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ON THE DISTRIBUTION OF SOME GENERALIZED SQUARE-FULL INTEGERS

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Let a and b be fixed positive integers. Let $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ be the canonical representation of $n>1$ and let $R_{a,b}$ denote the set of all n with the property that each exponent $a_i (1 \leqq i \leqq r)$ is either a multiple of a or is contained in the progression $at+b$, $t \geqq 0$. It is clear that $R_{2,3}=L$, the set of square-full integers; that is, the set of all n with property that each prime factor of n divides n to at least the second power. Thus the elements of $R_{a,b}$ may be called generalized square-full integers. This generalization of square-full integers has been given by E. Cohen in 1963, who also established asymptotic formulae for $R_{a,b}(x)$, the enumerative function of the set $R_{a,b}$, in various cases. In this paper, we improve the 0-estimates of the error terms in the asymptotic formulae for $R_{a,b}(x)$ established by E. Cohen in some cases and further improve them on the assumption of the Riemann hypothesis.

1. Introduction. An integer $n>1$ is called square-full if in the canonical representation of n into prime powers each exponent is $\geqq 2$. Let L denote the set of square-full integers. Let x denote a real variable $\geqq 1$ and let $L(x)$ denote the number of square-full integers $\leqq x$. For the work done on the asymptotic formula for $L(x)$ or for $L_k(x)$, the number of k-full integers $\leqq x$ (an integer $n>1$ is called k-full, if in the canonical representation of n each exponent $\geqq k$) we refer to the bibliography given by E. Cohen [2] and by E. Cohen and K. J. Davis [3]. In particular, for the best known results on the 0-estimates of the error term in the asymptotic formula for $L(x)$, we refer to the paper by the author and R. Sita Rama Chandra Rao [7] and also to the recent paper by the author [8].

In 1963, E. Cohen [1] generalized square-full integers in the following way: Let a and b be fixed positive integers. Let $n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ and let $R_{a,b}$ denote the set of all integers n with the property that each exponent a_i $(1 \leqq i \leqq r)$ is either a multiple of a or is contained in the progression $at+b$, $t \geqq 0$. It is clear that $R_{2,3}=L$. Let $r_{a,b}$ denote the characteristic function of the set $R_{a,b}$; that is, $r_{a,b}(n)=1$ or 0 according as $n \in R_{a,b}$ or $n \notin R_{a,b}$. Also, let $R_{a,b}(x)$ denote the number of integers $n \leqq x$ such that $n \in R_{a,b}$. The following results have been established by E. Cohen (cf. [1], Theorems 2.1, 3.1 and 3.2):

If $a < 2b$, $b \not\equiv o \pmod{a}$, then

$$R_{a,b}(x) = \alpha^* x^{1/a} + \beta^* x^{1/b} + \begin{cases} 0(x^{1/(a+b)}) \\ 0(x^{1/2b}) , \end{cases} \tag{1.1}$$

according as $b > a$ or $b < a$; if $a > 2b$, then

$$R_{a,b}(x) = \beta^* x^{1/b} + 0(x^{1/2b}) , \tag{1.2}$$

the constants α^* and β^* are defined by

$$\alpha^* = \zeta(b/a)/\zeta(2b/a) \quad \text{and} \quad \beta^* = \zeta(a/b)/\zeta(2) , \tag{1.3}$$

where $\zeta(s)$ in the Riemann Zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$ and

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} (t - [t])t^{-s-1}dt \quad \text{for} \quad 0 < s < 1 . \tag{1.4}$$

If $2a > b > a$, $b \not\equiv o \pmod{a}$, then for $x \geqq 2$,

$$R_{a,b}(x) = \alpha^* x^{1/a} + \beta^* x^{1/b} + \begin{cases} 0(x^{1/2b}) \\ 0(x^{1/2b} \log^2 x) \\ 0(x^{1/3a} \log x) , \end{cases} \tag{1.5}$$

according as $3a > 2b$, $3a = 2b$ or $3a < 2b$.

