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TORSION FREE ABELIAN GROUPS QUASI-PROJECTIVE OVER THEIR ENDOMORPHISM RINGS II

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Let R be a commutative ring with 1, and X an R-module. Then $M = X \oplus R$ is quasi-projective as an E-module, where E is either $\mathrm{Hom}_Z(M, M)$ or $\mathrm{Hom}_R(M, M)$. In this note it is shown that any torsion free abelian group G of finite rank, quasi-projective over its endomorphism ring, is quasi-isomorphic to $X \oplus R$, where R is a direct sum of Dedekind domains and X is an R-module.

Introduction. If R is a ring with identity, an R-module M is said to be quasi-projective if for any submodule N of M, and R-map $f: M \to M/N$, there is an R-map $\bar{f}: M \to M$ such that $\bar{f}$ followed by the factor map $M \to M/N$ is equal to f. Results on quasi-projective modules appear in [3], [6], and [7]. In this note, we will be concerned with the case where $M = G$ is a torsion free abelian group of finite rank and $R = \mathrm{Hom}_Z(G, G) = E(G)$, and call G "*Eqp*" if G is quasi-projective as an $E(G)$-module. The strongly indecomposable *Eqp* groups have been characterized in [6], so we will focus on those groups G (always torsion free abelian of finite rank) such that $nG \subseteqq G_1 \oplus G_2 \subseteqq G$ for some integer $n \neq 0$ and subgroups $G_1 \neq 0$, $G_2 \neq 0$ of G. In fact, any group G can be quasi-decomposed into a direct sum of strongly indecomposable summands, $nG \subset G_1 \oplus G_2 \oplus \cdots \oplus G_k \subset G$. It is well-known that such a decomposition is unique up to order and the quasi-isomorphism class of the summands. It is therefore desirable to work with a slightly more general notion of quasi-projectivity which is invariant under quasi-isomorphism:

DEFINITION. An R-module M is almost quasi-projective (*aqp*) if there exists an integer $n \neq 0$ such that given any submodule N of M, and R-map $f: M \to M/N$, there is an R-map $\bar{f}: M \to M$ such that $\bar{f}$ followed by the factor map $M \to M/N$ is equal to nf.

In case M is a group G and $R = E(G)$, G is called *almost E-quasi-projective* (*aEqp*).

PROPOSITION 1. *Let G and H be quasi-isomorphic groups* (notation: $G \sim H$). *If G is aEqp, then H is aEqp.*

Proof. Assume that $mG \subseteqq H \subseteqq G$ for some integer $m \neq 0$. Then if $\alpha \in E(G)$, $m\alpha|_H \in E(H)$; and if $\beta \in E(H)$, $\beta m \in E(G)$, so we say

$mE(G) \subseteqq E(H)$ and $mE(H) \subseteqq E(G)$. Now let K be a fully invariant subgroup ($E(H)$-submodule) of H, and $f: H \rightarrow H/K$. Then $K^* = E(G)(K)$ satisfies $mK^* \subset K \subset K^*$ and f induces $f^*: G \rightarrow G/K^*$ via $f^*(x) = f(mx) + K^*$. By assumption this lifts to a map $g \in E(G)$ such that $\pi g = nf^*$, where $\pi: G \rightarrow G/K^*$ is the natural factor map. Let $y \in H$. Then $g(y) = nf^*(y) \bmod K^* = nf(my) \bmod K^* = nmf(y) \bmod K^*$. This implies $mg(y) = nm^2 f(y) \bmod K$, so that mg is a lifting of $nm^2 f$ and H is $aEqp$.

By the preceding proposition we may, without loss of generality, work with a group $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ where each G_i is strongly indecomposable. The following notation is also used:

$E = E(G) = \mathrm{Hom}_Z(G, G)$.

$E_i = E(G_i)$.

$J_i = J(E_i) =$ Jacobson radical of E_i.

