteq \psi_p^{(i)}$. We require that if p is repaired with respect to W_i, then p is repaired with respect to W_j for $j \geqq i$. Essentially, W_{i+1} is obtained from W_i by selecting and repairing a finite set of primes. We will indicate how this finite set is chosen after the proof of the following lemma.

LEMMA 2.3. *Let F be a finite subset of W_i. Then we may choose W_{i+1} so that each element of F is a linear combination of permanent elements of W_{i+1}.*

Proof. First we note that any prime p can be repaired. As indicated previously, $\alpha^{-1}\psi_p^{-1}\phi\alpha$ divides W_i into equivalence classes consisting of countably infinite cycles, two cycles, and fixed points. Let $V = \{x_i\}_{i \in Z}$ be the elements of an infinite cycle, with $x_{i+1} = \alpha^{-1}\psi_p^{-1}\phi\alpha x_i$. Then Lemma 3.1 will show that V can be replaced by $V' = \{y_i\}_{i \in Z}$ such that $H_p(y_i) = H_p(x_{i+1})$ and $H_q(x_i) = H_q(y_i)$, for $q \neq p$. Additionally, $[V] = [V']$. Let $\gamma: V \to V'$ be defined so that $\gamma(x_i) = y_i$, and $\delta: V \to V'$ so that $\delta(x_i) = y_{i-1}$, and let $\mathbf{S}^*$ be the collection of height matrices of V', with canonical bijection $\alpha^*: V' \to \mathbf{S}^*$. Define $\phi^*: \mathbf{S}^* \to \mathbf{S}'$ by $\phi^* = \phi\alpha\gamma^{-1}(\alpha^*)^{-1}$, and $\psi_q^*: \mathbf{S}^* \to \mathbf{S}'$ by $\psi_q^* = \psi_q\alpha\gamma^{-1}(\alpha^*)^{-1}$, for $q \neq p$. We define $\psi_p^*: \mathbf{S}^* \to \mathbf{S}'$ by $\psi_p^* = \psi_p\alpha\delta^{-1}(\alpha^*)^{-1}$. We show that $\psi_p^* = \phi^*$. If $M \in \mathbf{S}^*$, then $M = \alpha^* y_i$, for some $y_i \in V'$. Then

$$\begin{aligned}\psi_p^* M &= \psi_p\alpha\delta^{-1}(\alpha^*)^{-1}M = \psi_p\alpha\delta^{-1}y_i = \psi_p\alpha x_{i+1} = \psi_p\alpha\alpha^{-1}\psi_p^{-1}\phi\alpha x_i \\ &= \phi\alpha\gamma^{-1}(y_i) = \phi\alpha\gamma^{-1}(\alpha^*)^{-1}M = \phi^* M.\end{aligned}$$

In a similar way, p can be repaired for any two-cycle, using Lemma 7 of **[6]**, which essentially says for two-cycles what Lemma 3.1 says for infinite cycles. We have thus established the claim in the first line of the proof.

Call a prime *good* if $\psi_p(\alpha x) = \phi(\alpha x)$ for all $x \in F$. The remaining primes are *bad* and form a finite set $\{q_1, q_2, \cdots, q_n\}$. Suppose q_1 were to be repaired immediately in the manner described above. Then there would be a new decomposition basis T, each of the elements of F would be linear combinations of elements of T which in turn would be a linear combination of a finite set G_1 of elements in W_i. Starting with W_i, repair all the good primes p for which there is an element x in G_1 such that $\phi(\alpha x) \neq \psi_p(\alpha x)$. A new decomposition basis U_1 results. Since only good primes were repaired, the elements of F remain in U_1. An element x of G_1 is replaced by a new element y such that

$$\phi(\alpha x) = \phi^*(\alpha^* y),$$

$$\psi_q(\alpha x) = \psi_q^*(\alpha^* y), \text{ whenever } q \text{ is a bad prime.}$$

$$\psi_p^*(\alpha^* y) = \phi^*(\alpha^* y), \text{ whenever } p \text{ is a good prime.}$$

(ϕ^*, ψ_p^*, ψ_q^* have the obvious meaning.)

