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PRODUCTS OF BANACH SPACES THAT ARE UNIFORMLY ROTUND IN EVERY DIRECTION

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It is shown that the product of a collection of Banach spaces that are uniformly rotund in every direction (URED) over a URED Banach space need not be URED; this answers a question raised by M. M. Day. A positive result under an additional hypothesis is also proved.

Introduction. A Banach space B is *uniformly rotund in every direction* (URED) if and only if, for every nonzero member z of B and $\varepsilon > 0$, there exists a $\delta > 0$ such that $||(1/2)(x + y)|| \leqq 1 - \delta$ whenever $||x|| = ||y|| = 1$, $x - y = \alpha z$ and

$$||x - y|| \geqq \varepsilon .$$

This generalization of uniform rotundity was introduced by Garkavi [3] to characterize Banach spaces in which every bounded subset has at most one Čebyšev center. Zizler [6] and Day, James, and Swaminathan [2] have investigated this geometrical notion more fully. The purpose of this note is to answer negatively the following question raised by M. M. Day [1, p. 148]: Is the product of a collection of URED Banach spaces over a URED Banach space still URED? In §1, a positive result is proved under an additional hypothesis; the counterexample, §2, is present exactly when this hypothesis fails.

Let S be an index set. A *full function space X on S* is a Banach space of real valued functions f on S such that for each f in X, each function g for which $|g(s)| \leqq |f(s)|$ for each s in S is again in X and $||g|| \leqq ||f||$.

Note that X has a natural Banach lattice structure with positive cone $\{f \in X: f(s) \geqq 0 \text{ for all } s \in S\}$ and that X is order complete by its fullness. It follows easily from theorems of Lotz [4, p. 121] and McArthur [5, p. 5] that the following are equivalent:

(1) X contains no closed sublattice order isomorphic to ℓ^∞.

(2) Each order interval in X is compact.

If for each s in S, a Banach space B_s is given, let $P_X B_s$, the *product of the B_s over X*, be the space of all those functions x on S such that (i) $x(s)$ is in B_s for each s in S, and (ii) if f is defined by $f(s) = ||x(s)||$ for all s in S, then f is in X. For each x in $P_X B_s$, define $||x|| = ||f||_X$. With the above definitions, $(P_X B_s; ||\cdot||)$ is a Banach space.

1. A positive result. The question of whether the product of a collection of URED spaces is isomorphic to a URED space was considered in [2, p. 1056]. There, it was shown that $P_X B_s$ is isomorphic to a URED space if each B_s is URED, and if either (i) S is countable or (ii) $X = \ell_p(S)$ for $1 \leqq p < \infty$. Here, the isometric question raised by Day is considered.

THEOREM. *The product space $P_X B_s$ is uniformly rotund in the direction z if each B_s and X is* URED *and the order interval $[0, \{||z(s)||\}]$ is compact.*

Proof. Let z be a nonzero member of $P_X B_s$ for which the order interval $[0, \{||z(s)||\}]$ is compact. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $P_X B_s$ such that $||x_n|| = ||y_n|| = 1$, $||x_n + y_n|| \rightarrow 2$ and $x_n - y_n = \alpha_n z$. Then

$$||x_n - \eta\alpha_n z|| \longrightarrow 1 \quad \text{if} \quad 0 \leqq \eta \leqq 1 .$$

Define sequences $\{f_n\}$ and $\{g_n^\theta\}$, for $\theta = (1/2), 1$, by letting

$$f_n(s) = ||x_n(s)|| \quad \text{and} \quad g_n^\theta(s) = ||x_n(s) - \theta\alpha_n z(s)||$$

for s in S. Then $||f_n|| = 1$ and $||g_n^\theta|| \rightarrow 1$. Since $||2x_n(s) - \theta\alpha_n z(s)|| \leqq f_n(s) + g_n^\theta(s)$ for each s and $||2x_n - \theta\alpha_n z|| \rightarrow 2$, we have

$$||f_n + g_n^\theta|| \longrightarrow 2 .$$

For each n and s, note that $|f_n(s) - g_n^\theta(s)| \leqq ||\theta\alpha_n z(s)||$. By the compactness hypothesis, there exist h^θ in X and a sequence $\{n_k\}$ such that

$$f_{n_k} - g_{n_k} \longrightarrow h^\theta .$$

Since X is URED, it follows by Theorem 1 of [2] that $h^\theta = 0$. Thus $||x_n(s)|| - ||x_n(s) - \theta\alpha_n z(s)|| \rightarrow 0$ for each s in S and $\theta = (1/2), 1$. Choosing s such that $z(s) \neq 0$ and using the fact that B_s is URED, we conclude that $\alpha_n \rightarrow 0$. This completes the proof.

