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INFINITE TENSOR PRODUCTS OF C^*-ALGEBRAS

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The infinite tensor product $A = \otimes(A_i, p_i)$ of a family of C^*-algebras A_i with respect to projections $p_i \in A_i$ is examined. The primitive ideal space and the characters of A are completely described in the case where each A_i is simple, or separable and nuclear. If A is not type I, an explicit construction is given of a factor representation of A generating an arbitrary hyperfinite factor. In addition, new results are obtained about primitive ideals and characters of a tensor product of two C^*-algebras. Examples are given of various phenomena, providing solutions to previously published problems.

In this paper, the structure of an infinite tensor product $A = \bigotimes(A_i, p_i)$ of a family of C^*-algebras A_i with respect to projections $p_i \in A_i$ is studied. Most structural questions about such algebras can be reduced to analogous questions about the structure of finite tensor products. In particular, the primitive ideal space and characters of A can be completely described in the case where each A_i is simple, or separable and nuclear. If A is not type I, an explicit construction is given of a factor representation of A generating an arbitrary infinite hyperfinite factor.

More detailed information about A is available if each A_i is type I. Conditions are given for when A is type I, CCR, GTC, or continuous trace, and examples are given of various phenomena, some of which are new. In addition, some new results are obtained about characters of finite tensor products.

There has been considerable work recently studying the struture of certain non-type-I C^*-algebras. The work of Glimm on UHF C^*-algebras and Dixmier on matroid C^*-algebras led to the study of AF algebras (inductive limits of finite-dimensional C^*-algebras) by Bratteli and others. It seems possible that a reasonable structure theory for inductive limits of type I C^*-algebras can be developed, although the situation can become quite complicated. In this paper, we discuss a particular type of inductive limit of C^*-algebras, the infinite tensor product. It is hoped that some of the results and methods of this paper will be useful in studying the general situation.

The results of this paper have immediate application to the representation theory of restricted direct product groups (such as adele groups), since the group C^*-algebra of such a group is an infinite tensor product of the C^*-algebras of the coordinate groups.

The organization of the paper is as follows:

Section 2 is a summary of the constructions and notation used. Section 3 contains results on finite tensor products. The two main results are a complete description of the primitive ideals (Theorem 3.3) and characters (3.7, 3.8) of a tensor product of two C^*-algebras under the hypothesis that one of the algebras be separable and nuclear, or that both be simple.

Section 4 is a description of the primitive ideal space of an infinite tensor product of type I algebras. Conditions are given for the product to be type I, CCR, GTC, or continuous trace and several examples are included, such as a CCR C^*-algebra with no finite composition series $\{J_n\}$ such that $(J_{n+1}/J_n)^\wedge$ is Hausdorff, solving an open question.

Section 6 contains an infinite product weight construction which is used to describe all characters of an infinite tensor product of C^*-algebras in certain cases, including products of type I algebras, simple algebras, or separable nuclear algebras. It is shown that a primitive ideal of an infinite tensor product of type I C^*-algebras is the kernel of a (necessarily unique) traceable factor representation if and only if it is locally closed.

Section 7 is an explicit construction of a large family of factor representations of an infinite tensor product of C^*-algebras. If A is an infinite tensor product of a countable number of separable type I C^*-algebras, and if A is not type I, then an explicit construction is given of a representation of A generating a given infinite hyperfinite factor.

The work of this paper generalizes results of Guichardet [6] and Moore [9]; many of the results extend those of Tomiyama [11] to infinite tensor products. Some of this paper was part of the author's doctoral dissertation at the University of California, Berkeley (1975), and he expresses his appreciation to his adviser, Calvin C. Moore, for a great deal of help and guidance. He is also grateful to the referee for pointing out a number of errors and obscurities in the original draft.

2. Definitions and notation. In this section we will briefly review some constructions which have appeared before in the literature, in order to establish notation.

(a) Let X_i $(i \in I)$ be locally compact topological spaces with compact open subspaces Y_i. Let $X = \{(\cdots x_i \cdots) \in \Pi X_i : x_i \in Y_i \text{ a.e.}\}$. (In this paper, "almost everywhere" will always mean "for all but a finite number.") If F is a finite subset of I, set $X_F = \prod_{i \in F} X_i$, $Y_F^* = \prod_{i \notin F} Y_i$, $Z_F = X_F \times Y_F^*$. Z_F is locally compact, and if $E \subseteqq F$, Z_E is an open subset of Z_F. Topologize $X = \cup Z_F$ by letting $S \subseteqq X$ be open if and only if $S \cap Z_F$ is open for all F. X is then locally compact, and each Z_F is open in X.

DEFINITION. $X = \prod'_{i \in I} (X_i, Y_i)$ is the restricted direct product of the X_i with respect to Y_i.

If G_i are locally compact topological groups with compact open subgroups K_i, then $G = \Pi'(G_i, K_i)$ is a locally compact topological group under coordinatewise multiplication, and $K = \Pi K_i$ is a compact open subgroup. If each G_i is second countable and I is countable, then G is also second countable.

(b) If B and C and C^*-algebras, the algebraic tensor product $B \odot C$ can be completed with respect to the least C^*-cross norm to give the C^*-tensor product $B \otimes C$. This is the only tensor product we will consider in this paper; we will mostly be concerned only with nuclear C^*-algebras, for which all C^*-cross norms coincide. If B and C are von Neumann algebras, we can also form the von Neumann algebra tensor product $B \overline{\otimes} C$. The symbol $\odot$ will always denote algebraic tensor product, $\otimes$ the C^* product, and $\overline{\otimes}$ the von Neumann product. Also, if B is a C^*-algebra, we will use the notation $\mathrm{Pr}(B)$ to denote the set of ideals of B which are kernels of factor representations of B. $\mathrm{Pr}(B) = \mathrm{Prim}(B)$ for most, if not all, C^*-algebras, including all separable or GCR C^*-algebras. We will also write $\tilde{B}$ for the smallest C^*-algebra with an identity containing B (i.e., $\tilde{B} = B$ if B has an identity, otherwise $\tilde{B}$ is B with identity adjoined.)

(c) Let A_i $(i \in I)$ be C^*-algebras, and let p_i be a nonzero projection in A_i. If $F \subseteqq I$ is finite, let $A_F = \bigotimes_{i \in F} A_i$ as above. Write $p_F = \bigotimes_{i \in F} p_i$. If $E \subseteqq F$, define an isomorphic embedding σ_{EF} of A_E into A_F by $\sigma_{EF}(a) = a \otimes p_{F \sim E}$. Then $\{A_F, \sigma_{EF}\}$ form a directed system of C^*-algebras; let $A = \varinjlim \{A_F, \sigma_{EF}\}$.

DEFINITION. $A = \bigotimes_{i \in I} (A_i, p_i)$ is the infinite tensor product of the A_i with respect to p_i.

If each A_i is separable and I is countable, then A is separable. There is a canonical embedding σ_F of A_F into A, such that if $E \subseteqq F$, $\sigma_E = \sigma_F \circ \sigma_{EF}$. Also, there is a projection $p \in A$ with $p = \sigma_F(p_F)$ for all F. For any F, we will use the notation $B_F = \bigotimes_{i \notin F} (A_i, p_i)$, q_F the distinguished projection of B_F; then $A \cong A_F \otimes B_F$, $p = p_F \otimes q_F$.

EXAMPLES. (1) If each A_i has an identity 1_i, then $\bigotimes (A_i, 1_i)$ is the ordinary infinite tensor product $\bigotimes A_i$.

(2) If X_i $(i \in I)$ are locallyc ompact Hausdorff spaces with compact open subspaces Y_i, let $A_i = C_0(X_i)$, continuous functions vanish-

ing at infinity; then $\bigotimes(C_0(X_i), \chi_{Y_i}) \cong C_0(\Pi'(X_i, Y_i))$ under the obvious identification of functions.

(3) Let $G = \Pi'(G_i, K_i)$; assume that $C^*(G_i)$ is nuclear for all i (see Theorem 3.2). Then we may identify $C^*(G_F)$ with $\bigotimes_{i \in F} C^*(G_i)$. Let $H_F = G_F \times K_F^* \subseteq G$; define an isomorphic embedding ϕ_F of $C^*(G_F)$ into $C^*(H_F) \subseteq C^*(G)$ by $\phi_F(a) = a \otimes \chi_{K_F^*}$. If $E \subseteq F$, $\phi_E = \phi_F \circ \sigma_{EF}$, so there is an embedding of $\bigotimes(C^*(G_i), \chi_{K_i})$ into $C^*(G)$, which is surjective since, for any F, the image contains all functions supported on H_F which depend on only a finite number of coordinates.

(d) If $\mathscr{H}_i$ $(i \in I)$ are Hilbert spaces with unit vectors $\xi_i \in \mathscr{H}_i$, write $\mathscr{H} = \bigotimes_{i \in I}(\mathscr{H}_i, \xi_i)$ for the infinite tensor product of the $\mathscr{H}_i$ with respect to ξ_i, as in [9]. If M_i is a von Neumann algebra on $\mathscr{H}_i$, write $M = \overline{\bigotimes}_{i \in I}(M_i, \mathscr{H}_i, \xi_i)$ for the von Neumann algebra on $\mathscr{H}$ generated by the images of the M_i. If each M_i is a factor, M is a factor.

The above constructions depend, of course, on the parameters (Y_i, p_i, ξ_i) chosen; however, the parameters may be changed or left undefined in a finite number of coordinates without changing the product. For convenience, we will usually assume they are defined everywhere.

