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FINITE GROUPS WITH CHEVALLEY-TYPE COMPONENTS

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This article contains the proof of one part of the unbalanced group conjecture of Aschbacher, Thompson and Walter.

1. Introduction. In [13] Thompson discussed simple groups X such that $O(C_X(\alpha)) \neq 1$ for some involution $\alpha \in \operatorname{Aut} X$ and sketched a proof of the theorem stated below in § 3. The key to this proof is a recent result of Aschbacher [2]. Some detailed properties of Chevalley type groups over finite fields of odd characteristic are also required.

The purpose of this article is to prove the necessary properties of Chevalley type groups (see §§ 5 and 6). To motivate these results it seemed worthwhile to review the arguments in [13]. This occupies §§ 3 and 4.

2. Notation. Let X denote a finite group, then

$$\operatorname{Inv} X = \{\alpha \in X: \alpha^2 = 1, \alpha \neq 1\}$$
$$\operatorname{Comp} X = \{Y: Y \lhd\lhd X,\ Y = Y',\ Y/0(Y) \text{ quasi-simple}\}$$
$$L(X) = \text{product of all } Y \in \operatorname{Comp} X .$$

For properties of L, in particular L-balance and its implications, see [5].

$$\mathscr{E} = \{X: L(X) \text{ simple, } C_X(L(X)) = 1\}$$
$$\mathscr{M} = \{X \in \mathscr{E}: O(C_X(\alpha)) \neq 1 \text{ some } \alpha \in \operatorname{Inv} X\} .$$

For convenience, the known groups in $\mathscr{M}$ are divided into four disjoint families.

$$\mathscr{M}_1 = \{X \in \mathscr{M}: L(X) \simeq L_2(q) \text{ or } A_7,\ q = \text{odd}\}$$
$$\mathscr{M}_2 = \{X \in \mathscr{M}: L(X) \simeq L_3(4) \text{ or Held's group [9]}\}$$
$$\mathscr{M}_3 = \{X \in \mathscr{M}: L(X) \simeq A_n,\ n \geqq 9 \text{ and odd}\}$$
$$\mathscr{M}_4 = \{X \in \mathscr{M}: L(X) \text{ a Chevalley type group of odd characteristic, but not an } L_2(q)\} .$$

In general, our notation follows Gorenstein [7].

3. The grand conjecture. This states that $\mathscr{M} = \mathscr{M}_1 \cup \mathscr{M}_2 \cup \mathscr{M}_3 \cup \mathscr{M}_4$. Thompson's attack on this conjecture starts with the following proposition. Its proof depends on several long and difficult results.

PROPOSITION. *Let* $G \in \mathscr{M}$ *and assume* $|G|$ *minimal subject to* $G \notin \mathscr{M}_1 \cup \mathscr{M}_2 \cup \mathscr{M}_3 \cup \mathscr{M}_4$. *For* $\alpha \in \operatorname{Inv} G$ *let* $\Gamma_\alpha = \{\beta \in \operatorname{Inv} C_G(\alpha)$: $O(C_G(\alpha)) \cap C_G(\beta) \nsubseteq O(C_G(\beta))\}$. *Then,*

(i) Γ_α *is nonempty for some* $\alpha \in \operatorname{Inv} G$.

(ii) *If* $\beta \in \Gamma_\alpha$ *and* $D = O(C_G(\alpha)) \cap C_G(\beta)$ *then there exists a* $Y \in \operatorname{Comp} C_G(\beta)$ *normalized by* D *and such that* $[D, Y/O(Y)] \neq 1$.

Proof. (i) If $SCN_3(2)$ of G is empty then by [11] G has sectional 2-rank at most 4. Hence by [8] G is a known group. Thus if (i) is false, since $SCN_3(2)$ is nonempty, the results of [4], [6], [1] may be used. They imply that G is known.

(ii) An extended form of L-balance, see [5], implies that D normalizes each element of Comp $C_G(\beta)$. Since $D \nsubseteq O(C_G(\beta))$ the result follows from elementary properties of L.

Let G, α, β, D, Y be defined as in the proposition. Then a lemma in [6] gives $Y^\alpha = Y$ and $[\alpha, Y/O(Y)] \neq 1$. Put $M = \langle \alpha, D, Y \rangle$ and $M^* = M/Z^*(M)$ then $M^* \in \mathscr{M}$ and so, by the choice of G, $M^* \in \mathscr{M}_i$ for some $i \in \{1, 2, 3, 4\}$.