If $b > a$, $b \not\equiv o \pmod{a}$, then for $x \geqq 2$,

$$R_{a,b}(x) = \alpha^* x^{1/a} + \beta^* x^{1/b} + \begin{cases} 0(x^{1/2b}) \\ 0(x^{1/2b} \log x) \\ 0(x^{2/(2b+5a)}) , \end{cases} \tag{1.6}$$

according as $2b < 5a$, $2b = 5a$ or $2b > 5a$.

The object of this paper is to improve the 0-estimates of the error terms in the above asymptotic formulae for $R_{a,b}(x)$ applying the method adopted in [6] or [7] and the results due to H.-E. Richert [5] on the divisor problem for $\tau_{a,b}(n)$, namely

$$\sum_{n \leqq x} \tau_{a,b}(n) = \zeta(b/a)x^{1/a} + \zeta(a/b)x^{1/b} + 0(x^{\theta}) , \tag{1.7}$$

where $\tau_{a,b}(n) = \sum_{d^a \delta^b = n} 1$, the summation being taken over all ordered pairs (d, δ) of positive integers d and δ such that $d^a \delta^b = n$. It is known that $\theta \leqq 2/(3a + 3b)$ or $\theta \leqq 2/(5a + 2b)$, according as $b < 2a$ or $b > 2a$. These results on the upper bound of θ have been established by H.-E. Richert (cf. [5], Satz 2). As for as the lower bound of θ is concerned, it is known that $\theta \geqq 1/(2a + 2b)$ and this result has been established by E. Krätzel (cf. [4], Satz 7).

The improvements in the 0-estimates of the error terms are given in Theorem 3.1, Remark 3.1, Theorem 3.3 and Remark 3.3, which are further improved on the assumption of the Riemann hypothesis in Theorem 3.2, Remark 3.2, Theorem 3.4 and Remark 3.4.

Finally, we mention that applying the method adopted here, together with the results of H.-E. Richert and E. Krätzel stated above, we can improve the 0-estimates of the error terms in Theorems 3.1 and 3.2 of E. Cohen [2] also. In fact, he [2] establishes in Theorem 3.1, an asymptotic formula for the enumerative function of the set $S_{a,b}$ (when $b > a > 1$, $(a, b) = 1$), which like $R_{a,b}$ reduces to L when $a = 2$ and $b = 3$.

2. Preliminaries. In this section we state some lemmas which have been established already and which we need in our present discussion. First, we state the following best known estimate concerning the average of the Möbious function $\mu(n)$ established by Arnold Walfisz [10]:

LEMMA 2.1 (cf. [10], Satz 3, p. 191). *For $x \geqq 3$,*

$$M(x) = \sum_{n \leqq x} \mu(n) = 0(x\delta(x)) , \tag{2.1}$$

where

$$\delta(x) = \exp\{-A \log^{3/5} x(\log\log x)^{-1/5}\} , \tag{2.2}$$

A being a positive absolute constant.

LEMMA 2.2 (cf. [6], Lemma 2.2). *For $x \geqq 3$ and any $s > 1$,*

$$\sum_{n \leqq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + 0\left(\frac{\delta(x)}{x^{s-1}}\right) . \tag{2.3}$$

LEMMA 2.3 (cf. [9], Theorem 14-26(A), p. 316). *If the Riemann hypothesis true, then for $x \geqq 3$,*

$$M(x) = \sum_{n \leqq x} \mu(n) = 0(x^{1/2}\omega(x)) , \tag{2.4}$$

where

$$\omega(x) = \exp\{A \log x(\log\log x)^{-1}\} , \tag{2.5}$$

A being a positive absolute constant.