3. Finite tensor products. In this section, we discuss the primitive ideal space and the characters of a tensor product of two C^*-algebras. Some of the most basic questions about finite tensor products are still unsolved, but the recent work of Connes, Effros and Choi, and Lance allows us to solve the relevant problems for separable nuclear C^*-algebras, which are by far the most important ones in applications.

Let B and C be C^*-algebras. If π is a factor representation of B and ρ of C, then $\pi \otimes \rho$ is a factor representation of $B \otimes C$. The map j: $(\ker \pi, \ker \rho) \to \ker(\pi \otimes \rho)$ gives a well-defined injective map from $\mathrm{Pr}(B) \times \mathrm{Pr}(C)$ into $\mathrm{Pr}(B \otimes C)$, which maps $\mathrm{Prim}(B) \times \mathrm{Prim}(C)$ homeomorphically onto a dense subspace of $\mathrm{Prim}(B \otimes C)$.

If $(I, J) \in \mathrm{Pr}(B) \times \mathrm{Pr}(C)$, $j(I, J)$ is the kernel of the composite map $B \otimes C \to (B \otimes C)/(I \otimes C + B \otimes J) \to (B/I) \otimes (C/J)$; if B/I or C/J is nuclear, then $j(I, J) = I \otimes C + B \otimes J$. (See [6, §§ 6 and 7].) There is also a map r: $\mathrm{Pr}(B \otimes C) \to \mathrm{Pr}(B) \times \mathrm{Pr}(C)$ defined as follows: if π is a factor representation of $B \otimes C$, π extends uniquely to $\tilde{B} \otimes \tilde{C}$, since $B \otimes C$ is an ideal in $\tilde{B} \otimes \tilde{C}$. Set $\pi_1(b) = \pi(b \otimes 1)$, $\pi_2(c) = \pi(1 \otimes c)$. π_1 and π_2 are factor representations of B and C respectively, called the restrictions of π to B and C. Set $r(\ker \pi) = (\ker \pi_1, \ker \pi_2)$. If $K \in \mathrm{Pr}(B \otimes C)$, $r(K) = (I, J)$, where I and J are the kernels of the composite maps $B \to B \otimes \tilde{C} \to (B \otimes \tilde{C})/K$ and $C \to \tilde{B} \otimes C \to (\tilde{B} \otimes C)/K$, so r is well defined. $r \circ j$ is the identity.

If ϕ is a character on B and ψ a character on C, then $\phi \otimes \psi$ is defined to be the character on $B \otimes C$ corresponding to the factor representation $\pi_\phi \otimes \pi_\psi$.

DEFINITION 3.1. A pair (B, C) of C^*-algebras is said to have property (Pr) if the map $j: \mathrm{Pr}(B) \times \mathrm{Pr}(C) \rightarrow \mathrm{Pr}(B \otimes C)$ is surjective. (B, C) is said to have property (Ch) if every character on $B \otimes C$ has the form $\phi \otimes \psi$, for characters ϕ and ψ on B and C respectively.

Properties (Pr) and (Ch) are closely related, although it is not clear that either one implies the other. Properties (Pr) was studied by Tomiyama [11], who called it property (F). Wassermann [12] has shown that if B is the group C^*-algebra of the free group on two generators, then (B, B) does not satisfy (Pr); no examples are known of C^*-algebras not satisfying (Ch). Itis shown below that if either B or C is separable and nuclear, or if both B and C are simple, then (B, C) satisfies both properties.

Recall that a C^*-algebra B is nuclear if the algebraic tensor product $B \odot C$ has a unique C^*-cross norm for every C^*-algebra C. We summarize some recent results of Connes [3], Choi and Effros [2], and Lance [7], in the following theorem.

THEOREM 3.2. *Let B be a separable C^*-algebra. Then B is nuclear if and only if every factor representation of B generates a hyperfinite factor. If J is an ideal of B, then B is nuclear if and only if both J and B/J are nuclear. Furthermore, the class of nuclear C^*-algebras is closed under finite tensor products, inductive limits* (hence under infinite tensor products), *and crossed products by arbitrary amenable groups. The group C^*-algebra of any locally compact* (second countable) *group which is amenable or connected is nuclear.*

THEOREM 3.3. *Let B be a nuclear C^*-algebra, C any C^*-algebra. Then (B, C) satisfies* (Pr).

Proof. Follows immediately from [11, Theorem 5] and [2].

It is reasonable to conjecture that, for a fixed C^*-algebra B, (B, C) satisfies (Pr) for every C if and only if B is nuclear.

We now turn to the property (Ch). We first need two lemmas, the first of which is closely related to Lemma 13 of [6].

LEMMA 3.4. *Let N_1 and N_2 be factors on a Hilbert space $\mathscr{H}$, with $N_2 \subseteq N_1'$; suppose the map $\Phi: N_1 \odot N_2 \rightarrow \mathscr{L}(\mathscr{H})$ given by $n_1 \otimes n_2 \rightarrow n_1 n_2$ extends to an isometry of the C^*-tensor product $N_1 \otimes N_2$*

into $\mathscr{L}(\mathscr{H})$. Suppose that the von Neumann algebra N generated by N_1 and N_2 is a semifinite factor, and that the C^*-algebra generated by N_1 and N_2 has nonzero intersection with the ideal J of N which is the norm-closure of the "trace-class" operators of N. Then N_1 and N_2 are semifinite, and if N has a cyclic and separatng vector, then $\mathscr{H}$ can be written $\mathscr{H}_1 \otimes \mathscr{H}_2$ with $N_1 = \bar{N}_1 \otimes \mathbf{1}$, $N_2 = \mathbf{1} \otimes \bar{N}_2$, $N = \bar{N}_1 \overline{\otimes} \bar{N}_2$.

Proof. There are two cases: (1) N is finite. Then N_1 and N_2 are finite, and the result is well known (see [6, Lemma 13]). (2) N is not finite, in which case $J \neq N$. $\Phi(N_1 \otimes N_2) \cap J \neq \{0\}$, so $\Phi^{-1}(J)$ is a nontrivial ideal in $N_1 \otimes N_2$. Therefore, either N_1 or N_2 is not simple. We will reduce to case (1) in two steps. First, write $\mathscr{H} = \mathscr{V} \otimes \mathscr{W}$ with $N_1 = \mathscr{L}(\mathscr{V}) \overline{\otimes} M_1$, with M_1 a factor which is a simple C^*-algebra. (Every factor can be written as a tensor product of a type I factor and a factor which is a simple C^*-algebra.) $N_2 = \mathbf{1} \otimes M_2$ for a factor M_2 on $\mathscr{W}$; $N = \mathscr{L}(\mathscr{V}) \overline{\otimes} M$, where M is the von Neumann algebra generated by M_1 and M_2. M is a semifinite factor; let I be the norm-closure of the ideal of "trace-class" elements of M. Let J_1, J_2, I_2 be the minimal nonzero norm-closed (not necessarily proper) ideals of N_1, N_2, and M_2 respectively. $J_1 \otimes J_2$ is a simple C^*-algebra and hence is the minimal nonzero ideal of $N_1 \otimes N_2$, so $\Phi(J_1 \otimes J_2) \subseteqq J$. Let $m_1 \in M_1$, $m_2 \in I_2$, be nonzero, and let s be a rank 1 projection in $\mathscr{L}(\mathscr{V})$. Then $s \otimes m_1 \in J_1$, $1 \otimes m_2 \in J_2$, and so $s \otimes m_1 m_2 = (s \otimes m_1)(1 \otimes m_2) \in \Phi(J_1 \otimes J_2) \subseteqq J$. Thus $m_1 m_2 \in I_2$, and $m_1 m_2 \neq 0$ by [6, Prop. 0], so the C^*-algebra generated by M_1 and M_2 has nonzero intersection with I_2, and so M_1, M_2, M, and I satisfy the hypotheses of the lemma. If M is finite, we are in case (1) and so we are finished; otherwise M_2 is not simple and we can reduce again in the same way. After the second reduction, we must be in case (1).

LEMMA 3.5. *Let N_1 and N_2 be semifinite factors with minimal ideals J_1 and J_2; let $N = N_1 \overline{\otimes} N_2$, J the minimal ideal of N, $K = J \cap (N_1 \otimes N_2)$. If $\varphi \in N_1^*$, $\psi \in N_2^*$ with $\varphi | J_1 \equiv 0$ or $\psi | J_2 \equiv 0$, then $(\varphi \otimes \psi) | K \equiv 0$.*

Proof. We use the language and notation of [10]. Let τ_1 and τ_2 be traces on N_1 and N_2, and $\tau = \tau_1 \otimes \tau_2$ a trace on N. (See § 5.) If $\varphi \in N_{1*}$, $0 \leqq \varphi \leqq \tau_1$, then $\varphi \otimes \tau_2 = \tau_2 \circ R_\varphi \leqq \tau$, and so R_φ maps J into J_2. Similarly, if $\psi \in N_{2*}$, $0 \leqq \psi \leqq \tau_2$, L_ψ maps J into J_1. Linear combinations of such functionals are weak-* dense in N_1^* and N_2^*; this can be seen as follows. Let π be the GNS representation of N_1 on $\mathscr{H}$ with respect to τ_1. Any normal state φ of N_1 is a vector state from $\mathscr{H}$ [$\pi_\varphi(N_1)$ has a cyclic and separating vector, thus is spatially isomorphic to $\pi(N_1)$]. If $\varphi = \varphi_a$ is the vector state corresponding to the vector $\eta(a)$ for some $a \in \mathfrak{N}_{\tau_1}$, then $\varphi_a(x) = \tau_1(a^*xa) \leqq \|a\|^2\tau_1(x)$ for $x \in N_1^+$, and so $0 \leqq 1/\|a\|^2 \varphi_a \leqq \tau_1$. Since $\{\eta(a): a \in \mathfrak{N}_{\tau_1}\}$ is dense in $\mathscr{H}$, $\{\varphi_a: a \in \mathfrak{N}_{\tau_1}\}$ is weak-* dense in $(N_1)_*^+$, and so linear combinations are dense in N_1^*; similarly for N_2^*. So it follows from the argument in [10, Prop. 3.8] that $J \subseteqq F(J_1, J_2)$.