THEOREM. $M^* \notin \mathscr{M}_4$.

The proof will be given in the following sections. The result of [2] will be used in the following form.

ASCHBACHER'S THEOREM. *Let* $X \in \mathscr{M}$, $\gamma \in \operatorname{Inv} X$, $L \in \operatorname{Comp} C_X(\gamma)$. *Suppose* L *has* 2-*rank equal to* 1 *and* $\gamma \in L$ *then* $X \in \mathscr{M}_4$.

The grand conjecture directly implies the B-conjecture, namely; $B(G) \supseteq B(N_G(T))$ for any finite group G and T any 2-subgroup of G, where $B(X)$ = product of all $Y \in \operatorname{Comp} X$ with Y *not* quasi-simple.

4. Proof of the theorem. Let $G, \alpha, \beta, D, M, M^*$ be defined as in § 3 and *assume* $M^* \in \mathscr{M}_4$. Using Aschbacher's theorem and the results proved in § 5 and § 6 we proceed, as in [13], to obtain a contradiction.

Let $E \subseteq D$ and let A/B be some section of G: we say that $\langle \alpha, E \rangle$ 'acts properly' on A/B if $\langle \alpha, E \rangle$ normalizes A and B and $C_{\langle \alpha, E \rangle}(A/B)$ is a proper subgroup of E (possibly 1). Thus $E \neq 1$ and, to begin, we know that $\langle \alpha, D \rangle$ acts properly on M^*.

Step 1. By Proposition A of § 5 (with $X = M^*$, $t = \alpha$ and $Y = E$) there exists $\gamma^* \in \operatorname{Inv} M^*$ and $S^* \in \operatorname{Comp} C_M^*(\gamma^*)$ with $S^* \simeq SL_2(q)$ for some odd q, $\langle \gamma^* \rangle = Z(S^*)$, and $\langle \alpha, D \rangle$ acts properly on S^*.

Choose S_1 to be the full inverse image of S^* in M and put $S_2 = S_1^{(\infty)}$. Choose $\gamma \in \operatorname{Inv} Z^*(S_2)$ so that $[\gamma, \alpha] = 1$ and put $S = C_{S_2}(\gamma)^{(\infty)}$. By construction, $\gamma \in C_G(\langle \alpha, \beta \rangle)$ and so γ normalizes D. Put $D_1 = C_D(\gamma)$ then, since $[D, \gamma] \subseteqq D \cap O(S) \neq D$, we see that $\langle \alpha, D_1 \rangle$ acts properly on $S/O(S)$.

Since $S \lhd\lhd C_M(\gamma)$ therefore $S \in \operatorname{Comp} C_G(\langle \beta, \gamma \rangle)$. Let K be the normal closure of S in $L(C_G(\gamma))$. Then by L-balance either,

(a) $K \in \operatorname{Comp} C_G(\gamma)$ and $K^\beta = K$, or

(b) $K = K_1 K_2$ with K_1, $K_2 \in \operatorname{Comp} C_G(\gamma)$, $K_1^\beta = K_2$ and $K/O(K) \simeq SL_2(q) \times SL_2(q)$.

Furthermore, $\langle \alpha, D_1 \rangle$ acts properly on $K/O(K)$ and, in case (b), on each $K_i/O(K_i)$. In case (a), since $\gamma \in K$, $K/O(K)$ is a nontrivial covering of $K/Z^*(K)$.

In the next two steps we will show that cases (a), (b) both lead to the following configuration:

(*) $W \simeq Z_2 \times Z_2$ is a subgroup of G with N_1, $N_2 \in \operatorname{Comp} C_G(W)$ such that, if $N = N_1 N_2$, then $N/O(N) \simeq SL_2(q) \times SL_2(q)$ for some odd q, $W \subseteqq Z(N)$, and $\langle \alpha, E \rangle$ acts properly on each $N_i/O(N_i)$ for some $E \subseteqq D$.