Let G_1^* be the set of elements replacing G_1. At this point we actually repair q_1, in the same manner as we did when obtaining T. The set G_1^* is replaced by a new set F_1. If $x \in G_1^*$, the corresponding y in F_1 has the properties

$$\phi^*(\alpha^* x) = \phi^{**}(\alpha^{**} y),$$

$$\psi_{q_1}^{**}(\alpha^{**} y) = \phi^{**}(\alpha^{**} y),$$

$$\psi_p^*(\alpha^* x) = \psi_p^{**}(\alpha^{**} y), \quad \text{for} \quad p \neq q_1.$$

By the construction of G_1, the elements of F are a linear combination of elements of F_1.

The same routine is continued, with F_1 replacing F, and the decomposition basis obtained after the repair of q_1 replacing W_i. The set of bad primes is now $\{q_2, \cdots, q_n\}$. At the last step, we obtain a set F_n, consisting of permanent elements, such that F is a linear combination of the elements of F_n. The decomposition basis at this step will be W_{i+1}.

We now complete the proof of Theorem 2.2. Write the elements of the countable set $X_\sigma = \{x_0, x_1, \cdots\}$, and order the set of all primes $\{p_1, p_2, \cdots\}$. Let $W_0 = X_\sigma$. To obtain W_1, let $F = \{x_0\}$ and use the lemma. If n is a positive integer, W_{2n} is obtained from W_{2n-1} by repairing p_i. The element x_n is a linear combination of elements of W_{2n}. Let the set of these elements be F, and again use Lemma 2.3 to obtain W_{2n+1}. Let Y_σ be the set of permanent elements of the sequence $\{W_i\}$.

Now $Y = \cup Y_\sigma$ is easily seen to be a decomposition basis of A, because of the above construction. The map $\chi: Y \to \mathbf{S}'$ defined by the appropriate $\phi^{(i)}\alpha^{(i)}$ for each $y \in Y$ manifests the fact that $H(Y) \sim \mathbf{S}'$.

3. Changing a countable decomposition basis. The following lemma completes the proof of Theorem 2.2.

LEMMA 3.1. *Let X be a decomposition basis of a group A, let $\{x_i\}_{i\in Z}$ be a countably infinite subset of X, in which all elements are pairwise compatible, and let p be a prime. Then there is a subset $\{y_i\}_{i\in Z}$ of A such that $H_p(y_i) = H_p(x_{i+1})$, and for $q \neq p$, $H_q(y_i) = H_q(x_i)$. Moreover, $Y = (X\backslash\{x_i\}) \cup \{y_i\}$ is a decomposition basis and $[X] = [Y]$.*

Proof. Let x and y be two elements of infinite order in A. Then the p-indicator of x is *superior* to the p-indicator of y, $H_p(x) \gg H_p(y)$, if and only if

(i) if $h_p(p^i y) = \infty$, then $h_p(p^i x) = \infty$

(ii) if $h_p(p^i y) \neq \infty$, then $h_p(p^i x) > h_p(p^i y)$.

For the remainder of the proof, p will be the prime specified in the lemma, while q will always be a prime not equal to p. We proceed inductively to define a set of coefficients which will be used to form the y_i.

A prime q is called *one-benign* if and only if $H_q(x_{-1}) = H_q(x_0) = H_q(x_1)$. We select integers $a_0, e_0, c_0, d_0, a_{-1}, b_{-1}, c_{-1}, e_{-1}$, to satisfy conditions to be listed later. (See (2)–(4).) In particular, e_0 and a_{-1} are divisible only by p, and e_{-1} and c_0 are not divisible by p. So there are integers u_0 and u_{-1} so that

$$u_0e_0a_{-1} - u_{-1}e_{-1}c_0 = 1. \tag{1}$$

Define $b_0 = u_0e_0$ and $d_{-1} = u_{-1}e_{-1}$.

If $n > 1$ is a positive integer, q is *n-benign* if

(1) q is $(n-1)$-benign,

(2) $H_q(x_{-n}) = H_q(x_0) = H_q(x_n)$,

(3) q does not divide b_{n-2} and d_{-n+1}.