If N_1 and N_2 are hyperfinite with separable preduals, then it may be proved that $K = J_1 \otimes J_2$. It would be interesting to know if this is true in general.

COROLLARY 3.6. *Let B and C be C^*-algebras with factor representations π and ρ respectively. If $\pi \otimes \rho$ is a traceable representation of $B \otimes C$, then π and ρ are traceable.*

Proof. Let $N_1 = \pi(B)''$, $N_2 = \rho(C)''$, J_1 and J_2 the minimal ideals of N_1 and N_2. If $\pi(B) \cap J_1 = \{0\}$ or $\rho(C) \cap J_2 = \{0\}$, then for any nonzero element x of $\pi(B) \otimes \rho(C) \subseteqq N_1 \otimes N_2$ there are linear functionals $\varphi \in N_1^*$, $\psi \in N_2^*$ with $\varphi | J_1 \equiv 0$ or $\psi | J_2 \equiv 0$, and $(\varphi \otimes \psi)(x) \neq 0$. But by Proposition 3.5, $(\varphi \otimes \psi) | K = 0$, so $x \notin K$. Thus $\pi(B) \otimes \pi(C) \cap J = \{0\}$.

The author is indebted to L. Brown for pointing out an error in an earlier proof of 3.6.

THEOREM 3.7. *Let B and C be C^*-algebras. If B is separable and nuclear, then (B, C) satisfies* (Ch).

Proof. Let π be a traceable factor representation of $B \otimes C$ on $\mathscr{H}$, and let $N = \pi(B \otimes C)''$, $N_1 = \pi(B \otimes 1)''$, $N_2 = \pi(1 \otimes C)''$. N, N_1, N_2 are factors, $N_2 \subseteqq N_1'$, and N_1 and N_2 together generate N. We may assume that N has a cyclic and separating vector. Since B is nuclear, N_1 is semidiscrete, so the map $\Phi: N_1 \odot N_1' \to \mathscr{L}(\mathscr{H})$ extends to an isometry of $N_1 \otimes N_1' \to \mathscr{L}(\mathscr{H})$, and by restriction to an isometry of $N_1 \otimes N_2 \to \mathscr{L}(\mathscr{H})$. So N, N_1, N_2 satisfy the hypotheses of Lemma 3.4, and thus we can write $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, $\pi = \pi_1 \otimes \pi_2 \cdot \pi_1$ and π_2 are traceable by Corollary 3.6.

THEOREM 3.8. *Let B and C be simple C^*-algebras. Then (B, C) satisfies* (Ch).

Proof. Let π be a traceable factor representation of $B \otimes C$; π is faithful since $B \otimes C$ is simple. Let $N = \pi(B \otimes C)''$, $J = \overline{\mathfrak{m}_\tau(N)}$. $\pi(B \otimes C) \cap J \neq \{0\}$, so $\pi(B \otimes C) \subseteqq J$. Let $b \in B^+$, $c \in C^+$; let f be a continuous function from $[0, \infty)$ to $[0, 1]$ with $f \equiv 0$ in a neighborhood of 0 and $f(b) \neq 0$, $f(c) \neq 0$. Then $\pi(f(b) \otimes f(c)) \in \mathfrak{m}_\tau(N)^+$, and

$\pi(f(b) \otimes f(c)) \neq 0$, so $0 < \tau(\pi(f(b) \otimes f(c))) < \infty$. Thus, Lemma 13 of [6] shows that $\pi \cong \pi_1 \otimes \pi_2$, and Corollary 3.6 shows that π_1 and π_2 are traceable.

4. Primitive ideals and irreducible representations. In this section, we characterize the primitive ideal space of an infinite tensor product of C^*-algebras in terms of the primitive ideals of the coordinate algebras.

Let B be a C^*-algebra, q a projection in B. Set $\mathrm{Pr}^q(B) = \{J \in \mathrm{Pr}(B): q \notin J\}$, $\mathrm{Prim}^q(B) = \{J \in \mathrm{Prim}(B): q \notin J\}$, $\hat{B}^q = \{\pi \in \hat{B}: \pi(q) \neq 0\}$.

PROPOSITION 4.1. *$\hat{B}^q$ is a compact open subset of $\hat{B}$; $\mathrm{Prim}^q(B)$ is compact and open in $\mathrm{Prim}(B)$.*

Proof. It suffices to prove the proposition for $\hat{B}^q$. $\hat{B}^q$ is open in $\hat{B}$ by definition of the topology of $\hat{B}$. $\hat{B}^q = \{\pi \in \hat{B}: \|\pi(q)\| = 1\}$, so $\hat{B}^q$ is compact by [5, Prop. 3.3.7.]

Let G be a locally compact group, and K a compact open subgroup of G. We may identify $\hat{G}$ with $C^*(G)^\wedge$. Let q be the characteristic function of K. Set $\hat{G}^K = \{\pi \in \hat{G}: \pi|K$ contains the trivial representation of $K\}$. $\hat{G}^K = \widehat{C^*(G)}^q$. If G is abelian, $\hat{G}^K = K^\perp$.

For the rest of this section, let A_i $(i \in I)$ be a collection of C^*-algebras with projections p_i, and $A = \bigotimes_{i \in I}(A_i, p_i)$. For each F, we write $A = A_F \otimes B_F$ as in § 2(c). We will assume that (A_F, B_F) satisfies (Pr) for each F; in particular, if each A_i is nuclear, this condition will be satisfied.

There is a map $r: \mathrm{Pr}(A) \to \Pi\, \mathrm{Pr}(A_i)$, defined as in § 3.

LEMMA 4.2. *Let $J \in \mathrm{Pr}(A)$, $r(J) = (J_i)$. Then $J_i \in \mathrm{Pr}^{p_i}(A_i)$ for almost all i.*

Proof. Let $F \subseteq I$ be finite, and suppose there is an $i \notin F$ with $p_i \in J_i$. Then if $E = F \cup \{i\}$, $\sigma_F(A_F) = A_F \otimes p_i \otimes q_E \subseteq A_F \otimes J_i \otimes B_E \subseteq J$. Since $\bigcup \sigma_F(A_F)$ is dense in A, there must be an F with $\sigma_F(A_F) \nsubseteq J$, so $p_i \notin J_i$ for all $i \notin F$.

The following lemma provides a general method for constructing representations of inductive limits of C^*-algebras.

LEMMA 4.3. *Let $\{B_\alpha, \sigma_{\alpha\beta}\}$ be a directed system of C^*-algebras, and $B = \varinjlim \{B_\alpha, \sigma_{\alpha\beta}\}$. Let α_0 be fixed, and let $\mathscr{K}$ be a Hilbert space with a set $\{\mathscr{K}_\alpha: \alpha > \alpha_0\}$ of closed subspaces, directed by inclusion, with $\bigcup \mathscr{K}_\alpha$ dense in $\mathscr{K}$. For $\alpha > \alpha_0$, let π_α be a repre-*

sentation of B_α *on* $\mathscr{K}_\alpha$, *with* $\pi_\alpha = (\pi_\beta \circ \sigma_{\alpha\beta})|\mathscr{K}_\alpha$ *for* $\alpha_0 < \alpha < \beta$. *Then there is a unique representation* π *of* B *on* $\mathscr{K}$ *with* $\pi_\alpha = (\pi \circ \sigma_\alpha)|\mathscr{K}_\alpha$ *for each* $\alpha > \alpha_0$, *where* σ_α *is the canonical embedding of* B_α *into* B. *If each* π_α *is nondegenerate,* π *is nondegenerate.*

Proof. Let $\alpha > \alpha_0$ be fixed, and let $b \in B_\alpha$; for $\beta > \alpha$, set $\pi(\sigma_\alpha(b)) = \pi_\beta(\sigma_{\alpha\beta}(b))$ on $\mathscr{K}_\beta$. If $\alpha < \beta < \gamma$, $\pi_\gamma(\sigma_{\alpha\gamma}(b)) = \pi_\beta(\sigma_{\alpha\beta}(b))$ on $\mathscr{K}_\beta$, so the definition of $\pi(\sigma_\alpha(b))$ is unambiguous. $\pi(\sigma_\alpha(b))$ is defined on $\bigcup \mathscr{K}_\beta$, and $\|\pi(\sigma_\alpha(b))\| \leqq \|b\|$, so $\pi(\sigma_\alpha(b))$ extends to an operator on $\mathscr{K}$. π defines a norm-decreasing homomorphism of $\bigcup \sigma_\alpha(B_\alpha)$ into $\mathscr{L}(\mathscr{K})$, hence extends to a homomorphism of B into $\mathscr{L}(\mathscr{K})$. The uniqueness of π is clear. If each π_α in nondegenerate, each $\mathscr{K}_\alpha$ is in the essential subspace of π, so π is nondegenerate.