Step 2. In case (a) put $J = \langle \alpha, \beta, D_1, K \rangle$ and $J^* = J/Z^*(J)$. Then $J^* \in \mathscr{M}$ and so $J^* \in \mathscr{M}_i$ for some $i \in \{1, 2, 3, 4\}$. If $J^* \in \mathscr{M}_1$ then Aschbacher's theorem (with $X = G$, $L = K$) contradicts our choice of G. If $J^* \in \mathscr{M}_2$ then, since Held's group has no proper covering, $L(J^*) \simeq L_3(4)$ and the calculation in § 6 yields a contradiction. If $J^* \in \mathscr{M}_3$ the results in [12] contradict the choice of G. Hence $J^* \in \mathscr{M}_4$. In this case we may use Proposition B of § 5 (with $X = J^*$, $t = \alpha$, $s = \beta$ and $\hat{L} = KO(J)/O(J)$). This gives configuration (*) in $KO(J)/O(J)$ and arguing as in the second paragraph of Step 1 we see that (*) also occurs in G.

Step 3. In case (b), since $\gamma \in Z(K)$, we may choose $\rho \in \operatorname{Inv} Z^*(K)$ with $\rho \neq \gamma$ and so that ρ normalizes $\langle \alpha, D_1 \rangle$. Hence $[\rho, \alpha] = 1$ and putting $D_2 = C_{D_1}(\rho)$ we see, as in Step 1, that $\langle \alpha, D_2 \rangle$ acts properly on each $K_i/O(K_i)$. Then $W = \langle \gamma, \rho \rangle$ and $N = C_K(W)^{(\infty)}$ give the configuration (*).

Step 4. We may assume (*). Put $\langle \delta_i \rangle = N_i \cap W$, so that $W = \langle \delta_1, \delta_2 \rangle$, and put $C = G_G(\delta_1)$. Then, by L-balance, we have $N \subseteqq L(C)$. Hence $W \subseteqq L(C)$ and so δ_2 normalizes each element of Comp C. Thus if H_i is the normal closure of N_i in $L(C)$, then $H_i \in \operatorname{Comp} C$.

Suppose $[H_1, \delta_2] \subseteqq O(H_1)$. Then $H_1 = O(H_1) C_{H_1}(\delta_2)$ and, since

$N_1 \triangleleft\triangleleft C_{H_1}(\delta_2)$, we have $H_1 = O(H_1)N_1$. By Aschbacher's theorem this contradicts our choice of G. Thus $[H_1, \delta_2] \nsubseteq O(H_1)$ and, since $\delta_2 \in H_2$ we must have $H_1 = H_2$.

Put $H = \langle H_1, \alpha, E\rangle$ and $H^* = H/Z^*(H)$ and put δ_2^*, N_i^* for the images of δ_2, N_i in H^*. Then $N_i^* \in \operatorname{Comp} C_H^*(\delta_2^*)$ and $N_1^* \simeq L_2(q)$ while $\langle \delta_2^* \rangle \in N_2^* \simeq SL_2(q)$. By Aschbacher's theorem $H^* \in \mathscr{M}_4$ and so Proposition C of § 5 applies (with $X = H^*$, $R_i = N_i^*$, and $r = \delta$). We have $H^* \simeq B_3(q)$ and, since $\langle \alpha, E\rangle$ acts properly on each N_i^*, a contradiction.

Hence $M^* \notin \mathscr{M}_4$.

This last step, the reduction to the seven dimensional orthogonal group $B_3(q)$, is at the heart of the argument. This point is made in the closely related work of Walter [14].

5. **Results on Chevalley-type groups.** We now apply the methods of [3] to prove Propositions A, B, C. Together with the arguments in § 6, this will complete the proof that $M^* \in \mathscr{M}_4$. At several points the proofs of the propositions reduce to case by case calculations. These are always straightforward applications of the theory in [3] and are therefore omitted.