LEMMA 2.4 (cf. [6], Lemma 2.5). *If the Riemann hypothesis is true, then for $s > 1$,* (2.6) $$\sum_{n\leqq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + 0(x^{1/2-s}\omega(x)) .$$

LEMMA 2.5 (cf. [1], Lemma 2.1). *If* $b \not\equiv o \pmod a$, *then*

(2.7) $$r_{a,b}(n) = \sum_{d^{2b}\delta=n} \mu(d)\tau_{a,b}(\delta) .$$

3. Main results. In this section, we improve the 0-estimates of the error terms. First we treat (1.5).

THEOREM 3.1. *For* $x \geqq 3$, $2a > b > a$, $b \not\equiv o \pmod a$, *we have*

(3.1) $$R_{a,b}(x) = \alpha^* x^{1/a} + \beta^* x^{1/b} + 0(x^{1/2b}\delta(x)) ,$$

where α^* *and* β^* *are given by* (1.3) *and* $\delta(x)$ *is given by* (2.2).

Proof. By Lemma 2.5, we have

(3.2) $$\begin{aligned} R_{a,b}(x) &= \sum_{n\leqq x} r_{a,b}(n) = \sum_{n\leqq x} \sum_{d^{2b}\delta=n} \mu(d)\tau_{a,b}(\delta) \\ &= \sum_{d^{2b}\delta\leqq x} \tau_{a,b}(\delta) . \end{aligned}$$

Let $z = x^{1/2b}$. Further, let $0 < \rho = \rho(x) < 1$, where the function ρ will be suitably chosen later. If $d^{2b}\delta \leqq x$, then both $d > \rho z$ and $\delta > \rho^{-2b}$ cannot simultaneously hold. Hence

(3.3) $$\begin{aligned} R_{a,b}(x) &= \sum_{\substack{d^{2b}\delta\leqq x \\ d\leqq\rho z}} \mu(d)\tau_{a,b}(\delta) + \sum_{\substack{d^{2b}\delta\leqq x \\ \delta\leqq\rho^{-2b}}} \mu(d)\tau_{a,b}(\delta) - \sum_{\substack{d\leqq\rho z \\ \delta\leqq\rho^{-2b}}} \mu(d)\tau_{a,b}(\delta) \\ &= R_1 + R_2 - R_3 , \end{aligned}$$

say.

Now, by (1.7),

(3.4) $$\begin{aligned} R_1 &= \sum_{\substack{d^{2b}\delta\leqq x \\ d\leqq\rho z}} \mu(d)\tau_{a,b}(\delta) = \sum_{d\leqq\rho z} \mu(d) \sum_{\delta\leqq x/d^{2b}} \tau_{a,b}(\delta) \\ &= \sum_{d\leqq\rho z} \mu(d)\left\{\frac{\zeta(b/a)x^{1/a}}{d^{2b/a}} + \frac{\zeta(a/b)x^{1/b}}{d^2} + 0\left(\frac{x^\theta}{d^{2b\theta}}\right)\right\} \\ &= \zeta(b/a)x^{1/a} \sum_{d\leqq\rho z} \frac{\mu(d)}{d^{2b/a}} + \zeta(a/b)x^{1/b} \sum_{d\leqq\rho z} \frac{\mu(d)}{d^2} \\ &\quad + 0\left(x^\theta \sum_{d\leqq\rho z} \frac{1}{d^{2b\theta}}\right) . \end{aligned}$$

Since $2a > b$ and $\theta \leqq 2/(3a + 3b)$, we have $2b\theta < 1$. Hence

$$\begin{aligned} x^\theta \sum_{d\leqq\rho z} \frac{1}{d^{2b\theta}} &= 0(x^\theta(\rho z)^{1-2b\theta}) \\ &= 0(\rho^{1-2b}\theta z) . \end{aligned}$$

Now, applying Lemma 2.2 for $s = 2b/a > 1$ and $s = 2$, we obtain from (3.4) that

$$\begin{aligned} R_1 &= \zeta(b/a)x^{1/a}\left\{\frac{1}{\zeta(2b/a)} + 0(\rho z)^{1-2b/a}\delta(\rho z))\right\} \\ &\quad + \zeta(a/b)x^{1/b}\left\{\frac{1}{\zeta(2)} + 0((\rho z)^{-1}\delta(\rho z))\right\} \\ &\quad + 0(\rho^{1-2b\theta}z) \\ &= \frac{\zeta(b/a)}{\zeta(2b/a)}x^{1/a} + \frac{\zeta(a/b)}{\zeta(2)}\ x^{1/b} + 0(\rho^{1-2b/a}z\delta(\rho z)) \\ &\quad + 0(\rho^{1-2b\theta}z)\ , \end{aligned} \tag{3.5}$$

since $b > a$ implies $\rho^{-1} < \rho^{1-2b/a}$.