LEMMA 4.4. *Let* $(J_i) \in \Pi \operatorname{Pr}(A_i)$, $J_i \in \operatorname{Pr}^{p_i}(A_i)$ *for almost all* i. *Then there is a* $J \in \operatorname{Pr}(A)$ *with* $r(J) = (J_i)$. *If* $J_i \in \operatorname{Prim}(A_i)$ *for all* i, *then* $J \in \operatorname{Prim}(A)$.

Proof. Let π_i be a factor representation of A_i on $\mathscr{H}_i$ with kernel J_i; choose π_i irreducible if $J_i \in \operatorname{Prim}(A_i)$. Let $E \subseteqq I$ be a finite set with $p_i \notin J_i$ for $i \notin E$. Let ξ_i be a unit vector in $\mathscr{H}_i$ with $\xi_i \in \operatorname{range} \pi_i(p_i)$ for $i \notin E$. Let $\mathscr{K} = \bigotimes(\mathscr{H}_i, \xi_i)$, and if $F \supseteqq E$ is finite, let $\mathscr{K}_F = \mathscr{H}_F \otimes (\bigotimes_{i \notin F} \xi_i)$. We may consider $\pi_F = \bigotimes_{i \in F} \pi_i$ as being defined on $\mathscr{K}_F$. If $E \subseteqq F \subseteqq D$, $\pi_F = (\pi_D \circ \sigma_{FD})|\mathscr{K}_F$, so we may form the representation π as in Lemma 4.3. $\pi(A)'' = \bigotimes(\pi_i(A_i)'', \mathscr{H}_i, \xi_i)$, so π is a factor representation, and if each π_i is irreducible, then π is irreducible. It is clear that $r(\ker \pi) = (J_i)$.

The following lemma is well known, but apparently does not appear in this general form in the literature. It is the most important tool in reducing questions about infinite tensor products to ones about finite products.

LEMMA 4.5. *Let* B *be a* C^*-*algebra,* $\{B_\alpha\}$ $(\alpha \in \Omega)$ *a set of* C^*-*subalgebras of* B *with* $\bigcup B_\alpha$ *dense in* B. *Let* J *be a closed* 2-*sided ideal of* B. *Then* J *is the closed ideal generated by* $\bigcup(J \cap B_\alpha)$. *If* $\bigcup B_\alpha$ *is an algebra* (in particular, if the B_α are nested), *then* $\bigcup(J \cap B_\alpha)$ *is dense in* J.

Proof. Let J_0 be the closed ideal generated by $\bigcup(J \cap B_\alpha)$ and let Φ be the quotient map of B onto $\bar{B} = B/J_0$. Set $\bar{B}_\alpha = \Phi(B_\alpha)$, $\bar{J} = \Phi(J)$. It suffices to show $\bar{J} = 0$. Let Ψ be the quotient map of $\bar{B}$ onto $\bar{B}/\bar{J}$. $\bar{J} \cap \bar{B}_\alpha = 0$, so $\Psi|\bar{B}_\alpha$ is injective, hence an isometry. So Ψ is an isometry on $\bar{B}$, hence injective.

COROLLARY 4.6. *If each B_α is simple, B is simple. So $\bigotimes(A_i, p_i)$ is simple if and only if each A_i is simple.*

COROLLARY 4.7. *Let $J \in \mathrm{Pr}(A)$, $r(J) = (J_i)$. Then J is generated by $\bigcup (J_i \otimes \bigotimes_{i \neq j}(A_j, p_j))$. So the map r is injective.*

COROLLARY 4.8. *If* $\mathrm{Pr}(A_i) = \mathrm{Prim}(A_i)$ *for each i, then* $\mathrm{Pr}(A) = \mathrm{Prim}(A)$.

Putting together 4.2, 4.4, 4.7 and 4.8, r gives a (set-theoretic) bijection between $\mathrm{Pr}(A)$ and $\Pi'(\mathrm{Pr}(A_i), \mathrm{Pr}^{p_i}(A_i))$.

Now assume that $\mathrm{Pr}(A_i) = \mathrm{Prim}(A_i)$ for each i (this assumption will be satisfied, for example, if each A_i is separable, simple, or type I). Then r gives a bijection between $\mathrm{Prim}(A)$ and $X = \Pi'(\mathrm{Prim}(A_i), \mathrm{Prim}^{p_i}(A_i))$. Give X the restricted direct product topology ($\mathrm{Prim}(A_i)$ is locally compact, and $\mathrm{Prim}^{p_i}(A_i)$ is compact and open by Prop. 4.1).

THEOREM 4.9. *$r\colon \mathrm{Prim}(A) \to X$ is a homeomorphism.*

Proof. $A \cong A_i \otimes B_i$ and $\mathrm{Prim}(A) = \mathrm{Prim}(A_i) \times \mathrm{Prim}(B_i)$ with the product topology, so the composite of r with each coordinate projection is continuous. Let $F \subseteq I$ be a finite set. If $J \in \mathrm{Prim}(A)$ with $r(J) = (J_i)$, then $(J_i) \notin Z_F = \prod_{i \in F} \mathrm{Prim}(A_i) \times \prod_{i \notin F} \mathrm{Prim}^{p_i}(A_i)$ if and only if $\sigma_F(A_F) \subseteq J$ (see Lemma 4.2).

So $r^{-1}(X \sim Z_F) = \{J \in \mathrm{Prim}(A)\colon \sigma_F(A_F) \subseteq J\}$, a closed set in $\mathrm{Prim}(A)$, so $r^{-1}(Z_F)$ is open. Therefore, r is continuous, since X has the weakest topology making each Z_F open and all the coordinate projections continuous. It remains to show that if $\{J^\alpha\} = r^{-1}(\{(J_i^\alpha)\})$ is a subset of $r^{-1}(Z_F)$, and $J = r^{-1}(J_i)$ with J_i in the closure of $\{J_i^\alpha\}$ for each i, then J is in the closure of $\{J^\alpha\}$. Set $J_0 = \bigcap_\alpha J^\alpha$; then J is in the closure of $\{J^\alpha\}$ if and only if $J_0 \subseteq J$. If $E \supseteq F$, then $J^\alpha \cap \sigma_E(A_E) = \sigma_E(J_E^\alpha)$, where J_E^α is the ideal of A_E corresponding to $\{J_i^\alpha\colon i \in E\}$. J_E is in the closure of $\{J_E^\alpha\}$ for each E, so $J_0 \cap \sigma_E(A_E) = \bigcap_\alpha \sigma_E(J_E^\alpha) \subseteq \sigma_E(J_E) = J \cap \sigma_E(A_E)$. So by Lemma 4.5, $J_0 \subseteq J$.

COROLLARY 4.10. *If each A_i is type* I, *then* $\mathrm{Prim}(A) \cong \prod'(\hat{A}_i, \hat{A}_i^{p_i})$.

COROLLARY 4.11. (a) *If $G = \prod'(G_i, K_i)$ with each G_i type* I, *then* $\mathrm{Prim}(G) \cong \prod'(\hat{G}_i, \hat{G}_i^{K_i})$.

(b) *If $G = \prod'(G_i, K_i)$ is abelian, then $\hat{G} = \prod'(\hat{G}_i, K_i^\perp)$.*

4.11(b) is undoubtedly well known, and is readily proved directly. It appears as Corollary 12 of [6]. It should also be noted that Example 2 of Section 2(c) is a special case of Theorem 4.9.

5. Infinite tensor products of type I algebras. This section describes some of the finer structure of an infinite tensor product of type I algebras. Theorems 5.1 and 5.3 are previously known results included for completeness.

If B is a C^*-algebra with projection q, and if n is a cardinal set $(\hat{B}^q)_n = \{\pi \in \hat{B}: \dim \pi(q) = n\}$.

THEOREM 5.1. *$A = \bigotimes(A_i, p_i)$ is type* I *if and only if each A_i is type* I *and $\hat{A}_i^{p_i} = (\hat{A}_i^{p_i})_1$* a.e.

Proof. See [6, Theorems 7 and 8].

A sharper form of one direction of this theorem is given next, while a strong form of the converse will be proved in Theorem 7.4.

THEOREM 5.2. *Suppose each A_i is type* I. *Let $(\pi_i) \in \prod' (\hat{A}_i, \hat{A}_i^{p_i})$ with $\pi_i \in (\hat{A}_i^{p_i})_1$* a.e. *Construct $\pi \in \hat{A}$ as in Lemma 4.4. Then if ρ is any factor representation of A with* $\ker \rho = \ker \pi$, *then ρ is a multiple of π.*

Proof. The proof is similar to the proof of Theorem 6 of [9].

THEOREM 5.3. *If, for each i, the elements of $\hat{A}_i^{p_i} \sim (\hat{A}_i^{p_i})_1$ separate the points of A_i, then $\bigotimes(A_i, p_i)$ is* NGCR.

Proof. See [6, Theorem 7.]

EXAMPLE. Let $\mathscr{H}$ be a separable Hilbert space, $B = \mathscr{L}\mathscr{C}(\mathscr{H}) + C1$, q a rank 1 projection in $\mathscr{L}\mathscr{C}(\mathscr{H})$. Let I be a countable index set, and let $A_i = B$, $p_i = q$ for each i, $A = \bigotimes(A_i, p_i)$. A is separable, and A is type I by Theorem 5.1. $\hat{A}$ contains no closed points, i.e., A has no maximal closed ideals. (Of course, other examples of such C^*-algebras are known: see [5, 4.7.17].)