For $n \geqq 1$, the integers $a_{n-1}, e_{n-1}, a_{-n}, b_{-n}$ are defined to be powers of p, and divisible by p. The following conditions must be satisfied.

$$\begin{aligned} &H_p(a_{n-1}x_{-n}) \gg H_p(x_n),\\ &H_p(e_{n-1}x_{n-1}) \gg H_p(x_n),\\ &H_p(a_{-n}x_n) \gg H_p(x_{-n+1}),\\ &H_p(b_{-n}x_{-n}) \gg H_p(x_{-n+1}). \end{aligned} \tag{2}$$

The integers c_{n-1}, d_{n-1}, c_{-n}, e_{-n} are divisible exactly by those primes q which are not n-benign. (There are finitely many such primes.) The following conditions must hold, whenever q is not n-benign.

$$
\begin{aligned}
H_q(c_{n-1}x_{-n}) &\gg H_q(x_{n-1}),\\
H_q(d_{n-1}x_n) &\gg H_q(x_{n-1}),\\
H_q(c_{-n}x_n) &\gg H_q(x_{-n}),\\
H_q(e_{-n}x_{-n+1}) &\gg H_q(x_{-n}).
\end{aligned}
\tag{3}
$$

The identities

$$
\begin{aligned}
b_{-n} &= a_{-n}a_{n-1},\\
d_{n-1} &= c_{-n}c_{n-1},
\end{aligned}
\tag{4}
$$

also must hold.

There are integers u_{n-1} and u_{-n} (for $n>1$) such that

$$
u_{n-1}e_{n-1}a_{n-2}a_{-n} + u_{-n}e_{-n}c_{n-1}c_{-n+1} \approx 1. \tag{5}
$$

Define $b_{n-1} = u_{n-1}e_{n-1}$ and $d_{-n} = u_{-n}e_{-n}$. Note that for $n>0$, and q not n-benign, $(u_{n-1}, q) = 1$, so primes that are not n-benign do not divide b_{n-1}. For $n>0$, $(u_{-n}, p) = 1$, so that p does not divide d_{-n}. If q is an n-benign prime, then q may divide b_{n-1} or d_{-n}, but not both.

We now define the new elements.

$$
\begin{aligned}
y_i &= b_i x_i + d_i x_{i+1} + a_i c_i x_{-i} \qquad (i<0),\\
y_i &= a_i c_i x_{-i-1} + b_i x_i + d_i x_{i+1} \qquad (i \geqq 0).
\end{aligned}
\tag{6}
$$

We check that y_i has the desired height properties.

$$
\begin{aligned}
H_p(y_i) &= H_p(d_i x_{i+1}) = H_p(x_i),\\
H_q(y_i) &= H_q(b_i x_i) = H_q(x_i),
\end{aligned}
$$

when $i \geqq 0$ and q is not $(i+1)$-benign, or when $i<0$ and q is not $(-i)$-benign.

$$
H_q(y_i) = H_q(d_i x_{i+1}) = H_q(x_{i+1}) = H_q(x_i)
$$

when $i \geqq 0$ and q is $(i+1)$-benign, or when $i<0$ and q is $(-i)$-benign.

We next show that each x_n is in the subgroup B generated by $\{y_i\}$. Routine calculations using (6), (4) and (1) show that $x_0 = a_{-1}y_0 - c_0y_{-1}$. We next assume that x_i is in B for $-k \leqq i \leqq k$, and show that both x_{k+1} and x_{-k-1} are in B. Using induction hypothesis on y_k and (6), the element $a_kc_kx_{-k-1} + d_kx_{k+1}$ is in B. Equations (6) and (4) and some computation yield

$$c_{k+1}y_{-k-2} - a_{-k-2}y_{k+1} = c_{k+1}d_{-k-2}x_{-k-1} - a_{-k-2}b_{k+1}x_{k+1}.$$

Combining terms and using (4) and (5), we find that the term

$$(d_kc_{k+1}d_{-k-2} + a_{-k-2}b_{k+1}a_kc_k)x_{-k-1} = c_kx_{-k-1}$$

is in B. Using y_{-k-1} in place of y_k in the above argument, we find that $a_{-k-1}x_{-k-1}$ is in B. Since $(c_k, a_{-k-1}) = 1$, we have $x_{-k-1} \in B$. Similarly, $x_{k+1} \in B$. Hence $B = [\{x_i\}]$.

Now it is easy to see that the elements of Y have infinite order, are independent, that $[X] = [Y]$, and that $A/[Y]$ is torsion. Hence Y will be a decomposition basis once we establish the height property. Clearly we need only consider linear combinations involving elements of $\{y_i\}$.