We have by Lemma 2.1,

$$\begin{aligned} R_2 &= \sum_{\substack{d^{2b}\delta \leqq x \\ \delta \leqq \rho^{-2b}}} \mu(d)\tau_{a,b}(\delta) = \sum_{\delta \leqq \rho^{-2b}} \tau_{a,b}(\delta) \sum_{d \leqq (x/\delta)^{1/2b}} \mu(d) \\ &= \sum_{\delta \leqq \rho^{-2b}} \tau_{a,b}(\delta) M\left(\left(\frac{x}{\delta}\right)^{1/2b}\right) \\ &= 0\left(x^{1/2b} \sum_{m \leqq \rho^{-2b}} \tau_{a,b}(m)m^{-1/2b}\delta\left(\left(\frac{x}{m}\right)^{1/2b}\right)\right) . \end{aligned}$$

Since $\delta(x)$ is monotonic decreasing and $(x/m)^{1/2b} \geqq \rho z$, we have $\delta((x/m)^{1/2b}) \leqq \delta(\rho z)$. Further, by (1.7) and partial summation, we get $\sum_{m \leqq \rho^{-2b}} \tau_{a,b}(m)m^{-1/2b} = 0(\rho^{1-2b/a})$. Hence,

$$R_2 = 0(\rho^{1-2b/a}z\delta(\rho z)) . \tag{3.6}$$

Also, we have by Lemma 2.1 and (1.7),

$$\begin{aligned} R_3 &= \sum_{d \leqq \rho z} \mu(d) \sum_{\delta \leqq \rho^{-2b}} \tau_{a,b}(\delta) = 0(\rho z\delta(\rho z)\rho^{-2b/a}) \\ &= 0(\rho^{1-2b/a}z\delta(\rho z)) . \end{aligned} \tag{3.7}$$

Hence by (3.3), (3.5), (1.3), (3.6) and (3.7), we obtain

$$R_{a,b}(x) = \alpha^* x^{1/a} + \beta^* x^{1/b} + o(\rho^{1-2b/a}z\delta(\rho z) + 0(\rho^{1-2b\theta}z) . \tag{3.8}$$

Now, we choose

$$\rho = \rho(x) = \{\delta(x^{1/4b})\}^{a/2b} , \tag{3.9}$$

and write

$$\begin{aligned} f(x) &= \log^{3/5}(x^{1/4b})\{\log\log(x^{1/4b})\}^{-1/5} , \\ &= \left(\frac{1}{4b}\right)^{3/5} u^{3/5}(v - \log 4b)^{-1/5} , \end{aligned} \tag{3.10}$$

where $u = \log x$ and $v = \log\log x$.

(3.11) For $v \geqq 2 \log 4b$, that is, $u \geqq 16b^2$, $x \geqq e^{16b^2}$,

we have $v^{-1/5} \leqq (v - \log 4b)^{-1/5} \leqq (v/2)^{-1/5}$, so that

$$\left(\frac{1}{4b}\right)^{3/5} u^{3/5} v^{-1/5} \leqq f(x) \leqq \left(\frac{1}{4b}\right)^{3/5} u^{3/5} \left(\frac{v}{2}\right)^{-1/5}, \tag{3.12}$$

We assume without loss of generality that the constant A in (2.2) is less than unity.