PROPOSITION 5.4. *$A = \bigotimes(A_i, p_i)$ is FD* (all $\pi \in \hat{A}$ finite-dimensional) *if and only if each A_i is FD and, for almost all i, every element of $\hat{A}_i^{p_i}$ is one-dimensional. A is BD* ($\dim \pi$ bounded for all $\pi \in \hat{A}$) *if and only if each A_i is BD and all but finitely many A_i are commutative.*

Now we examine when A is CCR, GTC, or has continuous trace. If B is a C^*-algebra and $\pi \in \hat{B}$, π is CCR if $\pi(b)$ is compact for every $b \in B$, as in [5, 4.7.12.]. Use the notation $J(B)$ to denote the ideal which is the closure of the ideal $\mathfrak{M}(B)$ of elements of continuous trace; denote by $K(B)$ the union of the transfinite sequence J_α where $J_{\alpha+1}/J_\alpha = J(B/J_\alpha)$; $K(B)$ is the smallest ideal of B such that $J(B/K) = 0$. B has continuous trace if $J(B) = B$; B is GTC if $K(B) = B$.

We assume from now on that each A_i is type I, and $A = \bigotimes(A_i, p_i)$.

THEOREM 5.5. *Let $\pi \in \hat{A}$, $r(\ker \pi) = (J_i)$, $\pi_i \in \hat{A}_i$ with $\ker \pi_i = J_i$. Then π is CCR if and only if each π_i is CCR and $\pi_i \in (\hat{A}_i^{p_i})_1$* a.e. *So A is CCR if and only if each A_i is CCR and $\hat{A}_i^{p_i} = (\hat{A}_i^{p_i})_1$* a.e.

Proof. See [9, Theorem 8]. The last assertion also follows from 4.10 and 5.1, since A is CCR if and only if A is type I and $\mathrm{Prim}(A)$ is a T_1 space.

Now assume that $\hat{A}_i^{p_i} = (\hat{A}_i^{p_i})_1$.

THEOREM 5.6. *If $p_i \in J(A_i)$* a.e., *then $J(A) = [\bigcup_F \sigma_F(J(A_F))]^-$, $K(A) = [\bigcup_F \sigma_F(K(A_F))]^-$; if $p_i \notin J(A_i)$ for infinitely many i, then $J(A) = K(A) = \{0\}$.*

Proof. Let E be a finite set such that $\hat{A}_i^{p_i} = (\hat{A}_i^{p_i})_1$ and $p_i \in J(A_i)$ for all $i \notin E$. $\mathfrak{M}(A_i)$ is a dense hereditary ideal of $J(A_i)$, and so contains all projections of $J(A_i)$. So $p_i \in \mathfrak{M}(A_i)$ for all $i \notin E$. If $F \supseteqq E$, it can then be readily verified that $q_F \in \mathfrak{M}(B_F)$. It follows from [11, Lemma 4] that $a \otimes q_F \in J(A)$ if and only if $a \in J(A_F)$; so $J(A) \cap \sigma_F(A_F) = J(A_F)$. By Lemma 4.5, $J(A) = [\bigcup_F (J(A) \cap \sigma_F(A_F))]^- = [\bigcup_F \sigma_F(J(A_F))]^-$. Similarly, it may be verified that $K(A) \cap \sigma_F(A_F) = \sigma_F(K(A_F))$, so $K(A) = [\bigcup_F \sigma_F(K(A_F))]^-$. Conversely, if $J(A) \neq 0$, by Lemma 4.5 there is a finite set F such that $J(A) \cap \sigma_F(A_F) \neq 0$. If $a \otimes q_F \in J(A)^+$, $a \neq 0$, let f be a continuous function from $[0, \infty)$ to $[0, 1]$, vanishing in a neighborhood of 0, with $f(a) \neq 0$; $f(a \otimes q_F) = f(a) \otimes q_F \in \mathfrak{M}(A)^+$, so $f(a) \in \mathfrak{M}(A_F)$ and $q_F \in \mathfrak{M}(B_F)$. But this implies $p_i \in \mathfrak{M}(A_i)$ for every $i \notin F$.

COROLLARY 5.7. *$A = \bigotimes(A_i, p_i)$ has continuous trace if and only if each A_i has continuous trace and $\hat{A}_i^{p_i} = (\hat{A}_i^{p_i})_1$ a.e.*

It is interesting and instructive to examine the situation where A has continuous trace. More generally, if each A_i is a C^*-algebra defined by a continuous locally trivial field of C^*-algebras over a locally compact space X_i, then p_i will correspond to a compact open subspace Y_i of X_i, and $A = \bigotimes(A_i, p_i)$ will be the C^*-algebra defined by a continuous field of C^*-algebras over $\prod'(X_i, Y_i)$, where the C^*-algebra at the point $(\cdots x_i \cdots) \in \prod'(X_i, Y)$ is $\bigotimes(A_i(x_i), p_i(x_i))$. (See [10, Lemma 3.3].)

COROLLARY 5.8. *$A = \bigotimes(A_i, p_i)$ is GTC if and only if* (1) *each A_i is GTC* (2) *$\hat{A}_i^{p_i} = (\hat{A}_i^{p_i})_1$ a.e.* (3) *$p_i \in J(A_i)$ a.e.*

EXAMPLES. (a) Let B be the C^*-algebra of sequences of 2×2 matrices converging to a diagonal matrix, as in [5, 4.7.19], and let q be a projection in B with $\hat{B}^q = (\hat{B}^q)_1$; for example,

$$q = \left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \cdots\right) \quad \text{or} \quad q = \left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \cdots\right).$$

Let I be a countable index set, and let $A = \bigotimes_{i \in I}(A_i, p_i)$ where $A_i = B$, $p_i = q$ for each i. A is CCR, but there cannot be a finite sequence $\{\mathscr{U}_n\}$ of increasing open sets in $\hat{A}$ with $\hat{A} = \cup \mathscr{U}_n$ and $\mathscr{U}_n \sim \mathscr{U}_{n-1}$ Hausdorff. For if B_k denotes the tensor product of k copies of B, $\hat{A}$ contains a copy of $\hat{B}_k$ for each k, and the restriction of $\{\mathscr{U}_n\}$ gives such a sequence for $\hat{B}_k$. But if $\mathscr{U}$ is a Hausdorff open subset of $\hat{B}_k$, $\hat{B}_k \sim \mathscr{U}$ contains a copy of $\hat{B}_{k-1}$, so such a sequence for $\hat{B}_k$ must contain at least $k+1$ elements by induction. In particular, there cannot be a finite composition series $\{J_n\}$ for A with $(J_{n+1}/J_n)^\wedge$ Hausdorff. This provides a solution to Problem 4.7.25 of [5]. A is GTC if and only if $q \in J(B)$, which is the set of sequences converging to 0. Thus, if $q = \left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \cdots\right)$, $J(A) = K(A) = 0$, and $\hat{A}$ contains a dense set of points which are not separated (see [5, 3.9.4 and 4.7.9]).

(b) Let B be the C^*-algebra of sequences of 2×2 matrices converging to a scalar multiple of the identity, and let q be a projection of B with $\hat{B}^q = (\hat{B}^q)_1$; for example, $q = \left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \cdots\right)$, and let A be the tensor product of a countable number of copies of B with respect to q. $\hat{A}$ is Hausdorff, so A is GTC (this also follows from Corollary 5.7), but the GTC composition series for A does not have finite length. In fact, A does not have a finite composition series $\{J_n\}$ such that J_{n+1}/J_n has continuous trace.

(c) In (a) and (b) above, let each $A_i = B \oplus C$, $p_i = (0, 1)$; then $A = \bigotimes(A_i, p_i)$ has the same properties as before, and has only finite-dimensional irreducible representations.

Remark. All of the above C^*-algebras are AF algebras.

Finally, we examine the question of when a primitive ideal of an infinite tensor product is locally closed (i.e. open in its closure). If B is a C^*-algebra and $K \in \operatorname{Prim}(B)$, set $K^0 = \bigcap \{J \in \operatorname{Prim}(B): J \supsetneqq K\}$. (set $K^0 = B$ if K is closed in $\operatorname{Prim}(B)$). Then K is locally closed if and only if $K^0 \neq K$; in this case K^0/K is a simple C^*-algebra, said to be the simple C^*-algebra lying above K. If K is the kernel of a traceable irreducible representation π on a Hilbert space $\mathscr{H}$, $K^0 = \pi^{-1}(\mathscr{L}\mathscr{C}(\mathscr{H})) \neq K$, so K is locally closed, and K^0/K is an elementary C^*-algebra. In particular, if B is type I, every element of $\operatorname{Prim}(B)$ is locally closed.

PROPOSITION 5.9. *The point $(\cdots x_i \cdots) \in \Pi'(X_i, Y_i)$ is locally closed if and only if x_i is locally closed in X_i for each i and x_i is closed in Y_i for almost all i. Hence, if $A = \bigotimes (A_i, p_i)$ such that (A_F, B_F) satisfies* (Pr) *for each F, and $J \in \operatorname{Prim}(A)$, $r(J) = (J_i)$, then J is locally closed in* $\operatorname{Prim}(A)$ *if and only if J_i is locally closed in* $\operatorname{Prim}(A_i)$ *for all i and $p_i \in J_i^0 \sim J_i$ for almost all i.*

Proof. The proof of the first assertion is straightforward, and is omitted. The second assertion follows from the fact that the closure of $\{J_i\}$ in $\operatorname{Prim}^{p_i}(A_i) = \{K \in \operatorname{Prim}(A): K \supseteq J, p_i \notin K\}$; for almost all i, $p_i \notin J_i$; and if $p_i \notin J_i$, $\{J_i\}$ is closed in $\operatorname{Prim}^{p_i}(A_i)$ if and only if $p_i \in J_i^0$.