Well order $\{y_i\}$ as follows:

$$\{y_0, y_{-1}, y_1, y_{-2}, y_2, \cdots\}$$

and let $y_{(k)}$ represent the k-th term of this sequence. We wish to show, for any k, that

$$H\left(\sum_{i=1}^{k} m_iy_{(i)}\right) = \min_{1 \leqq i \leqq k} \{H(m_iy_{(i)})\}, \tag{7}$$

where the m_i are integers. We induct on k. When $k = 1$, we are reduced to the trivial equation $H(m_0y_0) = H(m_0y_0)$. When $k = 2$ we have a special case, and the proof is nearly identical to that of Lemma 7 of **[6]**. Two more cases remain.

Case A. k is even, $k \geqq 4$.

Case B. k is odd, $k \geqq 3$.

In either case, we may assume as induction hypothesis:

$$H\left(\sum_{i=1}^{k-1} m_iy_{(i)}\right) = \min_{1 \leqq i \leqq k-1} \{H(m_iy_{(i)})\}. \tag{8}$$

Moreover it is only necessary to prove that $\leqq$ holds in (7), as the other inequality is trivial.

We begin work on Case A. Assume that $y_{(k)} = y_n$. Now each of the terms in (7) and (8) is a linear combination of terms in $\{x_i\}$. Since $\{x_i\}$ is part of a decomposition basis, each height matrix is the minimum of the height matrices of the $\{x_i\}$ terms. In going from (8) to (7), the terms $H(m_k b_n x_n)$, $H(m_k d_n x_{n+1})$ and $H(m_k a_n c_n x_{-n})$ are added to the right hand side of (8). The terms

(9a) $$(m_k b_n + m_{k-1} a_{-n-1} c_{-n-1}) x_n,$$

(9b) $$(m_k d_n + m_{k-2} b_{n+1} + m_{k-3} a_{-n-2} c_{-n-2}) x_{n+1},$$

(9c) $$(m_k a_n c_n + m_{k-1} d_{-n-1}) x_{-n},$$

must replace

(10a) $$(m_{k-1} a_{-n-1} c_{-n-1}) x_n,$$

(10b) $$(m_{k-2} b_{n+1} + m_{k-3} a_{-n-2} c_{-n-2}) x_{n+1},$$

(10c) $$(m_{k-1} d_{-n-1}) x_{-n},$$

in order to change the left side of (8) to the left side of (7). We may add (9a), (9b), and (9c) to the left side of (8) without disturbing the needed inequality.

It is sufficient to prove the inequality for each indicator, and we will start with p. It is easy to see that a term may be added to the right side of (8) if its p-indicator is greater than or equal to a p-indicator of a term on either side. A term may be deleted from the left side if its p-indicator is greater than or equal to the p-indicator of a term on the left side, or superior to the p-indicator of a term on the right side. Since

$$H_p(m_k b_n x_n) \gg H_p(m_k d_n x_{n+1})$$
$$H_p(m_k a_n c_n x_{-n}) \gg H_p(m_k d_n x_{n+1})$$
$$H_p((10a)) \gg H_p((10c))$$

by (2), it suffices to prove that the inequality is maintained if (10b) and (10c) are deleted from the left hand side of (8) and if $m_k d_n x_{n+1}$ is added to the right hand side.

We show that (10b) may be deleted. If r is an integer, r' denotes the p-factor of r. We say $0' = \infty$. Suppose

$$m'_{k-2} b'_{n+1} \leqq m'_{k-3} a'_{-n-2}.$$

Then

$$H_p(10b) \geqq H_p(m_{k-2} b_{n+1} x_{n+1}) \gg H_p(m_{k-2} d_{n+1} x_{n+2}).$$

Since the latter term is on the right hand side of (8) and $H_p(10\text{b})$ is superior to it, (10b) plays no role in determining the p-indicator of the left side, and so may be deleted.

Now suppose

$$m'_{k-3}a'_{n-2} < m'_{k-2}b'_{n+1} \quad \text{and} \quad m'_{k-3} < m'_{k-1}b'_{-n-1}.$$

These conditions, along with (4) imply that

$$H_p((m_{k-2}a_{n+1}c_{n+1} + m_{k-1}b_{-n-1} + m_{k-3}d_{-n-2})x_{-n-1}) = H_p(m_{k-3}d_{-n-2}x_{-n-1}),$$

where the first term is on the left hand side of (8). We also have

$$H_p(10\text{b}) = H_p(m_{k-3}a_{-n-2}c_{-n-2}x_{n+1}) \geqq H_p(m_{k-3}d_{-n-2}x_{-n-1}).$$

So again we may delete (10b).