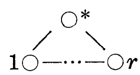
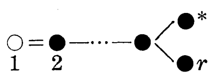
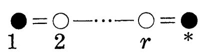
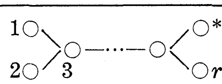
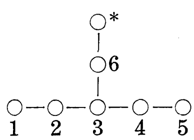
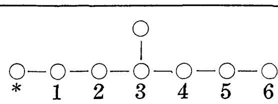
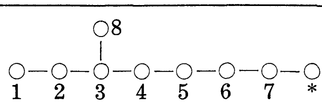
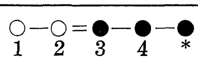
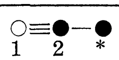
The notation of [3] is followed closely: thus G will now denote a connected, simple algebraic group over an algebraically closed field k. T is a maximal torus of G and $X(T)$, $\Gamma(T)$ are the associated lattices. Σ is the root system in $X(T)$ and $W = N_G(T)/T$ the Weyl group. We assume that rank $G = r$ is $\geqq 2$ and that the characteristic of k is p, an odd prime. Since G is simple we may take $X(T)$ to be the adjoint lattice, i.e., spanned by Σ. Let $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ be a simple root system in Σ and $\{\eta_1, \cdots, \eta_r\}$ the dual basis in $\Gamma(T)$. Let $\alpha_* = -(m_1\alpha_1 + \cdots + m_r\alpha_r)$ be the low root in Σ relative to Π and $\hat{\alpha}_* \in \Gamma(T)$ its co-root.

To avoid confusion with the above notation, the involutions $\alpha, \beta, \gamma, \delta, \cdots$ occurring in §§ 1-4 are replaced by lower case latin letters. Since the calculations of this section are completely independent of the earlier sections this should not cause any trouble. Note that if H is some connected reductive algebraic group then $E(H)$ is used to denote its maximal semi-simple subgroup and $F(H)$ to denote the largest central torus of H. Thus $[E(H), F(H)] = 1$ and $E(H)F(H) = H$ (see [3] § 2). Context should enable one to avoid confusion with the corresponding symbols in finite group theory.

In the following table we list (1) the simple Chevalley groups and their extended Dynkin diagrams. Each simple root is numbered and α_* is denoted by $*$, (2) a representative in T for each class of involutions in the group. Here "η_i" is short for $\eta_i(-1) \in T$, (3) the (quasi) simple components of the centralizer of each involution. Certain obvious conventions are used: $A_0 = B_0 = C_0 = 1$, $A_1 = B_1 = C_1$, $B_2 = C_2$, $A_3 = D_3$, and D_1 is not simple and should be omitted while D_2 has two components, each of type A_1.

Similar results for the graph automorphisms are tabulated in § 4.3 of [3].

The methods of [3] are to a certain extent based on the earlier work of Iwahori [10]. This useful paper contains several very detailed computations of classes and centralizers of involutions.

Group		Involutions	Components
A_r $r \geq 2$		η_i $1 \leq i \leq (r+1)/2$	A_{i-1}, A_{r-i}
B_r $r \geq 3$		η_i $1 \leq i \leq r$	B_{i-1}, D_{r-i+1}
C_r $r \geq 2$		η_1 η_i $2 \leq i \leq (r+2)/2$	A_{r-1} C_{i-1}, C_{r-i+1}
D_r $r \geq 4$		η_1, (η_2) η_i $3 \leq i \leq (r+2)/2$ η_r	A_{r-1} D_{i-1}, D_{r-i+1} D_{r-1}
E_6		η_1 η_6	D_5 A_1, A_5
E_7		η_1 η_6 η_7	A_1, D_6 E_6 A_7
E_8		η_1 η_7	D_8 A_1, E_7
F_4		η_1 η_4	B_4 A_1, C_3
G_2		η_2	A_1, A_1

● = long root. In D_r, η_1 and η_2 are (are not) conjugate if r is odd (even).

The finite groups corresponding to G are the fixed point sets $G(\rho) = \{g \in G: \rho g = g\}$ where ρ is a finite type endomorphism of G (§ 5.1 in [3]). We may assume that ρ stabilizes T and hence $\rho = i_n \sigma$, where $i_n g = ngn^{-1}$ with $n \in N_G(T)$, and σ is in standard form relative to T and Π, i.e.,

$$\sigma x_{\pm\alpha}(\xi) = x_{\pm\sigma\alpha}(\xi^{q_\alpha}) \quad \text{all} \quad \alpha \in \Pi, \quad \xi \in k$$

where $\alpha \rightarrow \sigma\alpha$ is a permutation of Π and q_α is a power of p. The distinct pairs G, σ produce all possible finite Chevalley type groups $G(\sigma)$ (and $G(\sigma) \simeq G(\rho)$ when $\rho = i_n\sigma$).