COROLLARY 5.10. *If $A = \bigotimes (A_i, p_i)$, each A_i type I, $J \in \operatorname{Prim}(A)$, $r(J) = (J_i)$, and $\pi_i \in \hat{A}_i$ with* $\ker \pi_i = J_i$, *then J is locally closed in* $\operatorname{Prim}(A)$ *if and only if $0 < \dim \pi_i(p_i) < \infty$ for almost all i.*

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6. Infinite product weights and characters. In this section, we define an infinite product weight construction on an infinite tensor product of C^*-algebras which is a generalization of the construction of an infinite product state, which is then used to determine all the characters on such a C^*-algebra. This construction was done independently for traces by Guichardet [6].

Let ϕ be a lower semicontinuous (lsc) weight on a C^*-algebra B. Write $\mathfrak{N}_\phi = \{x \in B: \phi(x^*x) < \infty\}$, $\mathfrak{M}_\phi = \mathfrak{N}_\phi^*\mathfrak{N}_\phi$, $N_\phi = \{x \in B: \phi(x^*x) = 0\}$, π_ϕ the representation from the GNS construction on $\mathscr{H}_\phi = \overline{\mathfrak{N}_\phi/N_\phi}$, η_ϕ the canonical map of $\mathfrak{N}_\phi$ into $\mathscr{H}_\phi$. ϕ extends a weakly lower semicontinuous weight $\bar{\phi}$ on the universal enveloping von Neumann algebra $\bar{B}$ by setting $\bar{\phi} = \sup\{\bar{f}: f \in B_+^*, f \leqq \phi\}$, where $\bar{f}$ is the canonical extension of f to $\bar{B}$. (See [3] for details.)

DEFINITION. ϕ is said to be weakly semifinite if $\bar{\phi}$ is semifinite.

For example, if ϕ is semifinite, or if $\mathfrak{N}_\phi$ is dense in B, then ϕ is weakly semifinite. If ϕ is weakly semifinite, π_ϕ, $\mathscr{H}_\phi$ can be identified with $\pi_{\bar{\phi}}$, $\mathscr{H}_{\bar{\phi}}$, and $\pi_{\bar{\phi}}(\bar{B})'' = \mathscr{L}(\mathscr{B})$, where $\mathscr{B}$ is the left Hilbert algebra $(\mathfrak{N}_{\bar{\phi}} \cap \mathfrak{N}_{\bar{\phi}}^*)/N_{\bar{\phi}}$.

If ϕ and ψ are lsc weights on B and C respectively, we can define a lsc weight $\phi \otimes \psi$ on $B \otimes C$ as follows. Let $F = \{f \in B_+^*: f \leqq \phi \text{ on } B^+\}$, $G = \{g \in C_+^*: g \leqq \psi \text{ on } C^+\}$, and for $x \in (B \otimes C)^+$, set $(\phi \otimes \psi)(x) = \sup_{f \in F, g \in G} (f \otimes g)(x)$. If ϕ and ψ are weakly semifinite, $\phi \otimes \psi$ can be described alternatively as the weight defined by the full left Hilbert algebra corresponding to $\mathscr{B} \odot \mathscr{C}$, where $\mathscr{B} = (\mathfrak{N}_{\bar{\phi}} \cap \mathfrak{N}_{\bar{\phi}}^*)/N_{\bar{\phi}}$, $\mathscr{C} = (\mathfrak{N}_{\bar{\psi}} \cap \mathfrak{N}_{\bar{\psi}}^*)/N_{\bar{\psi}}$, so that $\pi_{\phi\otimes\psi} \cong \pi_\phi \otimes \pi_\psi$.

Now let $A = \bigotimes (A_i, p_i)$, and let ϕ_i be a lsc weakly semifinite weight on A_i. Suppose that, for almost all i, $\phi_i(p_i) = 1$; let $E = \{i: \phi_i(p_i) \neq 1\}$. Let $\mathscr{A}_i$ be the left Hilbert algebra $(\mathfrak{N}_{\bar{\phi}_i} \cap \mathfrak{N}_{\bar{\phi}_i}^*)/N_{\phi_i}$; let $\eta_i = \eta_{\bar{\phi}_i}(p_i)$ for $i \notin E$; let $\mathscr{A} = \odot (\mathscr{A}_i, \eta_i)$ be the set of linear combinations of elementary tensors of the form $\bigotimes \xi_i$, where $\xi_i = \eta_i$ a.e. Then $\mathscr{A}$ has a natural structure as a left Hilbert algebra. Let ϕ be the weight on A defined by the corresponding full left Hilbert algebra.

DEFINITION. ϕ is called the infinite product weight of the ϕ_i, denoted $\bigotimes_{i \in I} \phi_i$.

ϕ is lsc and weakly semifinite; ϕ is a semifinite trace if and only if each ϕ_i is a semifinite trace. If each ϕ_i is a positive linear functional, ϕ is not necessarily a positive linear functional:

PROPOSITION 6.1. *ϕ is a positive linear functional if and only if each ϕ_i is bounded and $\Pi \|\phi_i\| < \infty$. In this case, $\|\phi\| = \Pi \|\phi_i\|$.*

If $\{\phi_i\}$, ϕ, and $\mathscr{A}$ are as above, then the completion of $\mathscr{A}$ is $\mathscr{H}_\phi = \bigotimes (\mathscr{H}_{\phi_i}, \eta_i)$ and the representation π_ϕ is the representation $\bigotimes \pi_{\phi_i}$ on $\mathscr{H}_\phi$ defined as in Lemma 4.4; $\pi_\phi(A)'' = \overline{\bigotimes} (\pi_{\phi_i}(A_i)'', \mathscr{H}_{\phi_i}, \eta_i)$. Thus ϕ is factorial if and only if each ϕ_i is factorial. In particular, ϕ is a character if and only if each ϕ_i is a character, so we have a way of constructing characters on A.

Now we will show that, if (A_F, B_F) satisfy (Ch) for each F (in particular, if each A_i is type I, or if each is separable and nuclear, or if each is simple), then every character on A is an infinite tensor product. Let ψ be a character on A; for each F, write $A = A_F \otimes B_F$, $\psi = \phi_F \otimes \psi_F$ for characters ϕ_F on A_F, ψ_F on B_F. $\phi_F = \bigotimes_{i \in F} \phi_i$, where ϕ_i is a character on A_i.

LEMMA 6.2. *For almost all i, $0 < \phi_i(p_i) < \infty$.*

Proof. Let $J_\psi = \overline{\mathfrak{N}_\psi}$; J_ψ is a closed 2-sided ideal of A, and $J_\psi \supsetneqq N_\psi$. By Lemma 4.5, there is a finite set $F \subseteq I$ for which $J_\psi \cap \sigma_F(A_F) \supsetneqq N_\psi \cap \sigma_F(A_F)$. Let $a \in A_F^+$ have $\sigma_F(a) \in J_\psi$, $\psi(\sigma_F(a)) > 0$. $a \otimes q_F = \sigma_F(a) \in J_\psi = J_{\phi_F} \otimes J_{\psi_F}$, so $a \in J_{\phi_F}$, $q_F \in J_{\psi_F}$ (see [6, Lemma 7]). For $i \notin F$, let $E = F \cup \{i\}$; $B_F \cong A_i \otimes B_E$, $\psi_F = \phi_i \otimes \psi_E$, $J_{\psi_F} = J_{\phi_i} \otimes J_{\psi_E}$. $q_F = p_i \otimes q_E \in J_{\phi_i} \otimes J_{\psi_E}$, so $p_i \in J_{\phi_i}$. But $\mathfrak{N}_{\phi_i}$ contains all projections of J_{ϕ_i}, so $\phi_i(p_i) < \infty$. Also, $\psi(\sigma_F(a)) > 0$, so $\psi_F(q_F) > 0$, and so $\phi_i(p_i) > 0$.

Let $F \neq \emptyset$ be a finite set with $0 < \psi_F(q_F) < \infty$ as above; then $0 < \phi_i(p_i) < \infty$ for $i \notin F$. For $i \notin F$, renormalize ϕ_i so that $\phi_i(p_i) = 1$, and form $\phi = \bigotimes \phi_i$. Let $a \in A_F^+$ with $0 < \phi_F(a) < \infty$; then $\psi(\sigma_F(a)) = \phi_F(a)\psi_F(q_F)$, so $0 < \psi(\sigma_F(a)) < \infty$. Also, $0 < \phi(\sigma_F(a)) = \phi_F(a) < \infty$. Renormalize ϕ so that $\phi(\sigma_F(a)) = \psi(\sigma_F(a))$ by renormalizing one of the ϕ_i, $i \in F$. Then ϕ is still $\bigotimes \phi_i$, and if $E \supseteq F$, ϕ and ψ agree on $\sigma_E(A_E)$.

LEMMA 6.3. $\phi = \psi$.

Proof. ϕ and ψ agree on $\mathfrak{N}_\phi \cap (\bigcup \sigma_E(A_E))$, which is dense in $\mathfrak{N}_\phi$ in its pre-Hilbert space structure, so the lemma follows from the argument in the proof of [5, Lemma 6.5.3].