The last possibility is

$$(11) \qquad m'_{k-3}a'_{-n-2} < m'_{k-2}b'_{n+1} \quad \text{and} \quad m'_{k-3} \geqq m'_{k-1}b'_{-n-1}.$$

Then

$$H_p(10\text{b}) = H_p(m_{k-3}a_{-n-2}c_{-n-2}x_{n+1}) \geqq H_p(m_{k-3}d_{-n-2}x_{-n-1})$$
$$\geqq H_p(m_{k-1}b_{-n-1}x_{-n-1}) \geqq H_p(10\text{c}).$$

Thus if (11) holds, and if (10c) may be deleted, then so may (10b).

We now delete (10c). If $m'_k a'_n \neq m'_{k-1}$, we have

$$H_p(10\text{c}) = H_p(9\text{c}),$$

so that (10c) may be deleted. If

$$(12) \qquad m'_k a'_n = m'_{k-1}$$

then

$$H_p(10\text{c}) = H_p(m_k a_n c_n x_{-n}) \gg H_p(m_k d_n x_{n+1}).$$

Therefore, if (12) holds, and if $H_p(m_k d_n x_{n+1})$ can be added to the right hand side of (8), then (10c) may be deleted from the left hand side.

We now show that $H_p(m_k d_n x_{n+1})$ may be added to the right hand side. If

$$m'_k \neq (m_{k-2}b_{n+1} + m_{k-3}a_{-n-2}c_{-n-2})',$$

then $H_p(m_k d_n x_{n+1}) \geqq H_p(9\text{b})$, so $H_p(m_k d_n x_{n+1})$ may be added. If $m'_k \geqq m'_{k-2} b'_{n+1}$, then

$$H_p(m_k d_n x_{n+1}) \geqq H_p(m_{k-2} b_{n+1} x_{n+1})$$

which is a term on the right hand side of (8), so again $H_p(m_k d_n x_{n+1})$ may be added. Finally, if

$$m'_k = (m_{k-2} b_{n+1} + m_{k-3} a_{-n-2} c_{-n-2})' \quad \text{and} \quad m'_k < m'_{k-2} b'_{n+1},$$

we have $m'_k = m'_{k-3} a'_{-n-2}$. Then

$$H_p(m_k d_n x_{n+1}) = H_p(m_{k-3} a_{-n-2} c_{-n-2} x_{n+1}),$$

which is one of the terms on the right hand side of (8).

For q-indicators, it suffices to show that $m_k b_n x_n$ may be added to the right hand side of (8) and that (10a), (10b), and (10c) may be deleted from the left hand side. For an integer n, n' now denotes the q-factor of n.

We add $m_k b_n x_n$ to the right side. If

$$m'_k \neq m'_{k-1} c'_{n-1},$$

then $H_q(m_k b_n x_n) \geqq H_q(9\text{a})$. If

$$m'_k = m'_{k-1} c'_{n-1},$$

then $H_q(m_k b_n x_n) = H_q(10\text{a})$. So if (10a) may be deleted from the left side, $m_k b_n x_n$ may be added to the right. Now (10b) will be deleted. If

$$(m_{k-2} b_{n+1} + m_{k-3} a_{-n-2} c_{-n-2})' \neq m'_k d'_n,$$

then $H_q(10\text{b}) \geqq H_q(9\text{b})$, so the former may be deleted. If

$$(m_{k-2} b_{n+1} + m_{k-3} a_{-n-2} c_{-n-2})' = m'_k d'_n,$$

then $H_q(10\text{b}) = H_q(m_k d_n x_n) \gg H_q(m_k b_n x_n)$, and this can be added to the right side provided that (10a) may be deleted.

We complete this case by showing that (10a) may be deleted. The argument for (10c) is similar to that for (10a), and will be omitted. If q is not $(-n)$-benign, then

$$H_q(10\text{a}) \gg H_q(m_{k-1} b_{-n-1} x_{-n-1}),$$

which is one of the terms on the right hand side of (8). Hence (10a) may be deleted.