The following result is an immediate consequence of the theorem and the above remarks concerning full function spaces.

COROLLARY. *The product space $P_X B_s$ is* URED *if each B_s and X is* URED *and X contains no closed sublattice order isomorphic to ℓ^∞.*

2. The counterexample. An equivalent full function space norm $|||\cdot|||$ on ℓ^∞ that is URED and a sequence $\{B_i\}$ of URED

Banach spaces are defined such that, for $X = (\ell^\infty; |||\cdot|||)$, the product space $P_X B_i$ is not URED.

Let $\{a_j\}_{j=2}^\infty$ be a sequence of positive real numbers such that $\sum_2^\infty a_j^2 = 1$. For $x = (x_j)_{j=1}^\infty$ an element of ℓ^∞, define

$$|||x||| = [||x||_\infty^2 + \sum_2^\infty a_j^2(|x_1| + |x_j|)^2]^{1/2} .$$

It is straightforward to verify that $|||\cdot|||$ is a norm on ℓ^∞ and that $||\cdot||_\infty \leqq |||\cdot||| \leqq \sqrt{5}\, ||\cdot||_\infty$. Also note that $|||x||| = |||\,|x|\,|||$ and that $0 \leqq x \leqq y$ implies $|||x||| \leqq |||y|||$ for all x and y in ℓ^∞. Therefore $|||\cdot|||$ is an equivalent full function space norm on ℓ^∞.

To show $(\ell^\infty; |||\cdot|||)$ is URED, let z be a member of ℓ^∞ such that $|||z||| = 1$. If $|||x||| = |||y||| = 1$, where $y = x + \alpha z$, then $x + y = 2x + \alpha z$ and

$$\begin{aligned}
|||2x + \alpha z|||^2 &= ||2x + \alpha z||_\infty^2 + \sum_2^\infty a_j^2(|2x_1 + \alpha z_1| + |2x_j + \alpha z_j|)^2 \\
&\leqq (||x||_\infty + ||x + \alpha z||_\infty)^2 + \sum_2^\infty a_j^2(|x_1| + |x_1 + \alpha z_1| + |x_j| + |x_j + \alpha z_j|)^2 \\
&= 4 - [(||x||_\infty - ||x + \alpha z||_\infty)^2 \\
&\quad + \sum_2^\infty a_j^2(|x_1 + \alpha z_1| + |x_j + \alpha z_j| - |x_1| - |x_j|)^2] ,
\end{aligned}$$

and hence

$$(1) \qquad \left[1 + \left|\left|\left|x + \frac{1}{2}\alpha z\right|\right|\right|^2\right]^{1/2} \geqq \frac{1}{2}[(||x||_\infty - ||x + \alpha z||_\infty)^2 + \sum_2^\infty a_j^2(|x_1 + \alpha z_1| + |x_j + \alpha z_j| - |x_1| - |x_j|)^2]^{1/2} .$$

Similarly, using $2(|||x|||^2 + |||x + (1/2)\alpha z|||^2) \leqq 4$, we obtain

$$(2) \qquad \left[1 - \left|\left|\left|x + \frac{1}{4}\alpha z\right|\right|\right|^2\right]^{1/2} \geqq \frac{1}{2}\left[\left(||x||_\infty - \left\|x + \frac{1}{2}\alpha z\right\|_\infty\right)^2 + \sum_2^\infty a_j^2\left(\left|x_1 + \frac{1}{2}\alpha z_1\right| + \left|x_j + \frac{1}{2}\alpha z_j\right| - |x_1| - |x_j|\right)^2\right]^{1/2} .$$

It is sufficient to show that for each $\varepsilon > 0$ the sum of the right members of (1) and (2) is bounded from zero, uniformly for all x such that $|||x||| = |||x + \alpha z||| = 1$ with $|\alpha| > \varepsilon$.