We summarize the previous considerations in a theorem.

THEOREM 6.4. *Let $A = \bigotimes (A_i, p_i)$; suppose, for each $F \subseteq I$ finite, (A_F, B_F) satisfies* (Ch). *Then every character of A is of the form $\bigotimes \phi_i$, where ϕ_i is a character on A_i with $\phi_i(p_i) = 1$* a.e.

If each A_i is type I, the situation is very nice, since there is a one-one correspondence between primitive ideals and characters. The result can be stated as follows:

THEOREM 6.5. *Let $A = \bigotimes(A_i, p_i)$ such that (A_F, B_F) satisfies* (Pr) *and* (Ch) *for each F; let $J \in$* Prim (A), *$r(J) = (J_i)$. Suppose, for each i, J_i is the kernel of a traceable irreducible representation π_i of A_i* (in particular, if A_i is type I). *Set $n_i = \dim \pi_i$, $r_i = \dim \pi_i(p_i)$. There is a finite set E with $r_i > 0$ for $i \notin E$. Then*

(a) *J is the kernel of at most one traceable factor representation.*

(b) *J is the kernel of a traceable factor representation if and only if J is locally closed in* Prim (A), i.e., *if and only if $r_i < \infty$ for almost all i. The corresponding character is $\bigotimes \phi_i$, where ϕ_i is the character of A_i corresponding to π_i, normalized so that $\phi_i(p_i) = 1$* a.e.

(c) *J is the kernel of a traceable factor representation with finite trace if and only if $n_i < \infty$ for all i and $\prod_{i \notin E} n_i/r_i < \infty$.*

(d) *J is the kernel of a traceble irreducible representation if* *and only if* $r_i = 1$ *for almost all* i.

Proof. (a) and (b) have been proved above and in 5.10. (c) follows from 6.1, since if ψ is the trace on a I_n factor normalized so that an r-dimensional projection has trace 1, then $\|\psi\| = n/r$. Proof of (d): ($\Leftarrow$) follows from (b) and 5.2; ($\Rightarrow$) follows from 6.7.5 and 4.1.10 of [5], since if $r_i > 1$ for infinitely many i, then two nonequivalent irreducible representations with kernel J can be constructed by the method of Lemma 4.4 (see [6], or [9, p. 170]).

Theorem 6.5(c) clarifies and generalizes Proposition 12 of [6].

It is worth noting that if J is locally closed in $\mathrm{Prim}(A)$ then $J^0/J = [\bigotimes (J_i^0, p_i) + J]/J \cong \bigotimes(J_i^0/J_i, \bar{p}_i)$. If J_i is the kernel of a traceable irreducible representation, then J_i^0/J_i is elementary. So under the hypotheses of Theorem 6.5(b), J^0/J is a matroid C^*-algebra. The existence and uniqueness of trace on a matroid C^*-algebra was proved by Dixmier. Thus, 6.5(a) and (b) can be restated as follows: J is the kernel of a (necessarily unique) traceable factor representation of A if and only if there is an ideal K of A containing J with K/J a matroid C^*-algebra. This observation is due to Philip Green.

Without the hypothesis that J_i be the kernel of a traceable irreducible representation, all four conclusions can fail. (a) can fail if one of the A_i's has two nonequivalent traceable factor representations with the same kernel; (d) will fail if one of the J_i's is not the kernel of a traceable irreducible representation. (b) can fail as follows. If J is not locally closed it can still be the kernel of a traceable factor representation: let B be a separable C^*-algebra with identity which is not simple, but which has a faithful II_1 factor representation (e.g., $B = C^*(G)$, where G is a countable discrete amenable group with infinite conjugacy classes), and let $A = \bigotimes(A_i, 1_i)$ with $A_i \cong B$. A has a faithful II_1 factor representation, but 0 is not locally closed in $\mathrm{Prim}(A)$. Conversely, a separable simple C^*-algebra need not have any characters, so J could be locally closed but not the kernel of a traceable factor representation. (c) can be rephrased as follows.

PROPOSITION 6.6. *Let* $A = \bigotimes (A_i, p_i)$ *such that* (A_F, B_F) *satisfies* (Pr) *and* (Ch) *for each* F. *Let* $J \in \mathrm{Prim}(A)$, $r(J) = (J_i)$. *Let* E *be a finite set for which* $p_i \notin J_i$ *for* $i \notin E$. *Then* J *is the kernel of a traceable factor representation of* A *with finite trace if and only if there is a traceable factor representation* π_i *of* A_i *with finite normalized trace* τ_i *for each* i, *with* $\ker \pi_i = J_i$, *such that* $\prod_{i \notin E} 1/\tau_i(p_i) < \infty$.

Theorem 6.5 shows that an infinite tensor product of type I C^*-algebras has only a limited number of characters. On the other hand, we will now show that such a C^*-algebra has enough characters to separate points. We first need a lemma which is almost certainly known.

LEMMA 6.7. *Let B be a type I C^*-algebra, q a projection in B. Then there is a $\pi \in \hat{B}$ with $0 < \dim \pi(q) < \infty$. Hence there is a* (suitably normalized) *character ϕ on B with $\phi(q) = 1$.*

Proof. Let $\{J_\alpha\}$ $(1 \leqq \alpha \leqq \sigma)$ be a composition series for B, with $J_{\alpha+1}/J_\alpha$ CCR. Let β be the first ordinal for which $q \in J_\beta$. If γ is a limit ordinal for which $q \in J_\gamma$, $\bigcup_{\alpha<\gamma} J_\alpha$ is dense in J_γ, so there is an $\alpha < \gamma$ and $r \in J_\alpha$ with $\|q - r\| < 1$. Let Φ be the quotient map of J_γ onto J_γ/J_α; $\|\Phi(q) - \Phi(r)\| = \|\Phi(q)\| < 1$. But $\Phi(q)$ is a projection, so $\Phi(q) = 0$, i.e., $q \in J_\alpha$. So β is not a limit ordinal. Let Ψ be the quotient map of J_β onto $J_\beta/J_{\beta-1}$; $\Psi(q) \neq 0$, so there is a $\pi_0 \in (J_\beta/J_{\beta-1})^\wedge$ with $\pi_0(\Psi(q)) \neq 0$. But $J_\beta/J_{\beta-1}$ is CCR, so $0 < \dim \pi_0(\Psi(q)) < \infty$. Let $\pi_1 = \pi_0 \circ \Psi$, π the extension of π_1 from J_β to B.

THEOREM 6.8. *Let $A = \bigotimes (A_i, p_i)$, each A_i type* I. *Then the characters of A separate the points of A,* i.e., *if $a \in A^+$, there is a character ϕ of A with $\phi(a) > 0$.*

Proof. Let ψ_i be a character on A_i with $\psi_i(p_i) = 1$. Let $a \in A_F^+$, and let ϕ_F be a character on A_F with $\phi_F(a) > 0$ (A_F is type I). Let $\phi = \phi_F \otimes (\bigotimes_{i \notin F} \psi_i)$; $\phi(\sigma_F(a)) > 0$. Let $J = \bigcap \{N_\phi : \phi \text{ a character on } A\}$. J is a closed 2-sided ideal of A, and $J \cap \sigma_F(A_F) = \{0\}$ for each F, so $J = 0$ by Lemma 4.5.

A particular application of 6.5 and 6.8 is to Adele groups.

THEOREM 6.9. *Let G be an algebraic group defined over $\boldsymbol{Q}$, G_p the points rational over $\boldsymbol{Q}_p$, G_A the adele group of G.* (It is known that each G_p is type I.)

(a) *An element of* Prim (G_A) *is the kernel of a traceable factor representation if and only if it is locally closed, and the representation, if it exists, is unique. The traceable factor representations of G_A separate the points of $C^*(G_A)$.*

(b) *If each G_p is* CCR (for example, if G is nilpotent), *then every ideal of* Prim (G_A) *is the kernel of a unique traceable factor representation.*

EXAMPLES. (a) Let $\mathscr{H}$ be a separable Hilbert space, $B = \mathscr{L}\mathscr{C}(\mathscr{H}) + \boldsymbol{C}1$, I a countable index set, $A_i = B$, $K_i = \mathscr{L}\mathscr{C}(\mathscr{H}) \subseteq$

A_i, $A = \bigotimes(A_i, 1_i)$. $\hat{A}_i = \{\lambda_i, \omega_i\}$, where ω_i is faithful and λ_i is one-dimensional; $\ker \lambda_i = K_i$. If $J \in \mathrm{Prim}(A)$, $r(J) = J_i$, $\pi_i \in \hat{A}_i$ with $\ker \pi_i = J_i$, then J is the kernel of a traceable factor representation if and only if $J_i = K_i$ a.e., i.e., if and only if π_i is one-dimensional for almost all i. Thus, $0 \in \mathrm{Prim}(A)$, but 0 is not the kernel of a traceable factor representation. From 6.5(d), every traceable factor representation (hence every traceable representation) is type I, and A has enough traceable irreducible representations to separate the points of A. But A is not type I; in fact, A is NGCR by Theorem 5.3.

(b) A slightly more complex version of the same phenomenon occurs in the context of group representations. Let $\{p_i\}$ be an arbitrary sequence of prime numbers; let G_i be the p_i-adic $ax + b$ group, K_i the compact open subgroup of integral points, and $G = G(\{p_i\}) = \Pi'(G_i, K_i)$. (See [1].) $\hat{G}_i$ consists of a family of one-dimensional representations and one faithful infinite-dimensional representation ω_i, and there is an infinite-dimensional space of vectors invariant under $\omega_i(K_i)$. As above, every traceable factor representation of G is type I, and there are enough to separate the points of $C^*(G)$, although $C^*(G)$ is NGCR.