Now assume q is $(-n)$-benign. If $m'_{k-1} \neq m'_k$, then $H_q(10\text{a}) = H_q(9\text{a})$, and (10a) may be deleted. Therefore assume that $m'_{k-1} = m'_k$. At least one of the terms b_{-n-1} or d_n is not divisible by q. If d_n is not divisible by q, then

$$H_q(10\text{a}) = H_q(m_k d_n x_{-n-1})$$

because $m'_{k-1} = m'_k$ and q is $(-n)$-benign. Unless

$$m'_k = (m_{k-3}a_{-n-2}c_{-n-2} + m_{k-2}b_{n+1})', \tag{13}$$

$H_q(m_k d_n x_{-n-1}) = H_q(9\text{b})$, and (10a) may be deleted. If (13) holds, then

$$m'_k \geqq \min\{m'_{k-3}, m'_{k-2}\}.$$

If d_n is divisible by q, then b_{-n-1} is not divisible by q, and a similar argument shows that either (10a) may be deleted or

$$m'_k = m'_{k-1} \geqq \min\{m'_{k-3}, m'_{k-2}\}.$$

Using the term on the right and continuing the above process we find that unless

$$\begin{aligned} m'_k = m'_{k-1} &\geqq \min\{m'_{k-3}, m'_{k-2}\} \geqq \min\{m'_{k-5}, m'_{k-4}\} \\ &\geqq \cdots \geqq \min\{m'_1, m'_2\}, \end{aligned} \tag{14}$$

we may delete (10a). (Note that we have assumed that k is even.) If

$$H_q((m_1 b_0 + m_2 d_{-1})x_0) = \min\{H_q(m_1 b_0 x_0), H_q(m_2 d_{-1} x_0)\},$$

then

$$H_q(10\text{a}) \geqq \min\{H_q(m_1 b_0 x_0), H_q(m_2 d_{-1} x_0)\} = H_q((m_1 b_0 + m_2 d_{-1})x_0)$$

with the latter term in the left hand side of (8), and so (10a) is deleted. So assume (14) and

$$H_q((m_1 b_0 + m_2 d_{-1})x_0) > \min\{H_q(m_1 b_0 x_0), H_q(m_2 d_{-1} x_0)\}, \tag{15}$$

are true. Because of (15), $m'_1 = m'_2$. Calculations using (1) show that $b_{-1}(m_1 b_0 + m_2 d_{-1}) - d_{-1}(m_1 c_0 + m_2 a_{-1}) = m_1$, and since $(m_1 b_0 + m_2 d_{-1})' > m'_1$, we must have

$$m_1' = (m_1c_0 + m_2a_{-1})'a_0' = (m_1a_0c_0 + m_2b_{-1})'$$

and

$$m_1' = (m_1c_0 + m_2a_{-1})'c_{-1} = (m_1d_0 + m_2a_{-1}c_{-1})'.$$

If $m_3' > m_1'$, we have

$$\begin{aligned} H_q((m_1d_0 + m_2a_{-1}c_{-1} + m_3b_1)x_1) &= H_q((m_1d_0 + m_2a_{-1}c_{-1})x_1) \\ &= H_q(m_1x_0) \leqq H_q(10\text{a}). \end{aligned} \tag{16}$$

Since the first term of (16) is in the left hand side of (8), we may delete (10a). In a similar way, we may delete (10a) if $m_4' > m_1'$, using

$$H_q((m_1a_0c_0 + m_2b_{-1} + m_4d_{-2})x_{-1}) = H_q(m_1x_0) \leqq H_q(10\text{a}). \tag{17}$$

So, considering (14), we are reduced to

$$m_1' = m_2' = m_3' = m_4'.$$

Furthermore, from the first terms of (16) and (17), we have no difficulties unless

$$(m_1a_0c_0 + m_2b_{-1} + m_4d_{-2})' > m_1'$$

and

$$(m_1d_0 + m_2a_{-1}c_{-1} + m_3b_1)' > m_1'.$$

Now

$$\begin{aligned} &a_0(m_1d_0 + m_2a_{-1}c_{-1} + m_3b_1) - c_{-1}(m_1a_0c_0 + m_2b_{-1} + m_4d_{-2}) \\ &\quad = m_3a_0b_1 - m_4c_{-1}d_{-2}, \end{aligned}$$

and, using (5),

$$a_0b_1(m_3c_1 + m_4a_{-2}) - c_1(m_3a_0b_1 - m_4c_{-1}d_{-2}) = m_4.$$

Arguing as before, we are finished unless $m_4' = m_5' = m_6'$. This continues until we have

$$m_1' = m_2' = \cdots = m_{k-2}' = m_{k-1}'b_{-n-1}' = m_k'd_n'.$$

Because $m_{k-1}' = m_k'$, we must have $b_{-n-1}' = d_n' = 1$. This implies that

$H_q(9\mathrm{a}) = H_q(m_1x_0) \leqq H_q(10\mathrm{a})$, and so (10a) may be deleted from the left side of (8).