By (3.9), (2.2) and (3.10), we have

$$\rho = \exp\left\{-\frac{Aa}{2b} f(x)\right\}. \tag{3.13}$$

By (3.11), we have $a/2b(1/4b)^{3/5}u^{3/5}(v/2)^{-1/5} \leqq u/4b$. Hence by (3.12), (3.13) and the above,

$$\rho \geqq \exp\left\{-\frac{Aa}{2b}\left(\frac{1}{4b}\right)^{3/5} u^{3/5} \left(\frac{v}{2}\right)^{-1/5}\right\} \geqq \exp\left\{-\frac{u}{4b}\right\},$$

so that $\rho \geqq x^{-1/4b}$. Hence

$$\rho z \geqq x^{1/2b - 1/4b} = x^{1/4b}. \tag{3.14}$$

Since $\delta(x)$ is monotonic decreasing $\delta(\rho z) \leqq \delta(x^{1/4b}) = \rho^{2b/a}$, and so by (3.12) and (3.13), we have

$$\rho^{1-2b/a}\delta(\rho z) \leqq \rho \leqq \exp\left\{-\frac{Aa}{2b}\left(\frac{1}{4b}\right)^{3/5} u^{3/5} v^{-1/5}\right\}. \tag{3.15}$$

Hence the first 0-term in (3.8) is equal to

$$0\left(x^{1/2b} \exp\left\{-\frac{Aa}{2b}\left(\frac{1}{4b}\right)^{3/5} u^{3/5} v^{-1/5}\right\}\right)$$
$$= 0\left(x^{1/2b} \exp\left\{-\frac{Aa(1-2b\theta)}{2b}\left(\frac{1}{4b}\right)^{3/5} u^{3/5} v^{-1/5}\right\}\right),$$

since $0 < 1 - 2b\theta < 1$. By (3.12) and (3.13), we see that the second 0-term in (3.8) is also of the above order. Thus, if $\Delta_{a,b}(x)$ denotes the sum of the two 0-terms in (3.8), we have

$$\Delta_{a,b}(x) = 0(x^{1/2b} \exp\{-B \log^{3/5} x (\log\log x)^{-1/5}\}), \tag{3.16}$$

where $B = B_{a,b} = Aa(1 - 2b\theta)/2b$ is a positive constant.

Hence Theorem 3.1 follows by (3.8) and (3.16).

REMARK 3.1. Following the same procedure adopted in the proof of Theorem 3.1 in the case $b > 2a$ also, we obtain improvements in the error term of $R_{a,b}(x)$ in the sub cases where we have $\theta < 1/2b$

(for example, when $2b < 5a$, we have $\theta \leqq 2/(5a + 2b) < 1/2b$) and in these cases, the asymptotic formula for $R_{a,b}(x)$ in given by

$$(3.17) \qquad R_{a,b}(x) = \alpha^* x^{1/a} + \beta^* x^{1/b} + 0(x^{1/2b}\delta(x)) ,$$

where α^* and β^* are given by (1.3) and $\delta(x)$ is given by (2.2).

In fact, we get an improvement in the 0-estimate of (1.6) above in the first case from $0(x^{1/2b})$ to $0(^{1/2b}\delta(x))$.

THEOREM 3.2. *If the Riemann hypothesis is true, then the error term* $\Delta_{a,b}(x)$ *in the asymptotic formula for* $R_{a,b}(x)$ *in* (3.1) *is* $0(x^{(2-a\theta)/(4b+a-4ab\theta)}\omega(x))$, *where* θ *is given by* (1.7) *and* $\omega(x)$ *is given by* (2.5).

Proof. Following the same procedure adopted in Theorem 3.1 and making use of (2.6) instead of (2.3), we obtain

$$(3.18) \qquad \Delta_{a,b}(x) = 0(\rho^{1/2-2b/a}z^{1/2}\omega(\rho z)) + 0(\rho^{1-2b\theta}z) .$$

Now choosing $\rho = z^{-a/(4b+a-4ab\theta)}$, we see that $0 < \rho < 1$ and $\rho^{1/2-2b/a}z^{1/2} = \rho^{1-2b\theta}z = x^{(2-a\theta)/(4b+a-4ab\theta)}$. Since $\omega(x)$ is monotonic increasing and $\rho z < z$, we have $\omega(\rho z) < \omega(z) = \omega(x^{1/2b}) < \omega(x)$.