7. Construction of factor representations.

In this section, we show how to construct representations of $A = \bigotimes (A_i, p_i)$ generating a given hyperfinite factor. We do not make any restrictions on the C^*-algebras A_i.

DEFINITION. Let M be a factor. A generating system for M is a family of mutually commuting type I subfactors of M which together generate M. M is said to be hyperfinite if it has a generating system.

Since any type I factor has a generating system consisting of finite-dimensional factors, every generating system for a factor M has a "refinement" consisting of finite-dimensional subfactors. Thus, if M has a separable predual, the above definition is easily seen to be equivalent to the usual definition of a hyperfinite factor.

For a factor M, we will use the notation $w(M)$ to denote the smallest cardinal of a σ-weakly dense subset of M (=topological weight of M_*). For notational convenience, we will alway assume that hyperfinite factors are infinite-dimensional, and that a generating system $\{M_i\}$ $(i \in I)$ for a factor M satisfies card $I = w(M)$.

DEFINITION. A von Neumann algebra M is called maximally infinite if $M \cong M \overline{\otimes} \mathscr{L}(\mathscr{H})$, where $\dim \mathscr{H} = w(M)$. If M has separable predual, "maximally infinite" is the same as "properly infinite".

DEFINITION. Let $\{P_i\}$ $(i \in I)$ be a set of projections in a von Neumann algebra M. Set $\liminf \{P_i\} = \sup_F (\inf_{i \notin F} P_i)$, where F runs over finite subsets of I. $\{P_i\}$ is said to be fundamental if $\liminf P_i = 1$.

LEMMA 7.1 *Let M be a hyperfinite factor with generating system $\{M_i\}$, and let $P_i \in M_i$ be a projection with* $\inf P_i \neq 0$. *Then $\{P_i\}$ is fundamental.*

Proof. For any finite F, $(\inf_{i \notin F} \{P_i\}) \in \{M_i: i \in F\}'$, so $\liminf \{P_i\}$ is in $\{M_i: i \in I\}' = M'$; also $\liminf \{P_i\} \geqq \inf \{P_i\} \neq 0$.

LEMMA 7.2. *Let M be a maximally infinite hyperfinite factor, I an index set with* $\operatorname{card} I = w(M)$. *For each i, let $A_i \cong \mathscr{L}\mathscr{C}(\mathscr{H}_i)$,* $\dim \mathscr{H}_i \leqq w(M)$. *Let p_i be a 2-dimensional projection in A_i, and let $A = \bigotimes (A_i, p_i)$. Then there is a representation π of A with $\pi(A)'' \cong M$.*

Proof. The proof is virtually identical to the proof of [8, p. 850], which is essentially Glimm's proof that a non-GCR C^*-algebra has a non-type-I factor representation.

LEMMA 7.3. *Let M be a maximally infinite hyperfinite factor; let $\{n_i\}$ $(i \in I)$ be cardinals with $n_i \leqq w(M)$,* $\operatorname{card} I = w(M)$. *Then M has a generating system $\{M_i\}$ with M_i a I_{n_i} factor, and a fundamental set of projections $\{P_i\}$ with $P_i \in M_i$,* $\dim P_i = 2$ (dimension in M_i).

Proof. Let $A = \bigotimes (A_i, p_i)$ with $A_i \cong \mathscr{L}\mathscr{C}(\mathscr{H}_i)$, $\dim \mathscr{H}_i = n_i$, $\dim p_i = 2$, and let π be as in Lemma 7.2. For each i, let π_i be the restriction of π to A_i as in § 3 (regarding $A = A_i \otimes [\bigotimes_{j \neq i} (A_j, p_j)]$), and set $M_i = \pi_i(A_i)''$. $\{M_i\}$ is a generating system for M. Put $P_i = \pi_i(p_i)$; then $\inf P_i = \pi(\bigotimes p_i) \neq 0$, so $\{P_i\}$ is fundamental by Lemma 7.1.

THEOREM 7.4. *Let $A = \bigotimes (A_i, p_i)$ $(i \in I)$,* $\operatorname{card} I = n$. *Let $\pi_i \in \hat{A}_i$, $\pi_i \in \hat{A}_i^{p_i}$* a.e., *with $n_i =$* $\dim \pi_i \leqq n$. Set $I_1 = \{i: \dim \pi_i(p_i) \geqq 2\}$; *suppose* $\operatorname{card} I_1 = n$. *Then, if M is any maximally infinite hyperfinite factor with $w(M) = n$, there is a representation π of A with $\pi(A)'' \cong M$, such that the restriction of to A_i is a multiple of π_i.*

Proof. Let $I_2 = \{i: \dim \pi_i(p_i) < 2\}$; $\operatorname{card} I_2 \leqq n$. Set $A_1 = \bigotimes_{i \in I_1} (A_i, p_i)$, $A_2 = \bigotimes_{i \in I_2} (A_i, p_i)$. Form $\pi_2 \in \hat{A}_2$ on $\mathscr{H}$ as in Lemma 4.4. If π_1 is a representation of A_1 with $\pi_1(A_1)'' \cong M$, then let $\pi =$

$\pi_1 \otimes \pi_2$ be the corresponding representation of $A = A_1 \otimes A_2$. $\pi(A)'' \cong M \overline{\otimes} \mathscr{L}(\mathscr{H}) \cong M$ since $\dim \mathscr{H} \leqq n$. Thus, we may assume $I_1 = I$. Let M_i and P_i be as in Lemma 7.3, and let $\mathscr{H}_i$ be the Hilbert space of π_i. Since $\dim \pi_i(p_i) \geqq 2$, we may identify $\mathscr{L}(\mathscr{H}_i)$ with M_i in such a way that $P_i \leqq \pi_i(p_i)$. Let M act on a Hilbert space $\mathscr{K}$; set $Q_F = \inf_{i \notin F} \{P_i\}$, and let $\mathscr{K}_F = Q_F \mathscr{K}$. Since $\{P_i\}$ is fundamental, $\bigcup \mathscr{K}_F$ is dense in $\mathscr{K}$. Let $M_F = \{M_i : i \in F\}'' \cong \overline{\bigotimes}_{i \in F} M_i$, a type I factor.

For each i, we may write $\mathscr{K} \cong \mathscr{H}_i \otimes \mathscr{V}_i$, $M_i \cong \mathscr{L}(\mathscr{H}_i) \otimes 1$ using the above identification of M_i and $\mathscr{L}(\mathscr{H}_i)$; more generally, we can write $\mathscr{K} \cong \mathscr{H}_F \otimes \mathscr{V}_F$, $M_F \cong \mathscr{L}(\mathscr{H}_F) \otimes 1$ with $\mathscr{H}_F = \bigotimes_{i \in F} \mathscr{H}_i$. Let ρ_F be the representation $(\bigotimes_{i \in F} \pi_i) \otimes 1$ of A_F on K. If $E \supseteq F$, $Q_E \in \{M_i : i \in F\}'$, so $\mathscr{K}_E$ is an invariant subspace for ρ_F. Also, if $a \in A_F$, $\rho_E(\sigma_{FE}(a))$ is of the form $\rho_F(a) \otimes (\bigotimes_{i \in E \sim F} \pi_i(p_i) \otimes 1)$. But $P_i \leqq \rho_i(p_i)$ for each i, so $\rho_E(\sigma_{FE}(a))$ and $\rho_F(a)$ agree on $\mathscr{K}_F = (\prod_{i \in E \sim F} P_i)\mathscr{K}_E)$. Thus, the subspaces $\mathscr{K}_F$ and the representations ρ_F satisfy the conditions of Lemma 4.3, so we may form the corresponding representation π of A on $\mathscr{K}$. The restriction of π to A_i is ρ_i, so $\pi(A)'' \supseteq \rho_i(A_i)'' = M_i$; Thus $\pi(A)'' \supseteq M$. But if $a \in A_F$, $\pi(\sigma_F(a)) = R_F \rho_F(a)$, where $R_F = \inf_{i \notin F} \rho_i(p_i)$, so $\pi(\sigma_F(a)) \in M$, $\pi(A)'' \subseteq M$.

This theorem is closely related to Marechal's theorem [8] which states that, if B is a separable C^*-algebra which is not type I, and if M is an infinite hyperfinite factor with separable predual, there is a representation π of B with $\pi(B)'' \cong M$. If A is a infinite tensor product, Theorem 7.4 gives an explicit construction of such a representation, whereas Marechal's method is somewhat nonconstructive.

If B is a C^*-algebra, we put an equivalence relation $\sim$ on $\hat{B}$ by letting $\pi \sim \rho$ if $\ker \pi = \ker \rho$ and $\pi(B)'' \cong \rho(B)''$. This equivalence relation is much weaker than quasi-equivalence (for example, it will not distinguish between irreducible representations with the same kernel), but if B is not type I this relation is probably the strongest one for which there is any reasonable hope of understanding the set of equivalence classes. If $A = \bigotimes (A_i, p_i)$, where each A_i is separable and type I and where there are only countably many i (so A is separable), then A is nuclear, so every factor representation of A generates a hyperfinite factor. Thus, Theorem 7.4 and Theorem 6.5 together give an explicit construction of a representative of each equivalence class of $\hat{A}$ under $\sim$.

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