$EG_i = E(G)G_i = E$-submodule of G generated by G_i.

Now, a sequence of lemmas leads to the main result.

LEMMA 2. *Suppose G is E-indecomposable. Then any E-map of G into G* (any map in the center of E) *is either monic or nilpotent.*

Proof. Let f be an E-map of G into G. Then $f = \bigoplus_{i=1}^n f_i$ where $f_i: G_i \rightarrow G_i$ is monic or nilpotent (see [4]). Let $H_1 = \bigoplus G_i f_i$ is nilpotent and $H_2 = \bigoplus G_j f_j$ is monic. Since G is E-indecomposable, there is a nonzero map between H_1 and H_2 (or H_2 and H_1), say $h: H_1 \rightarrow H_2$, $h \neq 0$. Letting $g_1 = \bigoplus f_i$ f_i nilpotent and $g_2 = \bigoplus f_j$ f_j monic, we have $g_2 h = h g_1$ so that $g_2^k h = h g_1^k$ for all $k > 0$. Since g_1 is nilpotent, $g_2^k h = 0$ for some $k > 0$. Since g_2 is monic, this says $h = 0$, a contradiction.

LEMMA 3. *Let G be $aEqp$ and E-indecomposable. Then for any nontrivial decomposition $G = H \oplus K$, either $EH \sim G$ or $EK \cap H \sim H$.*

Proof. Suppose the conclusion is false. Then the map given by the identity on $H/EK \cap H$ and zero on $K/K \cap EH$ is an E-map and can be quasi-lifted to an E-endomorphism of G. But the lifting can be neither monic nor nilpotent, contradicting Lemma 2,

PROPOSITION 4. *Let G be $aEqp$ and E-indecomposable. Then for each G_i, either G/EG_i is bounded or there is a $j \neq i$ such that $G_i/EG_j \cap G_i$ is bounded.*

Proof. Without loss of generality, assume $i = 1$. By Lemma

3, either G/EG_1 or $G_1/E(\bigoplus_{i=2}^n G_i) \cap G_1$ is bounded. In the latter case, let $H_1 = EG_2 \cap G_1$ and $H_2 = E(\bigoplus_{i=3}^n G_i) \cap G_1$. Then $(H_1 \cap H_2) \oplus [E(\bigoplus_{i=3}^n G_i) \cap G_2] \oplus (EG_2 \cap \bigoplus_{i=3}^n G_i) = K$ is an E-submodule of G, and if $G_1/H_1 \cap H_2$ has a nontrivial quasi-decomposition, then the quasi-projections can be extended to E-maps of G/K into G/K which can be quasi-lifted to E-endomorphisms of G. Again, the liftings can be neither monic nor nilpotent, contradicting Lemma 2. Therefore, $G/H_1 \cap H_2$ has no nontrivial quasi-decompositions, so that either $H_1 \cap H_2 \sim H_1$ or $H_1 \cap H_2 \sim H_2$, since $G_1/H_1 + H_2$ is bounded.

Case I. If $H_1 \cap H_2 \sim H_2$, then $H_1 \sim G_1$ and we are done.

Case II. If $H_1 \cap H_2 \sim H_1$, then $H_2 \sim G_1$.

In this case, let $H_1' = EG_3 \cap G_1$ and $H_2' = E(\bigoplus_{i=4}^n G_i) \cap G_1$ and let $K' = H_1' \cap H_2' \oplus E(\bigoplus_{i=4}^n G_i) \cap EG_3 \cap G_2 \oplus E(\bigoplus_{i=4}^n G_i) \cap G_3 \oplus EG_3 \cap \bigoplus_{i=4}^n G_i$. Then it is straightforward to check that K' is fully invariant and that, as in the first paragraph, quasi-projections of $G_1/H_1' \cap H_2'$ can be extended to E-maps of G/K' into G/K', which quasi-lift to maps in E. (It follows from Lemma 3 that $E(\bigoplus_{i=3}^n G_i) \cap G_2 \sim G_2$.) Thus as before, either $H_1' \cap H_2' \sim H_2'$ or $H_1' \cap H_2' \sim H_1'$. In Case I we are done, and in Case II we may repeat the above argument with slight modifications to eventually get $G_1/G_1 \cap EG_j$ bounded for some j.