By the lemma below, G of type G_2 will not occur, and hence, since $r \geqq 2$ and $p = \text{odd}$, we have $L(G(\sigma)) = O^{p'}(G(\sigma))$. Thus there is a natural embedding $L(G(\sigma)) \subseteqq G(\sigma) \subseteqq \operatorname{Aut} L(G(\sigma))$ with $G(\sigma)$ consisting of all the inner and diagonal automorphisms. The usual notation, e.g., $A_2(q)$, $B_3(q)$, ${}^2E_6(q), \cdots$ is used to denote the simple groups $L(G(\sigma))$. In all cases which occur below, $q_\alpha = q$ for all $\alpha \in \Pi$.

If $w \in W$ is the image of $n \in N_G(T)$ then $\rho = i_n\sigma$ induces the action $\rho = w\sigma$ on $\Gamma(T)$ (§ 2.3 in [3]). Since most calculations take place in $\Gamma(T)$ we usually describe ρ in this latter form.

Let $X \in \mathscr{M}_4$ then we may find G, ρ satisfying $L(X) = L(G(\rho))$.

LEMMA. *Let* $X \in \mathscr{M}_4$, $t \in \operatorname{Inv} X$ *and* $Y = O(C_X(t)) \neq 1$, *then*
(a) $\langle L(X), t, Y\rangle \subseteqq G(\rho)$
(b) $L(X)$ *is not one of* ${}^3D_4(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$, ${}^2G_2(q)$
(c) $|Y|$ *divides* $q-1$ *or* $q+1$ (in particular $q \neq 3$).

Proof. If t induces a field or graph automorphism then $O(C_X(t)) = 1$ as follow from §§ 5.5. and 4.3 in [3]. Hence $t \in G(\rho)$. Since $[Y, L(C_X(t))] = 1$ no element in Y can induce field automorphisms or (in the case of D_4) graph automorphisms and hence $Y \subseteqq G(\rho)$. This proves (a).

Using the classification in § 3 [3], with $\psi = 1$ (see the above table), we may assume $t = \eta_i(-1)$ for some $1 \leqq i \leqq r$. If $C_G(t)$ is semi-simple then, since $X(T)$ is adjoint, $Z(C_G(t))$ turns out to be a 2-group. This follows from inspection of the table (the only case that needs further calculation is the involution $\eta_6(-1)$ in E_6). Hence $F(C_G(t)) \neq 1$ (§ 2.1 in [3]) which implies that $m_i = 1$ in the expansion of α_*. This is immediate from the description of the centralizer subgroups as given in § 4 of [3], see also Proposition 8 of [10]. This eliminates groups of type E_8, F_4, G_2. If $L(G(\rho)) = {}^3D_4(q)$ then ρ permutes cyclically the three involutions with $m_i = 1$ and hence no conjugates of them can lie in $G(\rho)$. This proves (b).

Since $t = \eta_i(-1)$ and $m_i = 1$, $F(C_G(t)) = \langle \eta_i(\zeta) \colon \zeta \in k^*\rangle$, and so $Y \subseteqq \langle \eta_i(\zeta) \colon (\rho - 1)\eta_i(\zeta) = 1, \ \zeta \in k^*\rangle$. Since $\rho\eta_i = \pm q\eta_i$, this proves (c).

PROPOSITION A. *Let* $X \in \mathscr{M}_4$, $t \in \operatorname{Inv} X$ *and* $1 \neq Y \subseteqq O(C_X(t))$

then there exists $u \in \operatorname{Inv} X$ *such that* $S \in \operatorname{Comp} C_X(u)$ *where* $S \simeq SL_2(q)$, $\langle u \rangle = Z(S)$, *and* $\langle t, Y \rangle$ *acts properly on* S.

Proof. By the lemma we may suppose $X = G(\rho)$. We choose $u = \hat{\alpha}_*(-1)$ and put $S_* = \langle U_{\alpha_*}, U_{-\alpha_*} \rangle$ (§ 2.1 [3]). Since $\hat{\alpha}_*(-1) \in S_*$, we have $S_* \simeq SL_2(k)$. By inspection of the extended Dynkin diagram of G we see that S_* is always a factor of $E(C_G(u))$. Now $\eta_i(\zeta)x_{\alpha_*}(\xi)\eta_i(\zeta^{-1}) = x_{\alpha_*}(\xi\zeta^{-1})$ and hence $\langle t, Y \rangle$ acts properly on S_*. Thus it remains to show that ρ may always be chose to stabilize S_*, for then $\langle t, Y \rangle$ will act properly on $S = S_*(\rho)$.