Hence by (3.18) and the above discussion, we have $\Delta_{a,b}(x) = 0(x^{(2-a\theta)/(4b+a-4ab\theta)}\omega(x))$, so that Theorem 3.2 follows.

REMARK 3.2. If the Riemann hypothesis is true, then the error term $\Delta_{a,b}(x)$ in the asymptotic formula for $R_{a,b}(x)$ in (3.17) is $0(x^{(2-a\theta)/(4b+a-4ab\theta)}\omega(x))$, where $\omega(x)$ is given by (2.5).

THEOREM 3.3. *For* $x \geqq 3$, $b < a < 2b$, $b \not\equiv o \pmod a$, *we have*

$$(3.19) \qquad R_{a,b}(x) = \beta^* x^{1/b} + \alpha^* x^{1/a} + 0(x^{1/2b}\delta(x)) ,$$

where α^* *and* β^* *are given by* (1.3) *and* $\delta(x)$ *is given by* (2.2).

Proof. Starting with Lemma 2.5 as in the proof of Theorem 3.1, we obtain

$$R_{a,b}(x) = \beta^* x^{1/b} + \alpha^* x^{1/a} + 0(\rho^{-1}z\delta(\rho z)) + 0(\rho^{1-2b\theta}z) .$$

Now, choosing $\rho = \rho(x) = \{\delta(x^{1/4b})\}^{1/2}$ and $f(x)$ the same as in (3.10) and arguing on similar lines as in the proof of Theorem 3.1, we get Theorem 3.3.

REMARK 3.3. Following the same procedure adopted in the proof of Theorem 3.3, in case $a > 2b$ also, where $b \not\equiv o \pmod a$, we obtain the following:

(3.20) $$R_{a,b}(x) = \beta^* x^{1/b} + 0(x^{1/2b}\delta(x)) .$$

COROLLARY 3.3.1 ($b = 1$) (cf. [1], Corollary 2.1). *If $a > 1$, then the set $R_a = R_{a,1}$ of the integers each of whose exponents is $\equiv o$ or 1 (mod a) has asymptotic density $\zeta(a)/\zeta(2)$; more precisely*

(3.21) $$R_a(x) = \frac{\zeta(a)}{\zeta(2)}x + 0(x^{1/2}\delta(x)) .$$

THEOREM 3.4. *If the Riemann hypothesis is true, then the error term $\Delta_{a,b}(x)$ in the asymptotic formula for $R_{a,b}(x)$ given by (3.19) is $0(x^{(2-b\theta)/b(5-4b\theta)}\omega(x))$, where θ is given by (1.7) and $\omega(x)$ is given by (2.5).*

Proof. By making use of (2.6) instead of (2.3) in the proof of Theorem 3.3, we obtain $\Delta_{a,b}(x) = 0(\rho^{-3/2}z^{1/2}\omega(\rho z)) + 0(\rho^{1-2b\theta}z)$.

Now, choosing $\rho = z^{-1/(5-4b\theta)}$ and arguing as in Theorem 3.2, we get Theorem 3.4.

REMARK 3.4. If the Riemann hypothesis is true, then the error term $\Delta_{a,b}(x)$ in the asymptotic formula for $R_{a,b}(x)$, given by (3.20) is $0(x^{(2a-b)/b(5a-4b)}\omega(x))$.

This result follows from Theorem 3.4 above, if we show that $a > 2b$ implies $\theta = 1/a$. Now, $a > 2b$ implies that $b < 2a$, so that by H.-E. Richert's result mentioned in §1, $\theta \leqq 2/(3a + 3b) < 1/a$ and hence (1.7) reduces to $\sum_{n\leq x} \tau_{a,b}(n) = \zeta(a/b)x^{1/b} + 0(x^{1/a})$, which implies that $\theta = 1/a$.

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