COROLLARY 5. *There is a G_i such that G/EG_i is bounded.*

Proof. By the preceding proposition, either G/EG_1 is bounded or $EG_{i_1} \cap G_1 \sim G_1$ for some i_1. Then either G/EG_{i_1} is bounded or $EG_{i_2} \cap G_{i_1} \sim G_{i_1}$ for some i_2. Inductively obtain a sequence $1, i_1, i_2, \cdots, i_{n-1}$ such that $EG_{i_k} \cap G_{i_{k-1}} \sim G_{i_{k-1}}$. (Unless the process stops, in which case G/EG_{i_k} is bounded for some k.) It follows that $G/EG_{i_{n-1}}$ is bounded.

Henceforth the G_i of Corollary 5 will be denoted by G_0. That is, G/EG_0 is bounded.

LEMMA 6. *If $G_i/EG_0 \cap G_i$ is bounded, then either $G_i \sim G_0$ or $EG_i \cap G_0 \subseteqq J_0G_0$.*

Proof. Consider $G_0 \xrightarrow{f} G_i \xrightarrow{g} G_0$. If gf is monic, then $kg^{1-}f$ has an inverse in $E(G_0)$ for some $0 < k \in Z$. Then $G_0 \xrightarrow{f} G_i \xrightarrow{g} G_0 \xrightarrow{(k^{-1}gf)^{-1}} G_0$ gives a quasi-splitting of G_i. Since G_i is strongly indecomposable, $G_0 \sim G_i$.

On the other hand, if gf is nilpotent for all possible g and f, then $EG_i \cap G_0 \subseteqq J_0G_0$ since $EG_0 \cap G_i$ is of bounded index in G_i.

LEMMA 7. $G_0/J_0G_0 \sim E_0/J_0$.

Proof. Let $\bar{E}_0 = E_0/J_0$. Then $Q \otimes_Z \bar{E}_0$ is a division ring. Let $\bar{x}_1 = x_1 + J_0G_0, \cdots, \bar{x}_r = x_r + J_0G_0$ be a maximal $\bar{E}_0$-independent set in $G_0/J_0G_0 = \bar{G}_0$. Then $A = \bar{G}_0/\sum_{i=1}^r \bar{E}_0\bar{x}_i$ is torsion and furthermore must be bounded. If A were unbounded it would have uncountably many endomorphisms which would have to be induced by different endomorphisms of G. Now consider $\bar{G}_0/\bar{E}_0\bar{x}_1 \cap \sum_{i=2}^r \bar{E}_0\bar{x}_i$. If $r \geqq 2$, this group has a nontrivial quasi-decomposition, and the quasi-projections can be lifted to maps in E which are neither monic nor nilpotent, a contradiction. Thus $r = 1$ and $\bar{G}_0 \sim \bar{E}_0\bar{x}_1 \cong \bar{E}_0$.

LEMMA 8. *Let Z_0 be the center of E_0. Then $Z_0 + J_0 = E_0$.*

Proof. For any $x \in E_0$, right multiplication by x is an E_0-map $E_0/J_0 \xrightarrow{x_r} E_0/J_0$. Using the previous lemma, x_r quasi-lifts to an E_0 map of G_0, $\hat{x}_r$, which is in Z_0 since it is an E_0-map. Clearly $\hat{x}_r - x \in J_0$.

The next lemma is well-known but is included for completeness.

LEMMA 9. *Let E be a ring with identity and nilpotent ideal J. Let M be an E-module and L a submodule such that $L+JM=M$. Then $L = M$.* (This says JM is small in M.)