Let $v_i \in W$ be the unique element stabilizing the set $\{\alpha_1, \cdots, \alpha_r, \alpha_*\}$ and such that $v_i\alpha_* = \alpha_i$. Let $w_0 \in W$ be the unique element such that $w_0\Pi = -\Pi$, and let $w_0^{(i)}$ be the corresponding element for the simple root system $\Pi - \{\alpha_i\}$. A simple argument yields $v_i = w_0^{(i)}w_0$.

Let σ be in standard form relative to T and Π. The methods of § 5.3 in [3] show directly that all possible pairs (X, t) occur among $(G(\rho), \eta_i(-1))$ where $m_i = 1$ and $\rho = \sigma$ or $\rho = v_i\sigma$. When $\rho = \sigma$ it is clear that σ stabilizes S_* and we are done. However if $\rho = v_i\sigma$, then $\rho S_* \neq S_*$. In this case put $\rho' = w_0\sigma$. Let $n_0^{(i)} \in N_G(T)$ be any inverse image of $w_0^{(i)}$. By the definition of $w_0^{(i)}$ we see, by § 4.2 in [3], that $n_0^{(i)}$ lies in the connected component of $C_G(t)$. Hence, since $w_0 = w_0^{(i)-1}v_i$ and $G(\rho') \simeq G(\rho)$, all pairs (X, t) occur (up to isomorphism) among the pairs $(G(\rho'), \eta_i(-1))$ with $m_i = 1$ and $\rho' = \sigma$ or $\rho' = w_0\sigma$. Since $w_0\alpha_* = -\alpha_*$, ρ' stabilizes S_* and we are done.

Let q_1 be some power of q. In fact it will turn out that $q_1 \in \{q, q^2\}$.

PROPOSITION B. *Let* $X \in \mathscr{M}_4$ *and let* $\hat{L}$ *be a 2-fold covering of* $L(X)$ *and let* $t, s \in \operatorname{Inv}(\operatorname{Aut} \hat{L})$ *such that*

(i) $Y = O(C_{\hat{L}}(t)) \neq 1$ *and* $[s, \langle t, Y \rangle] = 1$,

(ii) *there exists* $\hat{Q} \in \operatorname{Comp} C_{\hat{L}}(s)$ *such that* $\hat{Q} \simeq SL_2(q_1)$, $Z(\hat{Q}) \subseteq Z(\hat{L})$ *and* $\langle t, Y \rangle$ *acts properly on* $\hat{Q}$.

Then there exists $u \in \operatorname{Inv} \hat{L}$ *such that* $\hat{S}_1, \hat{S}_2 \in \operatorname{Comp} C_{\hat{L}}(u)$ *where* $\hat{S}_1\hat{S}_2 \simeq SL_2(q) \times SL_2(q)$, $Z(\hat{S}_1\hat{S}_2) = \langle u, Z(\hat{Q}) \rangle$ *and* $\langle t, Y \rangle$ *acts properly on both* $\hat{S}_1$ *and* $\hat{S}_2$.

Proof. As before, we may suppose $L(X) = L(G(\rho))$ for suitable G, ρ. Let $\hat{G}$ denote the simply connected covering group of G and lift the action of ρ to $\hat{G}$, then $\hat{L} \subseteq \hat{G}(\rho)$. Since $|Z(\hat{G})|$ must be even G is not of type A_r ($r =$ even) or E_6.