In Case B, (k is odd, $k \geqq 3$), we add $m_ka_nc_nx_{-n-1}$ and $m_kd_nx_{n+1}$ to both sides of (8). These terms cause no problems. Additionally, $m_kb_nx_n$ is added to the right side, and

$$(18) \qquad (m_{k-2}d_{n-1} + m_{k-1}a_{-n}c_{-n})x_n$$

is replaced by

$$(19) \qquad (m_{k-2}d_{n-1} + m_{k-1}a_{-n}c_{-n} + m_kb_n)x_n$$

on the left side. As before, adding (19) to the left side presents no problems.

We begin with p-indicators. Since $H_p(m_kb_nx_n) \geqq H_p(m_kd_nx_{n+1})$, we may add $m_kb_nx_n$ to the right side. When

$$(m_{k-2}d_{n-1} + m_{k-1}a_{-n}c_{-n})' \neq m'_kb'_n,$$

$H_p(18) = H_p(19)$, so (18) may be deleted. When

$$(m_{k-2}d_{n-1} + m_{k-1}a_{-n}c_{-n})' = m'_kb'_n,$$

$H_p(18) = H_p(m_kb_nx_n) \geqq H_p(m_kd_nx_{n+1})$. The last term has been added to the left side of (8), so (18) may be deleted.

Finally, consider q-indicators. If $(m_{k-2}d_{n-1} + m_{k-1}a_{-n}c_{-n})' \neq m'_kb'_n$ we may add $m_kb_nx_n$ to the right and delete (18) from the left. So suppose $(m_{k-2}d_{n-1} + m_{k-1}a_{-n}c_{-n})' = m'_kb'_n$. If q is not n-benign, then q is also not $(n-1)$-benign and $H_q(m_{k-2}d_{n-1}x_n) \gg H_q(m_{k-2}b_{n-1}x_{n-1})$, $H_q(m_{k-1}a_{-n}c_{-n}x_n) \gg H_q(m_{k-1}b_{-n}x_n)$, follow from (3). So $H_q(m_kb_nx_n) = H_q(18) \geqq \min\{H_q(m_{k-2}d_{n-1}x_n), H_q(m_{k-1}a_{-n}c_{-n}x_n)\}$ and is equal to one of these terms, which in turn is superior to a term on the right hand side of (8). So both $m_kb_nx_n$ and (18) are dealt with in this case. If q is n-benign, the argument is similar to the deletion of (10a), and will be omitted. This completes the proof.

4. Ulm's Theorem for mixed abelian groups.

Theorem 2.2 plays a key role in the structure theory of mixed abelian groups. The following concept, due to Warfield **[10]**, (announced in **[9]**) is necessary.

DEFINITION. A decomposition basis X of A is a *lower decomposition basis* if, whenever the Ulm invariant $f^p_\sigma(A)$ is infinite, then $f^p_\sigma(A) = f^p_\sigma(A, [X])$.

LEMMA 4.1. *If A has a decomposition basis X, there is a decomposition basis Y subordinate to X which is a lower decomposition basis.*

Proof. The local version of this lemma, (i.e., for modules over a discrete valuation ring) was stated and proved by Warfield (**[10]**, Theorem 5.1). Warfield's proof, with some minor changes, may be used here.

DEFINITION. Let X be a decomposition basis of a group A. If, for every decomposition basis Y subordinate to X, $[Y]$ is p-nice in $A(p,[Y])$, then we call X a *strongly nice* decomposition basis.

We are now ready for Ulm's Theorem for mixed abelian groups. A slightly different formulation appears in **[6]**, Theorem 10.