Proof. $J(L + JM) = JM \Rightarrow JM \subseteqq L + J^2M \Rightarrow M = L + J^2M \Rightarrow M = L + J^kM$ for all $k > 0$ by induction. Since J is nilpotent, $M = L$.

PROPOSITION 10. *G_0 is strongly irreducible, and hence $G_0 \sim E_0 = Z_0$.*

Proof. Choose a subring S of E_0 maximal with respect to

$$(1)\quad S \cap J_0 = 0 \qquad (2)\quad S \subset Z_0 \qquad (3)\quad Q \cap E_0 \subset S\,.$$

Note that S is a pure subgroup of E_0 and is an integral domain. Suppose $z_0 \notin S + J_0$ for some $z_0 \in Z_0$. Then $S[z_0]$ properly contains S and satisfies (1), (2), and (3), a contradiction. Thus $Z_0 \subset S + J_0 \Rightarrow S \oplus J_0 = E_0$.

Now from the proof of Lemma 7 it follows that $G_0 \sim E_0x_1 + J_0G_0$ for some $x_1 \in G_0$. Hence, by Lemma 9, $G_0 \sim E_0x_1$, and $K = \operatorname{Ker}(E_0 \to E_0x_1) \subseteqq J_0$. Thus $G_0 \sim E_0/K = S \oplus J_0/K$. Since G_0 is strongly indecomposable, $G_0 \sim S$. Therefore G_0 is strongly irreducible, and hence $G_0 \sim E_0 = Z_0$ by the results of [5].

It now follows from Lemma 6 that for any G_i, either $G_i \sim G_0$

or $\mathrm{Hom}(G_i, G_0) = 0$. Thus, up to quasi-isomorphism and relabeling, $G = \bigoplus_{j=1}^{k} H_j \bigoplus_{i=1}^{l} G_i$ where $H_j = G_0$, $1 \leqq j \leqq k$ and $\mathrm{Hom}(G_i, G_0) = 0$, $1 \leqq i \leqq l$.

In the following, let $G' = \bigoplus_{i=1}^{l} G_i$, an E-submodule of G. For any map $\phi \in E_0 = E(G_0)$, $\underbrace{\phi \oplus \cdots \oplus \phi}_{k \text{ times}}$ is an E-map of G/G' into G/G' and hence can be quasi-lifted to an E-map, ψ, of G. The map ψ is unique, since if ψ' were another lifting $(\psi - \psi')(\bigoplus_{j=1}^{k} H_j) = 0$, so that $\psi - \psi' = 0$ because $EH_j \cap G_i \sim G_i$ for all i, j. Since ψ commutes with projections, $\psi(G_i) \subseteq G_i$ for each i. Thus a *ring isomorphism* $0 \rightarrow E_0 \rightarrow E(G_i)$ is obtained via $\phi \rightarrow \psi|_{G_i}$. This yields a unitary E_0-module structure on G_i.

Now if R_0 is the ring of integers in $Q \otimes E_0$, then $G_0 \sim E_0 \sim R_0 \otimes E_0$, and $R_0 \otimes E_0$ is a Dedekind domain.

We are now ready for the main result.

THEOREM 11. *If G is a torsion free abelian group of finite rank, then G is a Eqp if and only if $G \sim R \oplus X$, where R is a direct sum of Dedekind domains, and X is a unitary R-module.*

Proof. The "if" direction has been demonstrated.

If G is E-indecomposable, let $R = R_0 \otimes \bigoplus_{j=1}^{k} E_0$ and $X = R_0 \otimes \bigoplus_{i=1}^{l} G_i$ in the notation of the preceding lemma and remarks. The general case follows by taking direct sums.

REMARK. If $G = R \oplus X$ in the above, it is clear that G is actually *Eqp*. It would be nice to know exactly which quasi-isomorphic images of G were also *Eqp*.

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