Consider all $v \in \operatorname{Inv}(\operatorname{Aut} X)$ with $Q \in \operatorname{Comp} C_X(v)$ such that $Q \simeq L_2(q_1)$. Since $r \geqq 2$, by § 5.5 in [3], v cannot be a field-type automorphism. If G is of type A_r $(r \geqq 5)$, C_r $(r \geqq 3)$, or E_7 then the methods of § 4 in [3] show that v must be conjugate to $\hat{\alpha}_*(-1)$. Since $S_* = \langle U_{\alpha_*}, U_{-\alpha_*} \rangle$ is the unique simple rank 1 factor in $C_G(\hat{\alpha}_*(-1))$ and $S_*(\rho) \simeq SL_2(q)$ we conclude that G must be of type $B_r (r \geqq 2)$ or $D_r (r \geqq 3)$. For these cases we have, up to conjugacy in G, the following candidates for s:

$$B_r (r \geqq 2) \quad \underset{1}{\circ} = \underset{2}{\bullet} - \underset{3}{\bullet} - \cdots - \underset{r}{\bullet} \quad s = \hat{\alpha}_*(-1) \text{ or } \eta_2(-1)$$

$$D_r (r \geqq 3) \quad \begin{matrix} 1\circ \diagdown & & \\ & \underset{3}{\circ} - \cdots - \underset{r}{\circ} \\ 2\circ \diagup & & \end{matrix} \quad s = \hat{\alpha}^*(-1) \text{ or } \eta_r(-1)\psi$$

where ψ is the standard form graph automorphism interchanging α_1 and α_2.

Put $S_r = \langle U_{\alpha_r}, U_{-\alpha_r} \rangle$ then $S_r \simeq SL_2(k)$ is a factor of $E(C_G(\hat{\alpha}_*(-1))$ and $S_r S_* \simeq SL_2(k) * SL_2(k)$ with $\langle \hat{\alpha}_*(-1) \rangle = Z(S_r S_*)$. As in the proof of Proposition A, we see that if $t = \eta_i(-1)$ then $\langle t, Y \rangle$ acts properly on both S_r and S_* except in one case, namely $t = \eta_1(-1)$ (or $\eta_2(-1)$) and G of type D_r. However we can show that this case does not satisfy hypothesis (i) and (ii): Suppose $s = \eta_r(-1)\psi$, then a complete set of representatives for the classes of involutions in $C_G(s)^\circ$ are $\eta(-1)$ and $(\eta + \eta_r)(-1)$ in Γ_ψ (see § 4.2 in [3]) where $\eta \in \{\eta_1 + \eta_2, \eta_3, \cdots, \eta_{r-1}\}$. Using the algorithm in Appendix 2 of [3] one shows that none of these involutions are conjugate in G to either $\eta_1(-1)$ or $\eta_2(-1)$. For example $(\eta_1 + \eta_2 + \eta_r)(-1) \sim (\eta_1 + \eta_2 - \eta_{r-1} + \eta_r)(-1) \sim (\eta_1 + \eta_2 - \eta_{r-2} + \eta_{r-1})(-1) \sim \cdots \sim (\eta_1 + \eta_2 - \eta_3 + \eta_4)(-1) \sim \eta_3(-1)$ in G. Now classify the involutions in $C_G(\eta_1(-1))$. Up to conjugacy in $C_G(\eta_1(-1))$ we find that we may assume $s = \hat{\alpha}_*(-1)$. Hence if $\rho\eta_1(-1) = \eta_1(-1)$ and $s \in G(\rho)$, ρ must always stabilize both S_r and S_*. Hence $(S_r S_*)(\rho) \simeq SL_2(q) * SL_2(q)$ (if ρ flipped S_r and S_* then $(S_r S_*)(\rho) \simeq L_2(q^2) \times \langle s \rangle$) and so hypothesis (ii) is not satisfied.

Finally, note that we must have $\hat{L} = \hat{G}(\rho)$ since hypothesis (ii) is not satisfied for any intermediate covering when G is of type D_r. Let $\hat{S}_r, \hat{S}_* \in \operatorname{Comp} C_{\hat{G}}(\gamma)$ where $u = \hat{\alpha}_*(-1)$. Then $\hat{S}_r \hat{S}_* \simeq SL_2(k) \times SL_2(k)$ and $Z(\hat{S}_r \hat{S}_*) = \langle u, Z(\hat{Q}) \rangle$. As in the final step of the proof of Proposition A we can choose ρ to stabilize $\eta_1(-1)$ and $\hat{S}_r$ and $\hat{S}_*$ and hence are done.