THEOREM 4.2. *Let A and B be groups having strongly nice decomposition bases X and Y, respectively, such that $A/[X]$ and $B/[Y]$ are both totally projective. Then $A \cong B$ if and only if, for all p, σ, c and e, $f^p_\sigma(A) = f^p_\sigma(B)$ and $\mathrm{ST}(c,p,e,A) = \mathrm{ST}(c,p,e,B)$.*

Proof. By Lemma 4.1, there is a lower decomposition basis X_1 subordinate to X, and by Theorem 2.2 there is a decomposition basis X_2 of A such that $[X_2] = [X_1]$ and $H(X_2) \sim H(Y)$. Choose a lower decomposition basis Y_1 subordinate to Y, and decomposition bases X_3 subordinate to X_2 and Y_2 subordinate to Y_1, so that there is a bijection $\gamma\colon X_3 \to Y_2$ such that $H(x) = H(\gamma x)$ for all $x \in X_3$. Since X_3 and Y_2 are decomposition bases and $[X_3]$ and $[Y_2]$ are free, γ extends to a height preserving map $\delta\colon [X_3] \to [Y_2]$. (Note that $[X_3] = [X_4]$ for some lower decomposition basis X_4 of X, so $[X_3]$ has the required niceness properties.) $A/[X_3]$ and $B/[Y_2]$ are totally projective, and $f^p_\sigma(A,[X_3]) = f^p_\sigma(B,[Y_2])$ for all p and σ because X_3 and Y_2 are lower decomposition bases. Using Theorem 1.1, $A \cong B$. The converse is trivial.

We list four standard consequences in the following corollary.

COROLLARY 4.3. *Let A, B and C be groups having strongly nice decomposition bases X, Y and Z respectively such that $A/[X]$, $B/[Y]$ and $C/[Z]$ are all totally projective. Then:*

(i) *If $A \oplus A \cong B \oplus B$, then $A \cong B$.*

(ii) *If the Ulm invariants and* ST-*invariants of C are all finite, and $A \oplus C \cong B \oplus C$, then $A \cong B$.*

(iii) *If A is isomorphic to a summand of B and B is isomorphic to a summand of A, then $A \cong B$.*

(iv) *If $\mathrm{ST}(c,p,e,A) = \mathrm{ST}(c,p,e,B)$ for all c, p and e, then there are* *totally projective torsion groups T and T' such that $A \oplus T \cong B \oplus T'$.*

An important class of groups, defined in **[6]**, is the following.

DEFINITION. A group A is *affable* if it has a splitting decomposition basis X such that $A/[X]$ is totally projective.

It is easy to see that A is affable if and only if it is a totally projective torsion group, or if it is a direct sum $A = \bigoplus A_i$, where each A_i is of rank one and contains an element x_i of infinite order such that $A_i/[x_i]$ is totally projective. Since a splitting decomposition basis is strongly nice, the following is immediate from Theorem 4.2.

THEOREM 4.4. *Let A and B be affable groups. Then $A \cong B$ if and only if, for all p, σ, c, and e, $f_\sigma^p(A) = f_\sigma^p(B)$ and $\mathrm{ST}(c, p, e, A) = \mathrm{ST}(c, p, e, B)$.*

The next theorem demonstrates that an affable group has enough splitting decomposition bases. The proof is the same as that of Theorem 13 of **[6]**.

THEOREM 4.5. *For each decomposition basis X of an affable group A, there is a subordinate decomposition basis X' of X that is a splitting decomposition basis.*

The following follows from Corollary 4.3.

COROLLARY 4.6. *Let A be a group with a strongly nice decomposition basis X such that $A/[X]$ is totally projective. Then there is a totally projective group T such that $A \oplus T$ is an affable group.*

Proof. There is an affable group B such that $\mathrm{ST}(c, p, e, A) = \mathrm{ST}(c, p, e, B)$ for all c, p and e. By Corollary 4.3 (iv), there are totally projective groups T and T' such that $A \oplus T = B \oplus T'$, and the latter group is still affable.

It is known (see Rotman–Yen **[5]**, p. 251) that a summand of an affable group is not necessarily affable. It is an open question whether summands of affable groups can be classified by Theorem 4.2. If they could be so classified, then the class **A** consisting of summands of affable groups would be the largest class of groups that can be classified via Ulm invariants and ST-invariants. For suppose **C** is a larger class and $G \in \mathbf{C}$. Then there is an affable group H such that $\mathrm{ST}(c, p, e, H) = \mathrm{ST}(c, p, e, G \oplus H)$ and $f_\sigma^p(H) = f_\sigma^p(G \oplus H)$, for all c, p, e and σ. By the classification theorem we would have $G \oplus H \cong H$, so that $G \in \mathbf{A}$.

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