(i) If $z_1 = 0$, choose any k with $z_k \neq 0$. Then at least one of $|(|x_k + \alpha z_k| - |x_k|)|$ or $|(|x_k + (1/2)\alpha z_k| - |x_k|)|$ is as great as $2^{-2}|\alpha z_k|$, so either the right member of (1) or the right member of (2) is greater than $2^{-3}a_k\varepsilon|z_k|$.

(ii) If $z_1 \neq 0$ and $|z_k| < 2^{-3}|z_1|$ for some k, then either $|(|x_1 + \alpha z_1| - |x_1|)|$ or $|(|x_1 + (1/2)\alpha z_1| - |x_1|)|$ is as great as $2^{-2}|\alpha z_1|$, but

$|(|x_k + \alpha z_k| - |x_k|)| < 2^{-3}|\alpha z_1|$ and $|(|x_k + (1/2)\alpha z_k| - |x_k|)| < 2^{-4}|\alpha z_1|$, so either the right member of (1) or the right member of (2) is greater than $2^{-4}a_k\varepsilon|z_1|$.

(iii) If $z_1 \neq 0$ and $|z_j| \geqq 2^{-3}|z_1|$ for all j, then either

$$\left.\begin{aligned} &|(\|x\|_\infty - \|x + \alpha z\|_\infty)| > 2^{-5}\varepsilon|z_1| \\ \text{or} \quad & \\ &\left|\left(\|x\|_\infty - \left\|x + \frac{1}{2}\alpha z\right\|_\infty\right)\right| > 2^{-5}\varepsilon|z_1| \,, \end{aligned}\right\} \tag{3}$$

and so either the right member of (1) or the right member of (2) is greater than $2^{-6}\varepsilon|z_1|$. To prove (3), we need only observe that if $|(\|x\|_\infty - \|x + (1/2)\alpha z\|_\infty)| < 2^{-5}|\alpha z_1|$ and j is chosen so that $|(\|x\|_\infty - |x_j + (1/2)\alpha z_j|)| < 2^{-5}|\alpha z_1|$, then $|x_j + (1/2)\alpha z_j| > |x_j| - 2^{-2}|\alpha z_j|$ and hence $|x_j + \alpha z_j| = |x_j + (1/2)\alpha z_j| + (1/2)|\alpha z_j|$. Thus

$$\|x + \alpha z\|_\infty > \|x\|_\infty - 2^{-5}|\alpha z_1| + \frac{1}{2}|\alpha z_j| > \|x\|_\infty + 2^{-5}\varepsilon|z_1| \,.$$

This shows that $|||\cdot|||$ is URED.

Now, let $X = (\ell^\infty; |||\cdot|||)$ and for each positive integer i, let B_i be the two dimensional ℓ^{i+1} space. Note that each B_i is URED. Let z in $P_X B_i$ be defined by $z(i) = (1, 0)$ in B_i for each i. For each $n \geqq 2$, let x_n and y_n in $P_X B_i$ be defined by

$$x_n(i) = \begin{cases} (0, 0) & \text{if } i = 1 \\ \left(\frac{1}{2}, b_n\right) & \text{if } i = n \\ (1, 0) & \text{if } i \neq 1, n \end{cases}$$

and

$$y_n(i) = \begin{cases} (-1, 0) & \text{if } i = 1 \\ \left(-\frac{1}{2}, b_n\right) & \text{if } i = n \\ (0, 0) & \text{if } i \neq 1, n \end{cases}$$

where b_n is chosen such that $b_n > 0$ and $(1/2)^{n+1} + (b_n)^{n+1} = 1$. Then $\|x_n\| = \sqrt{2}$, $\|y_n\| = (2 + 3a_n^2)^{1/2}$,

$$\|x_n + y_n\| = [4b_n^2 + 4 + (4b_n^2 + 4b_n - 3)a_n^2]^{1/2} \,,$$

and $x_n - y_n = z$ for each $n \geqq 2$. Since $b_n \to 1$ and $a_n \to 0$, it follows that $\|y_n\| \to \sqrt{2}$ and $\|x_n + y_n\| \to 2\sqrt{2}$, and hence $P_X B_i$ is not URED.

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