PROPOSITION C. *Let* $X \in \mathscr{M}_4$, $r \in \operatorname{Inv} X$, $R_1, R_2 \in \operatorname{Comp} C_X(r)$ *such that* $R_1 \simeq L_2(q)$, $R_2 \simeq SL_2(q)$ *and* $\langle r \rangle = Z(R_2)$ *then*

(a) $L(X) \simeq B_3(q)$ *and*

(b) *there is no* $t \in \operatorname{Inv} X$ *with* $Y \subseteqq O(C_X(t)) \neq 1$ *such that* $\langle t, Y \rangle$ *acts properly on both* R_1 *and* R_2.

Proof. (a) follows from inspection of the centralizers of all elements in Inv (Aut X), ($L(X) = L(G(\rho))$, as before). For this, see the above table and related facts in [3].

So G is of Type B_3. We make take $r = \hat{\alpha}_*(-1)$ and since $[t, r] = 1$ can look for possible t's in $C_G(r)$. With $S_j = \langle U_{\alpha_j}, U_{-\alpha_j} \rangle$ we have $C_G(\delta)^0 = S_1 S_3 S_*$ where $S_1 \simeq L_2(k)$ and $S_3 S_* \simeq SL_2(k) * SL_2(k)$. Elements in $C_G(r)/C_G(r)^0$ flip S_3 and S_* and so $t \in C_G(r)^0$ (if it exists). There are 5 classes of involutions in $C_G(r)^0$ with representatives $\eta(-1)$ where $\eta \in \{\eta_1, \eta_2, \eta_1 + \eta_2, \eta_1 + \eta_3, \eta_2 + \eta_3\}$. Only $\eta(-1)$ with $\eta \in \{\eta_1 + \eta_2, \eta_2 + \eta_2\}$ are conjugate in G to $\eta_3(-1)$ (in B_3 only $m_3 = 1$). Since $\alpha_* = -(2\alpha_1 + 2\alpha_2 + \alpha_3)$ and since $Y \subseteqq \langle \eta(\zeta) : \zeta \in k^* \rangle$ we see that $\langle t, Y \rangle$ centralizes S_3 and S_* if $\eta = \eta_1 + \eta_2$ and S_1 if $\eta = \eta_2 + \eta_3$. Hence there is no such t and (b) is proved.

6. The $L_6(4)$ case. In Step 2 of the proof in § 4 the case $L(J^*) \simeq L_3(4)$ may be eliminated as follows:

The involutions inside $L_3(4)$ have solvable, core-free, centralizers and hence both α, β induce outer automorphisms on $L(J^*)$. Put $\hat{J} = J/O(J)$ then $L(\hat{J})$ is quasi-simple. Put $\hat{S} = SO(J)/O(J)$ then $\hat{S} \simeq SL_2(q)$ and $\hat{S} \in \mathrm{Comp}\, C_{\hat{J}}(\beta)$ (and $q = 5$ or 7). This implies, by a direct calculation on $L_3(4)$, that $L(\hat{J})$ is the full 2-fold covering of $L_3(4)$ and $C_{L(\hat{J})}(\beta) = \langle \hat{\rho} \rangle * \hat{S}$ where $\hat{\rho} \in Z(\hat{J})$, $\langle \hat{\rho}^2 \rangle = Z(\hat{S})$ and $\hat{\rho}^\alpha = \hat{\rho}^{-1}$.

Since $Z(S) = \langle \gamma \rangle$ we may choose $\rho \in C_J(\beta)$, an inverse image of $\hat{\rho}$, satisfying $\rho^2 = \gamma$. Then ρ normalizes but does not centralize M(see § 3). Put $M_1 = \langle M, \rho \rangle$ and $M_1^* = M_1/Z^*(M_1)$ and, for convenience, let α, ρ, γ also denote the images of these elements in M_1^*.

We may assume that α, γ are chosen as in Proposition A of § 5. Since $\hat{\rho} \in Z(\hat{J})$ therefore ρ centralizes every element of $\mathrm{Comp}\, C_{M_1^*}(\gamma)$. By the general structure of $C^*_{M_1}(\gamma)$ (see §§ 4 and 5 in [3]) we must have $\rho \in \langle \hat{\alpha}_*(\zeta) : \zeta \in k^* \rangle$ and hence ρ and $\alpha = \eta_i(-1)$ commute. This contradicts the fact that $\hat{\rho}$ is inverted by α.

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